数理解析研究所講究録
第 1856 巻 2013 年 113-122

Entire Solutions of Elliptic Equations with Exponential Nonlinearity*

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Abstract

We consider the elliptic equation $\Delta u+K(|x|)e^{u}=0$ in $\mathbf{R}^{n}\backslash\{0\}$ with $n>2$, when for $\ell>-2$, $r^{-\ell}K(r)$ behaves monotonically near 0 or ∞. The method of phase plane in [1] is useful in analyzing the structure of positive radial solutions, and the asymptotic behavior at ∞. The approach leads to the existence of singular solutions, and verifies the asymptotic behavior at 0.

Key Words: semilinear elliptic equations; exponential nonlinearity; entire solution; asymptotic behavior; separation; singular solution.

1. Introduction

We study the elliptic equation

$$\Delta u+K(|x|)e^{u}=0, \tag{1.1}$$

where $n>2$, $\Delta=\sum_{i=1}^{n}\frac{\partial^{2}}{\partial x_{i}^{2}}$ is the Laplace operator, and K is a continuous function in $\mathbf{R}^{n}\backslash\{0\}$. Radial solutions of (1.1) satisfy the equation

$$u_{rr}+\frac{n-1}{r}u_{r}+K(r)e^{u}=0 \tag{1.2}$$

where $r=|x|$. Under the following condition:

$$(\mathrm{K})\left\{\begin{array}{l}K(r)\text{ is continuous on }(0,\infty),\\ K(r)\geq 0\text{ and }K(r)\not\equiv 0\text{ on }(0,\infty),\\ \int_{0}rK(r)\,dr<\infty.\end{array}\right.$$

(1.2) with $u(0)=\alpha\in\mathbf{R}$ has a unique solution $u\in C^{2}(0,\varepsilon)\cap C[0,\varepsilon)$ for small $\varepsilon>0$. By $u_{\alpha}(r)$ we denote the unique local solution with $u_{\alpha}(0)=\alpha$. A typical example is the equation

$$u_{rr}+\frac{n-1}{r}u_{r}+cr^{\ell}e^{u}=0, \tag{1.3}$$

where $c>0$ and $\ell>-2$. The scale invariance of (1.3) is explained by

$$u_{\alpha}(r)=\alpha+u_{0}(e^{\frac{\alpha}{2+\ell}}r),$$

*This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2012-0002056).

and the invariant singular solution is given by

$$u_s(r) = -(2+\ell)\log r + \log(2+\ell)(n-2) - \log c.$$

We call this behavior self-similarity. In fact, for every α, $u_\alpha(r) = u_s(r) + o(1)$ at ∞. For more general equation (1.2), we look for entire solutions u_α of satisfying

$$\liminf_{r\to\infty}[u_\alpha(r) + (2+\ell)\log r] > -\infty. \tag{1.4}$$

We show the following existence result by making use of the method of phase plane [1].

Theorem 1.1. *Let $n > 2$ with $\ell > -2$. Assume that K satisfies* (K) *and $r^{-\ell}K(r)$ is non-increasing in $(0,\infty)$. For every $\alpha \in \mathbf{R}$, (1.2) has an entire solution u_α with (1.4).*

When $r^{-\ell}K(r)$ converges to a positive constant at ∞, $u_\alpha(r)$ is asymptotically self-similar.

Theorem 1.2. *Let $n > 2$ and $\ell > -2$. Assume that K satisfies* (K) *and $r^{-\ell}K(r) \to c$ as $r \to \infty$ for some $c > 0$. Then, every solution u of (1.2) near ∞ satisfies*

$$\lim_{r\to\infty}[u(r) - \log\frac{(2+\ell)(n-2)}{cr^{2+\ell}}] = 0, \tag{1.5}$$

provided that $u(r)+(2+\ell)\log r$ does not decrease to 0 near ∞. If $r^{-\ell}K(r)$ is non-increasing in $(0,\infty)$, every entire solution u_α satisfies (1.5).

There have been previous works on the asymptotic behavior. See [4, 6, 8].

The asymptotic behavior for $\ell = -2$ involves $-\log\log$ term.

Theorem 1.3. *Let $n > 2$. Assume that K satisfies* (K) *and $r^{-2}K(r) \to c$ as $r \to \infty$ for some $c > 0$. Then, every solution u of (1.2) near ∞ satisfies*

$$\lim_{r\to\infty}[u(r) - \log\frac{n-2}{c\log r}] = 0, \tag{1.6}$$

provided that $u(r) + \log(\log r)$ does not decrease to 0 near ∞.

It is interesting to ask whether two entire solutions intersect each or not. The property is closely related to stability of solutions as steady states. See [8] for the result on (1.3) with $c = 1$ and $\ell = 0$. We observe separation of solutions for $n \geq 10 + 4\ell$ while intersection for $2 < n < 10 + 4\ell$.

Theorem 1.4. *Let $2 < n < 10+4\ell$ and $\ell > -2$. Assume that K satisfies* (K) *and $r^{-\ell}K(r)$ is non-increasing in $(0,\infty)$. If $r^{-\ell}K(r) \to c$ at ∞ for some $c > 0$, then (1.2) possesses one singular solution and every u_α intersects the singular solution infinitely many times. Any two entire solutions u_α and u_β with $\alpha < \beta$ intersect each other infinitely many times.*

Define $\underline{\mathbf{k}}(r) = \inf_{0<s\leq r} s^{-\ell}K(s)$ and $\underline{\mathbf{K}}(r) = r^\ell\underline{\mathbf{k}}(r)$. Let $\mathbf{k}(0) = \lim_{r\to 0} r^{-\ell}K(r)$ if it exists.

Theorem 1.5. *Let $\ell > -2$ and $n \geq 10 + 4\ell$. Assume that K satisfies* (K) *and for $r > 0$,*

$$r^{-\ell}K(r) \leq \delta \underline{\mathbf{k}}(r) \tag{1.7}$$

where $\delta = \frac{n-2}{4(2+\ell)}$. Then, (1.2) possesses a singular solution U and any two entire solutions do not intersect each other. Moreover, $U(r)$ satisfies

$$e^{u_\alpha(r)} < e^{U(r)} \leq \frac{b}{r^2 \underline{\mathbf{K}}(r)}, \tag{1.8}$$

where $b = (2+\ell)(n-2)$ and $u_\alpha \to U$ as $\alpha \to \infty$. Moreover, if $r^{-\ell}K(r)$ is non-increasing in $(0, \infty)$, then for each α, $r^{2+\ell}e^{u_\alpha(r)}$ is strictly increasing in r.

Note that $\delta \geq 1$ iff $n \geq 10 + 4\ell$. In fact, (1.7) with $\delta = 1$, i.e., $n = 10 + 4\ell$, means that $r^{-\ell}K(r)$ is non-increasing in $(0, \infty)$.

The motivation of Theorems 1.4-5 is the separation structure for Lane-Emden equation

$$\Delta u + u^p = 0 \tag{1.9}$$

when $p > 1$ is sufficiently large. In fact, when $p \geq p_c$, where

$$p_c = p_c(n, \ell) = \begin{cases} \frac{(n-2)^2 - 2(\ell+2)(n+\ell) + 2(\ell+2)\sqrt{(n+\ell)^2 - (n-2)^2}}{(n-2)(n-10-4\ell)} & \text{if} \quad n > 10 + 4\ell, \\ \infty & \text{if} \quad n \leq 10 + 4\ell, \end{cases}$$

(1.9) has positive entire radial solutions and any two solutions among them do not intersect. Hence, it is natural to expect that (1.1) possesses the property for $n > 10 + 4\ell$. See [2, 3, 5, 7, 9] for the separation structure and the original paper [6] for p_c.

Our next goal is to study singular solutions which diverge to ∞ as r approaches 0. We take two steps for the existence singular solutions. At first, we consider the case that $k = r^{-\ell}K(r)$ is a positive constant near 0. Secondly, k has the positive limit at 0, but is strictly decreasing at 0. There exists a unique singular solution which has the similarity asymptotically.

Theorem 1.6. *Let $n > 2$ and $\ell > -2$. Assume that K satisfies* (K) *and $r^{-\ell}K(r) \to c$ as $r \to 0$ for some $c > 0$. Then, positive solution u (1.2) near 0 is unique and satisfies*

$$\lim_{r \to 0} [u(r) - \log \frac{(2+\ell)(n-2)}{cr^{2+\ell}}] = 0, \tag{1.10}$$

provided that $r^{2+\ell}e^{u(r)}$ does not increase from 0 near 0. Moreover, if $r^{-\ell}K(r)$ is non-increasing near 0, then (1.2) has a unique singular solution u_s near 0 which satisfies (1.10) and

$$r^{2+\ell}e^{u_s(r)} \geq \frac{(2+\ell)(n-2)}{c}.$$

2. Preliminaries

In this section, we recall basic facts on (1.2) under the assumption (K). For each $\alpha \in \mathbf{R}$, (1.2) has a unique solution $u \in C^2(0,\varepsilon) \cap C[0,\varepsilon)$ for some $\varepsilon > 0$. This local solution is decreasing and extended entirely.

Proposition 2.1. *Let $n > 2$ and $\ell > -2$. Assume that K satisfies* (K) *and*

$$\lim_{r\to 0} r^2 K(r) = 0. \tag{2.1}$$

Then, every solution u_α of (1.2) with $u_\alpha(0) = \alpha \in \mathbf{R}$ satisfies that

$$u_\alpha(r) - \log \frac{(2+\ell)(n-2)}{r^{2+\ell}}$$

is strictly increasing as long as the relation,

$$r^2 K(r) e^{u_\alpha(r)} < (2+\ell)(n-2). \tag{2.2}$$

holds from $r = 0$.

Let $V(t) := u_\alpha(r) - \log \frac{(2+\ell)(n-2)}{r^{2+\ell}}$, $t = \log r$. Then, V satisfies

$$V_{tt} + aV_t - b(1 - k(t)e^V) = 0, \tag{2.3}$$

where $a = n-2$, $b = (2+\ell)(n-2)$ and $k(t) := e^{-\ell t} K(e^t)$. It follows from (2.1) that

$$\lim_{t\to -\infty} k(t) e^{V(t)} = b^{-1} \lim_{r\to 0} r^2 K(r) e^{u_\alpha(r)} = 0$$

and thus, $ke^V < 1$ near $-\infty$. If $ke^V < 1$ on $(-\infty, T)$ for some T, then by (2.3), we have

$$V_{tt} + aV_t = b(1 - k(t)e^V) > 0 \quad \text{on} \quad (-\infty, T). \tag{2.4}$$

Multiplying (2.4) by e^{at} for $t < \tau \le T$ and integrating from t to τ, we obtain

$$e^{a\tau} V_t(\tau) > e^{at} V_t(t) = e^{at}(r u'_\alpha + 2 + \ell) \tag{2.5}$$

which converges to 0 as $r \to 0$. Hence, we have $e^{a\tau} V_t(\tau) > 0$. Therefore, $V_t > 0$ on $(-\infty, T]$.

In order to obtain the integral equation

$$V_t(t) = e^{-at} \int_{-\infty}^{t} b(1 - k(s)e^V) e^{as}\, ds \tag{2.6}$$

near $t = -\infty$, it suffices to have a sequence going to $-\infty$ in which $e^{at} V_t(t)$ tends to 0.

Lemma 2.2. *Let $n > 2$ with $\ell > -2$. Assume that K satisfies* (K)*. If u is a solution of (1.2) defined in a deleted neighborhood N of $r = 0$ such that $w(r) := u(r) - \log \frac{(2+\ell)(n-2)}{r^{2+\ell}}$ is bounded above and (2.2) holds on N, then $w(r)$ is strictly increasing as long as (2.2) holds.*

In fact, if $r^{-\ell} K(r) \ge c$ near 0 for some $c > 0$, then $w(r)$ is bounded above near 0 for any solution u in a deleted neighborhood of $r = 0$.

3. Entire solution

We consider (2.3) and let $q(V) := V_t(t)$. Then, it follows from Proposition 2.1 that $q(V) \to 2+\ell$ as $V \to -\infty$ and $q > 0$ near $t = -\infty$. Moreover, q satisfies

$$q\frac{dq}{dV} + aq = b(1 - k(V)e^V). \tag{3.1}$$

Here, we may define $k(V)$ as long as $q(V)$ does not change sign, and consider (3.1) on each region where V is defined and q has one sign.

Lemma 3.1. *Let $n > 2$ with $\ell > -2$. Assume K satisfies* (K).

- *If $q \le 0$ on $(\underline{v}, v)$ for some $v > \underline{v}$ and k is non-increasing and $q(\underline{v}) = 0$, then $\underline{v}$ is a local minimum point.*
- *If $q \ge 0$ on $(v, \bar{v})$ for some $\bar{v} > v$ and k is non-decreasing and $q(\bar{v}) = 0$, then $\bar{v}$ is a local maximum point.*

Remark. If $q(v) = 0$ and $1 - k(v)e^v \ne 0$, then v is an extremal point.

3.1. Existence

Let $u_\alpha(r)$ be a local solution of (1.2). Setting $V(t) := u_\alpha(r) - \log\frac{(2+\ell)(n-2)}{r^{2+\ell}}$, $t = \log r$, we see that V satisfies (2.3). By Proposition 2.1, V is defined in a neighborhood of $-\infty$, and V is strictly increasing as t increases as long as the relation $ke^V < 1$ holds. In case V is increasing as t increases from $-\infty$ to $+\infty$, u_α is an entire solution satisfying (1.4).

We consider the case that $T = \sup\{\tau \mid V_t(\tau) \ge 0 \text{ on } (-\infty, \tau)\} < +\infty$. Considering

$$V_t(t) = e^{-at}\int_{-\infty}^{t} b(1 - k(s)e^V)e^{as}\,ds,$$

we see that $1 - k(T)e^{V(T)} \le 0$ since $V_t > 0$ near $-\infty$ and $V_t(T) = 0$. Then, $V_{tt}(T) \le 0$ from (2.3). We first assume that $V_{tt}(T) < 0$. Now, we choose $t_1 > T$ where $t_1 = \sup\{\tau \in (T, +\infty) \mid V_t(\tau) \le 0 \text{ on } (T, \tau)\} \le +\infty$. Let $\bar{v} = V(T)$ and $\underline{v} = V(t_1)$. Suppose $\underline{v} = -\infty$. Since k is non-increasing, there exist T_1 and $c > 0$ such that

$$V_{tt} + aV_t = b(1 - ke^V) \ge c$$

for $t \ge T_1$. Hence, for $t > T_1$,

$$V_t(t) \ge e^{-a(t-T_1)}V_t(T_1) + \frac{c}{a}e^{-aT_1}$$

and thus, V_t should be positive eventually, a contradiction. Therefore, $\underline{v} > -\infty$ and $q(\underline{v}) = 0$. Then, $t_1 < +\infty$. Let $v_1 = \underline{v} = V(t_1) > -\infty$. Since $V_t(t_1) = 0$, it follows from Lemma 3.1 that $1 - k(t_1)e^{V(t_1)} > 0$ and $V_{tt}(t_1) > 0$. When V is increasing in (t_1, T_1) and decreasing in (T_1, t_2) for some $t_1 < T_1 < t_2$, we consider $q_+(V) = V_t(t)$ on $[t_1, T_1]$ and $V_1 = V(T_1)$. Then,

$$q_+\frac{dq_+}{dV} + aq_+ = b(1 - k_+(V)e^V) \quad \text{on } (v_1, V_1), \tag{3.2}$$

where $k_+(V) = k(t)$ for $t_1 \le t \le T_1$. Similarly, let $q_-(V) = V_t(t)$ on $[T_1, t_2]$. Then, $q_+(T_1) = q_-(T_1) = 0$, $q_-(v_2) \le 0$ where $v_2 = V(t_2)$, and

$$q_- \frac{dq_-}{dV} + aq_- = b(1 - k_-(V)e^V) \quad \text{on } (v_2, T_1), \tag{3.3}$$

where $k_-(V) = k(t)$ for $T_1 \le t \le t_2$.

Suppose $v_2 \le v_1$. Then, integrating (3.2) and (3.3) over (v_1, V_1) and subtracting, we have

$$\frac{1}{2} q_-^2(v_1) + a \int_{v_1}^{V_1} (q_+ - q_-)\, dV = b \int_{v_1}^{V_1} (k_- - k_+) e^V \, dV, \tag{3.4}$$

which is a contradiction since $k_+ \ge k_-$. Hence, $v_2 > v_1$. Applying the same arguments to any two consecutive local minimum points of V, we see that the global existence of the local solution u_α satisfying (1.4) since either V is increasing eventually or V is not monotone near $+\infty$. Moreover, in the latter case, setting $\{t_j\}$ be any set of consecutive increasing local minimum points of V, we conclude by employing the arguments that $v_j = V(t_j)$ is non-decreasing as $j \to \infty$. Therefore, u_α is an entire solution satisfying (1.4).

In fact, the monotonicity of local minima is valid even for singular solutions.

Proposition 3.2. *Assume that K satisfies* (K) *and $r^{-\ell}K(r)$ is non-increasing in* $(0, \infty)$. *If u is any solution of (1.2) on* $(0, \infty)$, *then local minima of* $u(r) - \log \frac{(2+\ell)(n-2)}{r^{2+\ell}}$ *can not be decreasing.*

3.2. Asymptotic behavior at infinity

We now study the asymptotic behavior of solutions.

3.2.1. $-(2+\ell)\log$ decay

Lemma 3.3. *Assume $c_1 \le k \le c_2$ for some $c_2 > c_1 > 0$. Then,*

$$\liminf_{t \to +\infty} ke^V \le 1 \le \limsup_{t \to +\infty} ke^V. \tag{3.5}$$

Moreover, if $k \to c > 0$, then we have

$$\liminf_{t \to +\infty} V(t) \le -\log c \le \limsup_{t \to +\infty} V(t). \tag{3.6}$$

Case 1: V is monotone near $+\infty$.
Then, it follows from (3.6) that $1 - ce^d = 0$, and thus $d = -\log c$.

Lemma 3.4. *If $k \to c > 0$ and V is monotone, then V converges to $-\log c$ at $+\infty$.*

Case 2: V_t oscillates near $+\infty$. We argue similarly as in the proof of Theorem 1.1.

Remark. If $k = r^{-\ell}K(r) \ge c > 0$ near ∞, $u(r) + (2+\ell)\log r$ is bounded above near ∞.

3.2.2. $-\log\log$ decay

Let $V(t) := u(r) + \log(\log r), t = \log r$. Then, V satisfies

$$V_{tt} + aV_t - \frac{1}{t}[a - \frac{1}{te^t} - k(t)e^V] = 0, \tag{3.7}$$

where $a = n - 2$ and $k(t) := e^{-2t}K(e^t)$.

Lemma 3.5. *Assume $c_1 \leq k \leq c_2$ for some $c_2 > c_1 > 0$. Then,*

$$\liminf_{t\to+\infty} ke^V \leq a \leq \limsup_{t\to+\infty} ke^V.$$

Therefore, if $k \to c > 0$, then we have

$$\liminf_{t\to+\infty} V(t) \leq \log\frac{a}{c} \leq \limsup_{t\to+\infty} V(t).$$

Case 1: V is monotone near $+\infty$.
Then, $ce^d = a$, and thus $d = \log a - \log c$.

Lemma 3.6. *If $k \to c > 0$ and V is monotone, then V converges to $\log\frac{a}{c}$ at $+\infty$.*

Case 2: V_t oscillates near $+\infty$. We argue in a similar way.

4. Intersection and Separation

When $2 < n < 10 + 4\ell$, we observe the structure of intersection.

Proposition 4.1. *Let $2 < n < 10 + 4\ell$ with $\ell > -2$. Assume that K satisfies* (K) *and $r^{-\ell}K(r) \to c > 0$ as $r \to \infty$. Let u be a solution of (1.2) satisfying (1.5). If ψ is a super-solution (or sub-solution) near ∞ of (1.2) and $\psi \geq$ (or $\leq$)u_α, then $\psi \equiv u$ near ∞.*

When n is large enough, the monotonicity of $u + (2+\ell)\log r$ in r may happen. We consider not only the existence of entire solutions but also their separation property.

If (2.2) is true on $[0, \infty)$, then u_α is a positive solution and $u_\alpha(r) + (2+\ell)\log r$ is increasing as r increases. In fact, the condition that $r^{-\ell}K(r)$ is non-increasing guarantees that this relation is satisfied in the entire space.

Theorem 4.2. *Let $n \geq 10 + 4\ell$ and $\ell > -2$. Suppose that $K(r)$ satisfies* (K) *and $r^{-\ell}K(r)$ is non-increasing. Then, for each α, (1.2) possesses a entire solution u_α with $u_\alpha(0) = \alpha$ such that $u_\alpha(r) + (2+\ell)\log r$ is strictly increasing and (2.2) holds on $[0, \infty)$.*

Proposition 4.3. *Let $n > 2$ and $\ell > -2$. Assume that K satisfies* (K) *and (2.1). Then, for every solution u_α of (1.2) with $u_\alpha(0) = \alpha \in \mathbf{R}$,*

$$r^2\underline{\mathbf{K}}(r)e^{u_\alpha(r)} < b \tag{4.1}$$

holds from $r = 0$.

5. Singular solution

We study the existence of singular solutions of (1.2) when $r^{-\ell}K(r)$ is non-increasing and

$$\lim_{r\to 0} r^{-\ell}K(r) = c$$

for some $0 < c < \infty$. Before discussing the existence, we consider the asymptotic behavior of singular solutions.

5.1. Asymptotic behavior at zero

The arguments of this subsection is similar to those of Subsection 4.2. But, we consider the issue for the completeness.

Lemma 5.1. *Assume $c_1 \le k \le c_2$ for some $c_2 > c_1 > 0$. Then,*

$$\liminf_{t\to-\infty} ke^V \le 1 \le \limsup_{t\to-\infty} ke^V. \tag{5.1}$$

Moreover, if $k \to c > 0$, then we have

$$\liminf_{t\to-\infty} V(t) \le -\log c \le \limsup_{t\to-\infty} V(t). \tag{5.2}$$

Case 1: V is monotone near $+\infty$.
Then, it follows from (5.2) that $1 - ce^d = 0$, and thus $d = -\log c$.

Lemma 5.2. *If $k \to c > 0$ and V is monotone, then V converges to $-\log c$ at $-\infty$.*

Case 2: V_t oscillates near $-\infty$.

Remark. From the proof, we see that if $k = r^{-\ell}K(r) \ge c > 0$ near 0, $u(r) + (2+\ell)\log r$ is bounded above near 0.

5.2. Asymptotically self-similar solution

We look for singular solutions with the behavior

$$\lim_{r\to 0}[u(r) - \log \frac{b}{cr^{2+\ell}}] = 0. \tag{5.3}$$

Setting $\varphi(r) = u(r) - \log \frac{b}{r^{2+\ell}}$, we have

$$\varphi_{rr} + \frac{a+1}{r}\varphi_r - \frac{b}{r^2} + \frac{b}{r^2}k(r)e^{\varphi} = 0, \tag{5.4}$$

where $a+1 = n-1$ and $k(r) = r^{-\ell}K(r)$. If $k \equiv c$, then the obvious solution is $\varphi \equiv -\log c$. Hence, we assume $k \not\equiv c$. In order to confirm the existence of a local positive solution with $\varphi(0) = -\log c$, we first construct the solution when $k(r)$ is constant near 0. Then, we utilize the obtained solutions to verify the existence for the case $r_c = 0$, where $r_c = \inf\{r > 0 \,|\, k(r) < c\}$.

Let $0 < c < \infty$. If k is constant near 0, the obvious solution is $\varphi = -\log c$ near 0 and the existence of local solution near $r = r_c$ is rather standard.

Step 1. Assume that $k(r) := r^{-\ell}K(r) = c > 0$ near 0.

Let $r_c = \sup\{r \geq 0 \,|\, k(r) = c\}$. For given $\delta > 0$, there exists $r_\delta > r_c$ such that $0 < k(r_\delta) < c$ and $|\log k(r_\delta) - \log c| < \delta$.

Theorem 5.3. *Let $n > 2$ and $l > -2$. Assume that $r^{-l}K(r)$ is continuous and $0 < r_c < \infty$ for some $c > 0$. Then, (5.4) with (5.3) has a unique local positive solution $u \in C^2((0, r_c + \varepsilon)) \cap C([0, r_c + \varepsilon))$ for small $\varepsilon > 0$.*

In order to make the local singular solution to be defined on the whole space, we apply the same arguments as in Theorem 1.1 and then conclude the existence of a solution with slow decay.

Now, we consider $V(t) = \varphi(r)$ with $t = \log r$. Then, we claim the orbit of $q(V)$ proceeds to the right in the phase plane.

Lemma 5.4. *Let $n > 2$ and $\ell > -2$. Assume* (K) *and $r^{-\ell}K(r)$ is non-increasing from $c > 0$ at 0. If u_s is a singular solution, then*

$$u_s(r) \geq \log \frac{b}{cr^{2+\ell}} \tag{5.5}$$

and (5.3) holds.

Lemma 5.5. *Let $n > 2$ and $\ell > -2$. Assume* (K) *and $r^{-\ell}K(r)$ is non-increasing from c_2 at 0 to c_1 at $R > 0$ for some $c_2 > c_1 > 0$. Then,*

$$u_s(r) < \log \frac{b}{r^{2+\ell}} + M(c_1, c_2) \tag{5.6}$$

on $(0, R)$, where $M(c_1, c_2)$ is defined by $c_1 e^M - M = \frac{c_1}{c_2} + \log c_2$.

Step 2. Assume that $k(r) \to c > 0$ at $r = 0$ and $r_c = 0$.

Define k_j by

$$k_j(r) = c_j = k(\frac{1}{2^j})$$

for $0 \leq r \leq \frac{1}{2^j}$, and $k_j(r) = k(r)$ for $r \geq \frac{1}{2^j}$. Set $V_j(t) = u_j - \log \frac{b}{r^{2+\ell}}$, where $u_j - \log \frac{b}{r^{2+\ell}} = -\log c_j$ on $(0, \frac{1}{2^j}]$ and $u_j - \log \frac{b}{r^{2+\ell}}$ are local solutions of (5.4) with $k = k_j$ satisfying (5.3) with $c = c_j$. Then, V_j satisfies

$$V_j'' + aV_j' = b(1 - k_j e^{V_j}).$$

Since k_j is decreasing and $V_j = L_j$ on $(-\infty, -j\log 2]$, there exists $r_j > \frac{1}{2^j}$ such that $V_j' \geq 0$ on $(-j\log 2, \log r_j)$ and $V_j(\log r_j) > -\log c_j$. Note that k_j is increasing in j and $-\log c_j$ decreases to $-\log c$ as $j \to \infty$. Setting $u_j := \varphi_j + \log \frac{b}{r^{2+\ell}}$, we have

$$-u_j' = mr^{-m-1}\varphi_j - r^{-m}\varphi_j',$$

and thus

$$\lim_{r\to 0} r^{n-1}u_j' = m \lim_{r\to 0} r^{n-2-m}\varphi_j = 0.$$

Let $c_R = R^{-l}K(R)$ and $K_j = r^l k_j$. Then, for j large, $k_j \geq c_R$ on $(0, R)$ and for $r \in (0, R)$,

$$\begin{aligned} -u_j'(r) &= \frac{1}{r^{n-1}} \int_0^r K_j(s) e^{u_j(s)} s^{n-1}\, ds \\ &\leq \frac{bce^M}{r^{n-1}} \int_0^r s^{n-3}\, ds = \frac{bce^M}{n-2} r^{-1}. \end{aligned} \tag{5.7}$$

where $M = M(c_R, c)$. Hence, u_j' is uniformly bounded on any compact subset of $(0, R)$ in j and consequently, $\{u_j\}$ is equicontinuous on any compact subset of $(0, R)$. Hence, by applying Arzelà-Ascoli Theorem and adapting a diagonal argument, $u(r) := \lim_{j\to\infty} u_j(r)$ is well-defined and continuous on $(0, \infty)$ and satisfies

$$u'' = -\frac{n-1}{r}u' - Ke^u \qquad \text{on} \quad (0, \infty).$$

Since $u_j(r) - \log \frac{b}{r^{2+l}} \geq -\log c_j \geq -\log c$, we conclude that $u(r) \geq \log \frac{b}{cr^{2+l}}$ and u is a singular solution.

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