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On the behavior of radial solutions to a parabolic-elliptic system related to biology
(走化性方程式の解の挙動について)

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§0 Introduction.

In the present paper, we consider the behavior of radial solutions to the following problem.

$$\text{(PE)} \qquad \begin{cases} U_t = \nabla\cdot(\nabla U - U\nabla V) \ \text{ in } \mathbf{R}^n \times (0,\infty), \\ 0 = \Delta V + U \ \text{ in } \mathbf{R}^n \times (0,\infty), \quad V(0,\cdot) = 0 \quad \text{in } (0,\infty), \\ U(\cdot,0) = U^{\mathcal{I}} \geq 0 \ \text{ in } \mathbf{R}^n. \end{cases}$$

Here, $n = 1, 2, 3, \cdots$.

In two dimensional case, the system (PE) is a simplified version of so called Keller-Segel system, and is also a model of self-interacting particles. In the Keller-Segel model, U represents density of cells, and V represents the concentration of a chemoattractant secreted by themselves. In the physical model, U represents the density of particles, and V represents the potential.

We consider the behavior of radial solutions to (PE).

§1 Time local existence and uniqueness of radial solutions

In this paper, we consider radial solutions. The radial solutions exists uniquely under some conditions.

If $U^{\mathcal{I}}$ is radial, positive and

$$U^{\mathcal{I}}(x) = \begin{cases} O(1)/|x|^2 \ (n \geq 3), \\ O(1)/|x|^4 \ (n = 2), \end{cases}$$

as $|x| \to \infty$, there exists a unique solution (U, V) as follows.

$$U(x,t) = \int_{\mathbf{R}^n} \mathcal{G}(x-\tilde{x},t)U^{\mathcal{I}}(\tilde{x})d\tilde{x} \\ - \int_0^t \int_{\mathbf{R}^n} \left\{ \nabla_{\tilde{x}}\mathcal{G}(x-\tilde{x},t-\tilde{t}) \cdot \frac{\tilde{x}}{\omega_n|\tilde{x}|^n} \int_{|\hat{x}|<|\tilde{x}|} U(\hat{x},\tilde{t})d\hat{x} \right\} U(\tilde{x},\tilde{t})d\tilde{x}d\tilde{t}$$

in $\mathbf{R}^n \times [0,T)$ with a constant $T \in (0,\infty]$. And we defined the function V as

$$V(x,t) = -\int_0^{|x|} \frac{1}{\omega_n r^{n-1}} \int_{|\tilde{x}|<r} U(\tilde{x},t)d\tilde{x}dr \quad \text{in } \mathbf{R}^n \times [0,T),$$

since we define $V = 0$ at the origin.

Here, $\mathcal{G}$ is the Gauss kernel of $\partial_t - \Delta$ in $\mathbf{R}^n$ and $\omega_n = |S^{n-1}|$.

§2 Fundamental properties of solutions

In this section, we explain some fundamental properties of solutions.

Lemma 1 *The following hold.*
(i) U is non-negative in $\mathbf{R}^n \times (0,T)$.
(ii) In the case where $n \geq 2$, for any $\alpha > 0$ there exists a unique radial stationary solutions (U_α, V_α) satisfying $U_\alpha(0) = \alpha$,

$$(SPE) \begin{cases} 0 = \Delta V_\alpha + \alpha e^{V_\alpha} & \text{in } \mathbf{R}^n, \\ V_\alpha(0) = 0, \quad U_\alpha = \alpha e^{V_\alpha} & \text{in } \mathbf{R}^n. \end{cases}$$

(iii) In the case where $n = 2$, for $\alpha > 0$ the function U_α satisfies

$$U_\alpha(x) = \frac{\alpha}{(1 + (\alpha/8)|x|^2)^2} \quad \text{and} \quad \int_{\mathbf{R}^2} U_\alpha(x)dx = 8\pi.$$

(iv) In the case where $n \geq 10$, the function U_α satisfies

$$U_\alpha(x) = \frac{O(1)}{|x|^2} \quad \text{as } |x| \to \infty.$$

(v) In the case where $n \geq 10$ and $n = 2$, the function U_α is continuous with respect to α and satisfies

$$\lim_{\alpha \to 0} U_\alpha = 0 \quad \text{and} \quad \lim_{\alpha \to \infty} U_\alpha = U_\infty.$$

Here,

$$U_\infty(x) = \begin{cases} \dfrac{2(n-2)}{|x|^2} & \text{if } n \geq 3, \\ 8\pi\delta_0 & \text{if } n = 2. \end{cases}$$

Sketch of proof. (i) comes from the comparison theorem, since we assume that $U^{\mathcal{I}} \geq 0$ in $\mathbf{R}^n$.
(ii) radial stationary solutions satisfies $\nabla U_\alpha - U_\alpha \nabla V_\alpha = 0$, $V_\alpha(0) = 0$ and $U_\alpha(0) = \alpha$. These ensure $U_\alpha = \alpha e^{V_\alpha}$, which together with the second equation of (PE) implies (SPE).
(iii) The straightforward calculation gives us this property.
(iv) This property is shown in [8, Lemma 2.1].
(v) This property is shown in the proof of [8, Theorem 3.1]. □

§3 Known results ∼ radial case ∼

- Finite time blowup solutions.

There exist radial solutions to (PE) satisfying

$$\limsup_{t\to T} \|U(\cdot,t)\|_{L^\infty(\mathbf{R}^n)} = \infty.$$

Many persons contribute to this problem (see [3]).

- Time-global solutions.

If the initial function $U^{\mathcal{I}}$ is radial and satisfies

$$\begin{cases} 0 \le U^{\mathcal{I}} \le U_\infty, \quad U^{\mathcal{I}} \not\equiv U_\infty \quad (n \ge 3), \\ U^{\mathcal{I}} \ge 0, \quad \Lambda = \displaystyle\int_{\mathbf{R}^2} U^{\mathcal{I}}(x)dx \le 8\pi \quad (n = 2), \end{cases}$$

the radial solution exists globally in time.

In the case where $n \ge 3$, the property is shown by the comparison theorem for the mass function $M(r,t) = \int_{|x|,r} U(x,t)dx$. In the case where $n = 2$, the property is shown in [1]. In the non-radial case, there exists many open problems.

- Infinite time blowup solution.

There exist solutions satisfying

$$\limsup_{t\to\infty} \|U(\cdot,t)\|_{L^\infty(\mathbf{R}^n)} = \infty.$$

In two dimensional case, these solutions are found in [2, 4]. In [2], non-radial solutions are treated. In [4], radial solutions in a disk are treated and investigated blowup rate. Moreover, such radial solutions are found also in the case where $n \ge 11$ (see [7]).

§4 Oscillating solutions in two dimensional case

Although system (PE) has several solutions, the behavior of each solution is not so complicated. However, there exists solutions having complicate behavior. We define ω-limit set as

$$\begin{aligned} \omega(U^{\mathcal{I}} : C(\mathbf{R}^2)) \quad = \quad & \Big\{ F \in C(\mathbf{R}^2) \cap L^\infty(\mathbf{R}^2) \ : \ \lim_{n\to\infty} t_n = \infty, \\ & \lim_{n\to\infty} \|U(\cdot,t_n) - F\|_{L^\infty(\mathbf{R}^2)} = 0 \quad \text{for some } \{t_n\} \subset (0,\infty) \Big\}. \end{aligned}$$

Theorem 1 *[5]*
(i) For a and d with $0 < a < d$ there exists a radial solution (U, V) with $U(\cdot, 0) = U^{\mathcal{I}}$ satisfying

$$\{U_b\}_{b\in[a,d]} \subset \omega(U^{\mathcal{I}} : C(\mathbf{R}^2)), \quad \int_{\mathbf{R}^2} U(x,t)dx = 8\pi.$$

(ii) For $\{b_j\}_{j=1}^{\infty} \subset (0,\infty)$ *with* $\lim_{j\to\infty} b_j = \infty$ *there exists a radial solution* (U,V) *with* $U(\cdot,0) = U^{\mathcal{I}}$ *satisfying*

$$\{U_{b_j}\}_{j=1}^{\infty} \subset \omega(U^{\mathcal{I}} : C(\mathbf{R}^2)), \qquad \int_{\mathbf{R}^2} U(x,t)dx = 8\pi$$

According to the definition of ω-limit set, these solutions satisfies the following.

Concerning the solution in (i), for each $b \in [a,d]$ there exists a sequence $\{t_k\}_{k\geq 1} \subset (0,\infty)$ satisfying

$$\lim_{k\to\infty} \|U(\cdot,t_k) - U_b\|_{L^\infty(\mathbf{R}^2)} = 0, \quad \lim_{k\to\infty} t_k = \infty.$$

Then, the solution oscillates among any stationary solutions between U_a and U_d.

Concerning the solution in (ii), for each $j = 1,2,3,\cdots$ there exists a sequence $\{t_k\}_{k=1}^{\infty} \subset (0,\infty)$ satisfying

$$\lim_{k\to\infty} \|U(\cdot,t_k) - U_{b_j}\|_{L^\infty(\mathbf{R}^2)} = 0, \quad \lim_{k\to\infty} t_k = \infty.$$

Since $\lim_{b\to\infty} U_b = 8\pi\delta_0$, there exists a sequence $\{t_k\}_{k=1}^{\infty} \subset (0,\infty)$ satisfying

$$\lim_{k\to\infty} \|U(\cdot,t_k)\|_{L^\infty(\mathbf{R}^2)} = \infty, \quad \lim_{k\to\infty} t_k = \infty.$$

§5 Idea of proof of Theorem 1

Essentially, using stability of stationary solutions, layer of stationary solutions and Poláčik and Yangida's argument in [9], we construct oscillating solutions.

The stability of radial stationary solutions are shown in [1]. The following proposition is a modified version of the result.

Proposition 1 *Let* $U^{\mathcal{I}}$ *be nonnegative and radial,* $\|U^{\mathcal{I}}\|_{L^1(\mathbf{R}^2)} = 8\pi$ *and*

$$\sup_{x\in\mathbf{R}^2} (1+|x|)^5 |U^{\mathcal{I}}(x) - U_b(x)| < \infty$$

with some $b > 0$. *Then,* $\lim_{t\to\infty} \|U(\cdot,t) - U_b\|_{L^\infty(\mathbf{R}^2)} = 0$.

In two dimensional case, radial stationary solutions layer in the following sense.

$$\begin{cases} \lim_{a\to b} \|U_a - U_b\|_{L^\infty(\mathbf{R}^2)} = 0 \quad (b > 0) \\ \displaystyle\int_{|x|<r} U_a(x)dx \leq \int_{|x|<r} U_b(x)dx \quad (r > 0), \qquad \text{if } a \leq b. \end{cases}$$

Poláčik and Yanagida [9] show stability of radial stationary solutions to the problem

$$\begin{cases} U_t = \Delta U + U^p & \text{in } \mathbf{R}^n \times (0,\infty), \\ U(\cdot,0) = U^{\mathcal{I}} & \text{in } \mathbf{R}^n \end{cases}$$

with $n \geq 11$ and $p \geq p_{JL} = \{(n-2)^2 - 4n + 8\sqrt{n-1}\}/\{(n-2)(n-10)\}$. Moreover, radial stationary solutions to this problem layer in the case where $n \geq 11$ and $p \geq p_{JL}$. Using the stability and the layer, they construct oscillating solutions to this problem.

In order to describe the idea of proof of Theorem 1, we consider a special case.

Theorem 2 (Special case of our problem) *There exists a radial solution (U,V) to (PE) with $U(\cdot,0) = U^{\mathcal{I}}$ such that $\{U_a, U_d\} \subset \omega(U^{\mathcal{I}} : C(\mathbf{R}^2))$ with $0 < a < d < \infty$.*

Let (U,V) be a solutions to (PE). Put

$$u(r,t) = \frac{1}{2\pi r^2}\int_{|x|<r} U(x,t)dx.$$

The function u satisfies

$$(IPE)\begin{cases} \mathcal{L}(u) = u_t - u_{rr} - \frac{3}{r}u_r - u\{ru_r + 2u\} = 0 \quad (0 < r < \infty,\ t > 0), \\ u_r(0,t) = 0 \quad (t > 0), \\ u(x,0) = u^{\mathcal{I}} \quad (0 \leq r < \infty). \end{cases}$$

Put

$$u_\alpha(r) = \frac{1}{2\pi r^2}\int_{|x|<r} U_\alpha(x)dx.$$

The function u_α is a stationary solution to (IPE).

Sketch of Theorem 2. For two positive constants $\tilde{L}_1 \gg L_1 \gg 1$ put

$$u_1^{\mathcal{I}}(r) = u_d(r)\ (\ r \leq L_1), \quad u_1^{\mathcal{I}}(r) = u_a(r)\ (\ \tilde{L}_1 < r).$$

Let u_1 be the solution to (IPE) with $u_1(\cdot,0) = u_1^{\mathcal{I}}$. By the continuity with respect to initial data, there exists $C(T_1) > 0$ such that

$$\begin{aligned} \|u_1(\cdot,t) - u_d\|_{L^\infty((0,\infty))} &\leq C(T_1)\|u_1(\cdot,0) - u_d\|_{L^\infty((0,\infty))} \\ &\leq C(T_1)L_1^{-2}\sup_{L_1<r} r^2|u_a(r) - u_d(r)| \\ &\leq C(T_1,a,d)L_1^{-2} \quad \text{for } t \in [0,T_1]. \end{aligned}$$

Therefore, for $0 < \varepsilon \ll 1$ and $T_1 > 0$ the solution u_1 satisfies

$$\|u_1(\cdot,t) - u_d\|_{L^\infty((0,\infty))} < \varepsilon \quad \text{for } t \in [0, T_1],$$

if $1 \ll L_1 \ll \tilde{L}_1$. On the other hand, Proposition 1 guarantees

$$\lim_{t\to\infty} \|u_1(\cdot,t) - u_a\|_{L^\infty((0,\infty))} = 0,$$

since $u_1(\cdot,0) - u_a$ has a compact support.

For $\tilde{L}_2 > L_2 \gg \tilde{L}_1$, putting initial data

$$u_2^{\mathcal{I}}(r) = u_1(r,0) \ (\ r \le L_2), \quad u_2^{\mathcal{I}}(r) = u_d(r) \ (\ \tilde{L}_2 < r).$$

Let u_2 be a solution to (IPE) with $u_2(\cdot,0) = u_2^{\mathcal{I}}$. Taking $T_2 \gg T_1$ and $\tilde{L}_2 \gg L_2 \gg \tilde{L}_1$ such that

$$\|u_1(\cdot,t) - u_a\|_{\beta,(0,\infty)} < \varepsilon/2 \quad \text{for } t \in [T_2 - 1, \infty)$$

and

$$\|u_2(\cdot,t) - u_1(\cdot,t)\|_{\beta,(0,\infty)} < \varepsilon/2 \quad \text{for } t \in [0, T_2 + 1],$$

we get

$$\|u_2(\cdot,T_2) - u_a\|_{\beta,(0,\infty)} < \varepsilon \quad \text{for } t \in [T_2 - 1, T_2 + 1],$$
$$\lim_{t\to\infty} \|u_2(\cdot,t) - u_d\|_{\beta,(0,\infty)} = 0.$$

Since the initial function $u_2^{\mathcal{I}}$ satisfies the property having the function $u_1^{\mathcal{I}}$, then the solution u_2 satisfies

$$\|u_2(\cdot,t) - u_d\|_{\beta,(0,\infty)} < \varepsilon \quad \text{for } t \in [T_1 - 1, T_1 + 1].$$

Repeating this argument, we find a solution u with $u(\cdot,0) = u^0$ such that

$$\{u_a, u_d\} \subset \omega(u^{\mathcal{I}} : C([0,\infty))). \tag{1}$$

Moreover, the parabolic regularity method guarantees the following. There exists a constant C such that

$$\|U(\cdot,t) - U_\alpha\|_{\beta,\mathbf{R}^n} \le C \max_{t-\frac{1}{2}\le s\le t+\frac{1}{2}} \|u(\cdot,s) - u_\alpha\|_{\beta,[0,\infty)} \quad \text{for } t \ge 1.$$

Then, for solution u satisfying (1) we obtain that

$$(U(x,t), V(x,t)) = \left(\frac{1}{|x|} u_r(|x|,t), -\int_0^{|x|} u(r,t)dr\right)$$

is the desired solution to (PE). □

§6 High dimensional case

As mentioned in the previous section, stability of stationary solutions and layer of stationary solutions guarantee the existence of oscillating solutions.

In the case where $n \geq 11$, stationary solutions are stable in the following sence.

Theorem 3 *[6] Let $n \geq 11$. $\beta_- = \{n+2-\sqrt{(n-2)(n-10)}\}/2 \in (2,n)$. Suppose $0 \leq U^{\mathcal{I}} \leq U_\infty$ in $\mathbf{R}^n$ and*

$$\lim_{|x|\to\infty} (1+|x|)^{\beta_-}|U^{\mathcal{I}}(x) - U_\alpha(x)| = 0$$

with some $\alpha > 0$. Then, the solution (U,V) to (PE) satisfies

$$\lim_{t\to\infty} \|U(\cdot,t) - U_\alpha\|_{\beta_-,\mathbf{R}^n} = 0,$$

where $\|F\|_{\beta,\mathbf{R}^n} = \sup_{x\in\mathbf{R}^n}(1+|x|)^\beta|F(x)|$.

Moreover, stationary solutions layer in the case where $n \geq 11$ in the following sense.

Proposition 2 *[8] Let $n \geq 11$. For $\alpha > 0$, there exists a unique stationary solutions (U_α, V_α) to (PE) satisfying $U_\alpha(0) = 0$ and (SPE). Moreover, the set of functions $\{U_\alpha\}_{\alpha>0}$ satisfies the following.*
(i) $\lim_{a\to b} \|U_a - U_b\|_{\beta_-,\mathbf{R}^n} = 0 \quad (b > 0)$
(ii) $U_b(x) = \dfrac{2(n-2)}{|x|^2} - \dfrac{A(b)}{|x|^{\beta_-}}$ as $|x| \to \infty$.
(iii) $U_a < U_b$ in $\mathbf{R}^n$, if $a < b$.
Here, $A(b)$ is continuous and strictly decreasing with respect to $b > 0$.

In order to describe our result, we define some functional spaces and ω-limit sets.

For a non-negative constant β, put

$$C_\beta(\mathbf{R}^n) = \left\{F \in C(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n) : \sup_{x\in\mathbf{R}^n} (1+|x|)^\beta|F(x)| < \infty\right\}.$$

Let (U,V) be a solution to (PE) with initial data $U^{\mathcal{I}}$ satisfying $U \in C([0,\infty) : C(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n))$. We put

$$\omega(U^{\mathcal{I}} : C_\beta(\mathbf{R}^n)) = \Big\{F \in C(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n) \ : \ \lim_{n\to\infty} t_n = \infty,$$
$$\lim_{n\to\infty} \|U(\cdot,t_n) - F\|_{\beta,\mathbf{R}^n} = 0 \quad \text{for some } \{t_n\} \subset (0,\infty)\Big\}.$$

Using Theorem 3 and Proposition 2, we construct the following solutions.

Theorem 4 *[6] Let* $n \geq 11$ *and let* Λ *be a set of* $[0, \infty)$. *Then, there exists a radial and continuous function* $U^{\mathcal{I}}$ *such that*

$$0 \leq U^{\mathcal{I}} \leq U_\infty \equiv \frac{2(n-2)}{|x|^2} \quad \text{in } \mathbf{R}^n.$$

and

$$\{U_a\}_{a \in \Lambda} \subset \omega(U^{\mathcal{I}} : C_\beta(\mathbf{R}^n)) \quad \text{for any } \beta \in [0, 2).$$

Moreover, suppose $\inf \Lambda > 0$. *Then, we can take* $\beta \in [0, \beta_-)$.

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