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Free probability theory and infinitely divisible distributions

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1 Summary of free probability theory

1.1 Noncommutative probability theory

Elements in a noncommutative operator algebra can be regarded as noncommutative random variables from a probabilistic viewpoint. Such understanding has its origin in quantum theory. Theory of operator algebras focusing on the probabilistic aspect is called *noncommutative probability theory.*

Noncommutative probability theory is divided into several directions. Some groups perform mathematical research, and others do physical research. The main focus of this article is *free probability*, a mathematical aspect of noncommutative probability. The name of free probability theory might sound strange for non-experts. This name was chosen because free probability fits in the analysis of the free product of groups or algebras. Free probability has been developed in terms of operator algebras to solve problems related to von Neumann algebras generated by free groups [HP00, VDN92]. From a probabilistic aspect, when one considers random walks on free groups, free probability is useful to analyze the recurrence/transience of the random walks [W86].[1]

In addition, Voiculescu [V91] found that free probability has application to the analysis of the eigenvalues of random matrices (see also [HP00, VDN92]). Why eigenvalues of random matrices interest researchers? The original motivation is to model the energy levels of nucleons of nuclei. Then subsequent studies revealed many relations of random matrices to other mathematics as well as physics, e.g. integrable systems (such as Peinlevé equations), the Riemann zeta function and representation theory [M04]. All these applications are based on the analysis of eigenvalue distributions of random matrices.

In this article, we are going to present the basics of free probability, and then describe the summary of results obtained so far on *freely infinitely divisible distributions*, the author's recent main subject. A purpose of free probability is to analyze *free convolution* which describes the eigenvalue distribution of the sum of independent large random matrices. The set of freely infinitely divisible probability measures is the central subject associated to free convolution.

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[1]The paper [W86] was written independently of Voiculescu's pioneering papers [V85, V86] on free probability, but Woess also used the Voiculescu transform to analyze random walks.

1.2 Algebraic probability space, random variable and probability distribution

Let $\mathcal{A}$ be a *$*$-algebra* over $\mathbb{C}$ with unit $1_{\mathcal{A}}$, that is, $\mathcal{A}$ is an algebra over $\mathbb{C}$ equipped with an antilinear mapping $* : \mathcal{A} \to \mathcal{A}$, $X \mapsto X^*$, which satisfies $X^{**} = X$ $(X \in \mathcal{A})$. A typical $*$-algebra is the set of bounded linear operators $\mathcal{A} := \mathbb{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$. If its inner product is denoted by $\langle \cdot, \cdot \rangle$, the antilinear mapping $*$ is the usual conjugation defined by

$$\langle u, Xv \rangle = \langle X^* u, v \rangle, \quad X \in \mathbb{B}(\mathcal{H}), \quad u, v \in \mathcal{H}.$$

A linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ is called a *state* on $\mathcal{A}$ if it satisfies $\varphi(1_{\mathcal{A}}) = 1$ and $\varphi(X^*X) \geq 0$, $X \in \mathcal{A}$. A state plays the role of expectation in probability theory. A pair $(\mathcal{A}, \varphi)$ is called an *algebraic probability space* and elements $X \in \mathcal{A}$ are called *random variables*.

The $*$-algebra $\mathbb{B}(\mathcal{H})$ is basic because any $*$-algebra $\mathcal{A}$ can be realized as a sub $*$-algebra of $\mathbb{B}(\mathcal{H})$ for some $\mathcal{H}$. A universal construction of such an $\mathcal{H}$ is known and is called the *GNS* construction. So, from now on $\mathcal{A}$ is assumed to be a sub $*$-algebra of a $\mathbb{B}(\mathcal{H})$, and moreover to be closed with respect to the strong topology (i.e., $\mathcal{A}$ is a von Neumann algebra). We further assume that φ is *normal*, a certain continuity condition on φ.

If X is self-adjoint, i.e. $X = X^*$, let E_X denote the spectral decomposition of X. Because φ is normal, $\mu_X(B) := \varphi(E_X(B))$ for B Borel sets of $\mathbb{R}$ becomes a probability measure on $\mathbb{R}$, and is called the *probability distribution*[2] of X.

In the above, random variables are assume to be bounded, but unbounded operators also fit in this probabilistic aspect. A possibly unbounded self-adjoint operator X on $\mathcal{H}$ is said to be *affiliated to* $\mathcal{A}$ if its spectral projections $E_X(B)$ (B is an arbitrary Borel set) all belong to $\mathcal{A}$. In this case, the probability distribution μ_X can be defined by $\mu_X(B) := \varphi(E_X(B))$ similarly to the bounded case.

Example 1.1. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{A} := L^\infty(\Omega, \mathcal{F}, P) \otimes \mathrm{M}_n(\mathbb{C})$ be the set of random matrices. The algebra $\mathcal{A}$ acts on the set of $\mathbb{C}^n$-valued square integrable random vectors. The antilinear mapping $*$ is the conjugation of complex matrices, and φ is defined to be $E \otimes (\frac{1}{n}\mathrm{Tr}_n)$, that is,

$$\varphi(X) := \frac{1}{n} \sum_{j=1}^{n} E[X_{jj}]$$

for random matrices $X = (X_{ij})_{1 \leq i,j \leq n}$. The set of self-adjoint operators affiliated to $\mathcal{A}$ is now equal to $\{X \in \mathcal{A} : \text{Hermitian}, \mathcal{F}\text{-measurable}\}$. The distribution μ_X coincides with the *mean eigenvalue distribution* of X:

$$\mu_X = E\left[\frac{1}{n} \sum_{j=1}^{n} \delta_{\lambda_j}\right],$$

where λ_j are random eigenvalues of X. In other words, $\mu_X(B) = E\left[\frac{\sharp\{1 \leq j \leq n : \lambda_j \in B\}}{n}\right]$ for Borel sets $B \subset \mathbb{R}$. When $n = 1$, the measure μ_X is the usual probability distribution of $\mathbb{R}$-valued random variable X.

[2]Sometimes we say simply *distribution* or *law* instead of probability distribution.

1.3 Tensor independence and free independence

Independence is a central concept in probability theory; almost all the concepts and results in probability theory are based on independence. However, more than one independences are known in noncommutative probability theory. From one aspect, independences can be classified into four or five [M03], but now we consider two of them. For $X \in \mathcal{A}$, let $\mathbb{C}[X, 1_{\mathcal{A}}]$ denote the polynomials generated by X and the unit $1_{\mathcal{A}}$.

First we are going to extend the usual independence to noncommutative random variables; such an independence is called tensor independence.

Definition 1.2. Random variables $X \in \mathcal{A}$ and $Y \in \mathcal{A}$ are said to be *tensor independent* if for any finite number of $X_i \in \mathbb{C}[X, 1_{\mathcal{A}}]$, $Y_i \in \mathbb{C}[Y, 1_{\mathcal{A}}]$, it holds that

$$\varphi(\cdots X_1Y_1X_2Y_2X_3Y_3\cdots) = \varphi\Big(\prod_i X_i\Big)\varphi\Big(\prod_i Y_i\Big).$$

The product $\prod_i X_i$ is assumed to preserve the order of random variables.

This definition can easily be extended for more than two variables.

Because tensor independence is just an extension of the usual concept, it can appear on commutative algebras. The following free independence, by contrast, cannot appear on commutative algebras, so it is a purely noncommutative concept.

Definition 1.3 (Voiculescu [V85]). Random variables X and Y are *free* (or freely independent) if for any finite number of $X_i \in \mathbb{C}[X, 1_{\mathcal{A}}]$, $Y_i \in \mathbb{C}[Y, 1_{\mathcal{A}}]$ satisfying $\varphi(X_i) = \varphi(Y_i) = 0$, it holds that

$$\varphi(\cdots X_1Y_1X_2Y_2X_3Y_3\cdots) = 0.$$

Free independence can be extended for more than two variables too.

Example 1.4. Let X, Y be free, then the following computations can be verified.

$$\varphi(XY) = \varphi(X)\varphi(Y), \quad \varphi(XYX) = \varphi(X^2)\varphi(Y),$$
$$\varphi(XYXY) = \varphi(X^2)\varphi(Y)^2 + \varphi(X)^2\varphi(Y^2) - \varphi(X)^2\varphi(Y)^2.$$

Let us prove the first identity. Set $X_1 := X - \varphi(X)1_{\mathcal{A}} \in \mathbb{C}[X, 1_{\mathcal{A}}]$, $Y_1 := Y - \varphi(Y)1_{\mathcal{A}} \in \mathbb{C}[Y, 1_{\mathcal{A}}]$. These random variables are centered, i.e., $\varphi(X_1) = \varphi(Y_1) = 0$, and so $\varphi(X_1Y_1) = 0$ by definition, or equivalently $\varphi((X - \varphi(X)1_{\mathcal{A}})(Y - \varphi(Y)1_{\mathcal{A}})) = 0$. After some calculations, this leads to $\varphi(XY) = \varphi(X)\varphi(Y)$. The other identities are proved similarly.

Thus independence gives calculation rule for random variables.

1.4 Free convolution

If $X, Y \in \mathcal{A}$ are free self-adjoint random variables, the distribution μ_{X+Y} is called the *free convolution* of μ_X and μ_Y, and is denoted by $\mu_X \boxplus \mu_Y$. Moreover if $X \geq 0$ ($Y \geq 0$), then the distribution $\mu_{X^{1/2}YX^{1/2}}$ ($\mu_{Y^{1/2}XY^{1/2}}$, respectively) is called the *free multiplicative convolution* of μ_X and μ_Y, and it is denoted by $\mu_X \boxtimes \mu_Y$. It is known that $\mu_{X^{1/2}YX^{1/2}} =$

$\mu_{Y^{1/2}XY^{1/2}}$ when both $X \geq 0, Y \geq 0$ hold. Because the random variable XY is not self-adjoint in general, the random variable $X^{1/2}YX^{1/2}$ or $Y^{1/2}XY^{1/2}$ is used instead.

How can we calculate free convolution? While classical convolution can be calculated in terms of the characteristic function (or the Fourier transform), free convolution is calculated with the Stieltjes transform. Given a probability distribution μ on $\mathbb{R}$, its *Stieltjes transform* is defined by

$$G_\mu(z) := \int_{\mathbb{R}} \frac{1}{z-x}\mu(dx), \quad z \in \mathbb{C}^+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\},$$

and its reciprocal is by

$$F_\mu(z) := \frac{1}{G_\mu(z)} \quad z \in \mathbb{C}^+.$$

Moreover, we define the *Voiculescu transform*

$$\phi_\mu(z) := F_\mu^{-1}(z) - z$$

in a suitable domain.

Theorem 1.5 (Voiculescu-Bercovici [BV93]). *For probability measures μ, ν on $\mathbb{R}$,*

$$\phi_{\mu \boxplus \nu}(z) = \phi_\mu(z) + \phi_\nu(z).$$

The domain can be taken as $\{z \in \mathbb{C}^+ : \operatorname{Im} z > \beta,\ \alpha|\operatorname{Re} z| \leq \operatorname{Im} z\}$ for some $\alpha, \beta > 0$.

Free multiplicative convolution has a similar characterization, but we omit it because free multiplicative convolution is not the main subject of this article. The interested readers can refer to [VDN92]. Research on free multiplicative convolution is still on progress, and the author thinks it will be developed more in future.

1.5 Random matrix and free probability

Free convolution and free multiplicative convolution are investigated partially because they have application to random matrices. Such application is based on the following result of Voiculescu. Note that recently this result has been extended to rectangular matrices by Benaych-Georges [B09] and more generally to random matrices divided into sub blocks by Lenczewski [L].

Theorem 1.6 (Voiculescu [V91]). *Suppose A_n, B_n are (tensor) independent $n \times n$ Hermitian matrices ($n \geq 1$), and moreover, suppose:*

(1) *For any $n \geq 1$, the distribution of A_n is rotationally invariant, i.e. for any $n \times n$ unitary U, the distributions of A_n and U^*A_nU on $M_n(\mathbb{C})$ are the same;* [3]

(2) *The mean eigenvalue distributions of A_n, B_n weakly converge to μ, ν, respectively, as $n \to \infty$.*

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[3]Since the random matrix A_n is regarded as a $M_n(\mathbb{C})$-valued random variable, it induces a probability measure on the vector space $M_n(\mathbb{C})$.

Then the mean eigenvalue distributions of $A_n + B_n$ weakly converge to $\mu \boxplus \nu$ as $n \to \infty$. Moreover, if $A_n \geq 0$ ($B_n \geq 0$), then the mean eigenvalue distributions of $\sqrt{A_n} B_n \sqrt{A_n}$ (of $\sqrt{B_n} A_n \sqrt{B_n}$ respectively) weakly converge to $\mu \boxtimes \nu$.

Thus free probability can describe the eigenvalues of large random matrices, and hence, understanding of the convolutions $\boxplus, \boxtimes$ becomes the main problem in free probability. In the next section, we state limit theorems on $\boxplus$ to get a better understanding of $\boxplus$.

2 Infinitely divisible distributions

The concept of infinitely divisible distributions are introduced by extending the well known central limit theorem.

Definition 2.1 ([S99, SH03]). A probability measure μ on $\mathbb{R}$ is said to be *infinitely divisible (ID)* if for any $n \geq 1$ there exist identically distributed, (tensor) independent (i.i.d.) $\mathbb{R}$-valued random variables $X_1^{(n)}, \cdots, X_n^{(n)}$ such that the distribution of $X_1^{(n)} + \cdots + X_n^{(n)}$ weakly converge to μ.

Example 2.2. (1) Suppose $(X_i)_{i \geq 1}$ be i.i.d. random variables and $\varphi(X_i) = 0$, $\varphi(X_i^2) = 1$. By defining $X_i^{(n)} := \frac{X_i}{\sqrt{n}}$, the situation is the central limit theorem, so the distribution of $X_1^{(n)} + \cdots + X_n^{(n)}$ converge to the standard Gaussian

$$\mathbf{g}(dx) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} 1_{\mathbb{R}}(x)\, dx.$$

The Gaussian is the most important ID law.

(2) Let $\lambda > 0$ be real and $n > \lambda$ be natural numbers. Assume $\mathbb{R}$-valued random variables $X_i^{(n)}$ take 0 at probability $1 - \frac{\lambda}{n}$, and take 1 at probability $\frac{\lambda}{n}$, and they are independent with respect to i for each n. Then the distribution of $X_1^{(n)} + \cdots + X_n^{(n)}$ weakly converge to the Poisson distribution[4]

$$\mathbf{p}_\lambda = \sum_{n=0}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} \delta_n.$$

In terms of convolution,

$$\left(\left(1 - \frac{\lambda}{n} \right) \delta_0 + \frac{\lambda}{n} \delta_1 \right)^{*n} \to \mathbf{p}_\lambda \quad (n \to \infty).$$

Hence the Poisson distribution is ID for any parameter $\lambda > 0$.

The above definition emphasizes on the aspect of the limit theorem, but it coincides with the usual definition of ID distributions.

[4]This limit theorem is called *Poisson's law of small numbers*.

Proposition 2.3. *A probability measure μ on $\mathbb{R}$ is ID if and only if for each $n \in \{1,2,3,\cdots\}$ there exists a probability measure μ_n such that*

$$\mu = \mu_n^{*n} := \underbrace{\mu_n * \cdots * \mu_n}_{n \text{ times}}.$$

Every ID distribution appears as the distribution of a *Lévy process.* This extends the fact that the Gaussian is the distribution of a Brownian motion. The reader can consult [S99] for Lévy processes.

Now we are going to define a free version of ID distributions. This concept is hopefully useful for a better understand of free convolution ⊞ and the sum of random matrices.

Definition 2.4 ([BV93]). A probability measure μ on $\mathbb{R}$ is said to be *freely infinitely divisible (FID)* if for any $n \geq 1$ there exist identically distributed, free random variables $X_1^{(n)}, \cdots, X_n^{(n)}$ such that the distribution of $X_1^{(n)} + \cdots + X_n^{(n)}$ weakly converge to μ.

The free analogue of Proposition 2.3 is also the case. This fact was proved by Bercovici and Pata [BP99]. Note that a more general limit theorem was proved by Chistyakov and Götze [CG08].

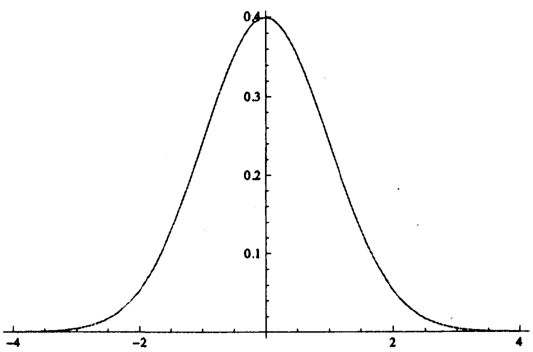

Figure 1: Probability density of the standard Gaussian $\mathbf{g}$

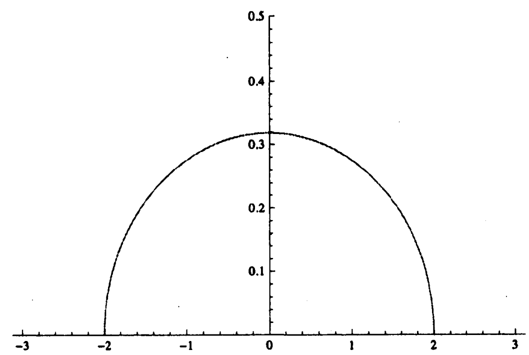

Figure 2: Probability density of Wigner's semicircle law $\mathbf{w}$

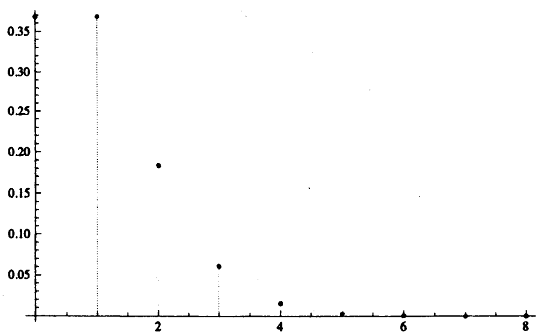

Figure 3: Poisson distribution $\mathbf{p}_1$

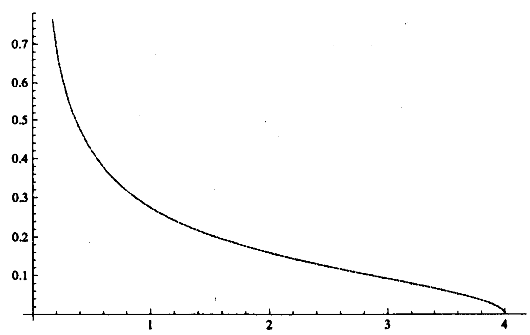

Figure 4: Probability density of free Poisson distribution $\boldsymbol{\pi}_1$

Example 2.5. (1) Suppose $(X_i)_{i\geq 1}$ be identically distributed, free random variables and $\varphi(X_i) = 0$, $\varphi(X_i^2) = 1$. Define $X_i^{(n)} := \frac{X_i}{\sqrt{n}}$, then as $n \to \infty$, the distributions of random variables $X_1^{(n)} + \cdots + X_n^{(n)}$ weakly converge to *Wigner's semicircle law*

$$\mathbf{w}(dx) = \frac{1}{2\pi}\sqrt{4-x^2}\,dx.$$

Therefore Wigner's semicircle law is FID. This measure appears as the limiting eigenvalue distribution of GUE ensemble. This distribution was found by Wigner in his approach to modeling statistics of energy levels of nucleons in nuclei. Recently Wigner's result has been refined by some research groups. Tao and Wu wrote a summary on this subject [TV].

(2) What should be called the *free Poisson distribution* is defined as follows:

$$\boldsymbol{\pi}_\lambda := \lim_{n\to\infty}\left(\left(1-\frac{\lambda}{n}\right)\delta_0 + \frac{\lambda}{n}\delta_1\right)^{\boxplus n} \qquad \lambda > 0.$$

When $\lambda = 1$, the probability density function of $\boldsymbol{\pi}_1$ can be written as

$$\frac{1}{2\pi}\sqrt{\frac{4-x}{x}}1_{[0,4]}(x)\,dx.$$

This distribution is also called the *Marchenko-Pastur distribution* that is known to appear as the eigenvalue distribution of the square of GUE; one can check that if a random variable X follows $\mathbf{w}$, then X^2 follows $\boldsymbol{\pi}_1$.

Because free convolution is linearized by the Voiculescu transform, it is expected that FID distributions can be characterized by the Voiculescu transform, and it is indeed the case.

Theorem 2.6 (Bercovici-Voiculescu [BV93]). *The following are equivalent.*

(1) *μ is FID.*

(2) *$-\phi_\mu$ analytically continues to $\mathbb{C}^+$ and it maps $\mathbb{C}^+$ into $\mathbb{C}^+ \cup \mathbb{R}$.*[5]

(3) *Constants $\eta \in \mathbb{R}, a \geq 0$ and nonnegative measure ν exist satisfying $\nu(\{0\}) = 0$, $\int_{\mathbb{R}} \min\{1, x^2\}\nu(dx) < \infty$, and*

$$z\phi_\mu(z^{-1}) = \eta z + az^2 + \int_{\mathbb{R}}\left(\frac{1}{1-xz} - 1 - xz1_{[-1,1]}(x)\right)\nu(dx), \quad z \in i(-\infty, 0). \quad (2.1)$$

The integral representation in (3) corresponds to the Lévy-Khintchine representation in probability theory [S99]. The measure ν is called the *free Lévy measure* of μ. In the classical case, a probability measure μ is ID if and only if

$$\begin{aligned}\log\widehat{\mu}(z) &= \log\left(\int_{\mathbb{R}} e^{izx}\,\mu(dx)\right)\\ &= i\eta z - \frac{1}{2}az^2 + \int_{\mathbb{R}}\left(e^{ixz} - 1 - ixz1_{[-1,1]}(x)\right)\nu(dx), \quad z \in \mathbb{R},\end{aligned} \qquad (2.2)$$

[5]Such a function is called a Pick-Nevanlinna function.

where η, a, ν satisfy the same conditions as in (2.1). If we replace e^z by $\frac{1}{1-z}$ and then iz by z in (2.2), we obtain (2.1) except the difference of the coefficient of z^2. Thus the two representations are quite similar, but the proofs are totally different. This similarity was investigated in [BP99] from a viewpoint of limit theorems. The correspondence between $\frac{1}{1-z}$ and e^z was discussed in [BT06].

What kind of distributions are FID? In probability theory, a lot of ID distributions are known and they appear in many applications. There are several sufficient conditions for a measure to be ID. If a measure has a probability density function that is completely monotone or log convex, then the measure is ID. Note that a function $f:(0,\infty)\to\mathbb{R}$ is *completely monotone* if there exists a Borel measure σ such that

$$f(x)=\int_0^\infty e^{-xt}\,\sigma(dt).$$

Or if the density function is HCM (hyperbolic completely monotone), the measure is ID. The book [SH03] contains the summary of past results, including the sufficient conditions explained in the above.

By contrast, existing FID distributions with concrete density functions are not so many in free probability, nor useful sufficient conditions. The author's recent work is mainly on finding examples of FID distributions, which hopefully leads to sufficient conditions for a probability measure to be FID.

3 Research achievements on FID distributions

3.1 Explicit probability density and explicit Voiculescu transform

When ϕ_μ is computable, Theorem 2.6(2) is useful to see whether μ is FID or not. Wigner's semicircle law has the Voiculescu transform $\phi_{\mathbf{w}}(z)=\frac{1}{z}$, and the free Poisson law has $\phi_{\pi_\lambda}(z)=\frac{\lambda z}{z-1}$, but there are not many examples. Recently, Arizmendi, Barndorff-Nielsen and Pérez-Abreu [ABP10] found that the symmetrized beta distribution with parameters $\frac{1}{2},\frac{3}{2}$

$$\mathbf{b}_s(dx):=\frac{1}{\pi\sqrt{s}}|x|^{-1/2}(\sqrt{s}-|x|)^{1/2}dx,\quad -\sqrt{s}\le x\le\sqrt{s}$$

has explicit Stieltjes and Voiculescu transforms:

$$\begin{aligned} G_{\mathbf{b}_s}(z)&=-2^{1/2}\left(\frac{1-\left(1-s\left(-\frac{1}{z}\right)^2\right)^{1/2}}{s}\right)^{1/2},\\ \phi_{\mathbf{b}_s}(z)&=-\left(\frac{1-\left(1-\frac{s}{2}\left(-\frac{1}{z}\right)^2\right)^2}{s}\right)^{-1/2}-z. \end{aligned}\tag{3.1}$$

It is then easy to see that $\mathbf{b}_s$ is a FID distribution from Theorem 2.6.

We can see many powers in (3.1), and so we try to extend these powers following the paper [AHb].

Definition 3.1. For $0 < \alpha \le 2,\ r > 0,\ s \in \mathbb{C}\backslash\{0\}$, define the function $G^{\alpha}_{s,r}$ as follows:

$$G^{\alpha}_{s,r}(z) = -r^{1/\alpha}\left(\frac{1-(1-s(-\frac{1}{z})^{\alpha})^{1/r}}{s}\right)^{1/\alpha}.$$

Also we denote its reciprocal by $F^{\alpha}_{s,r}(z) := \frac{1}{G^{\alpha}_{s,r}(z)}$. It turns out easily that $F^{\alpha}_{s,1}(z) = z$.

The reader probably wonders why such a deformation appears. The reason is summarized in the following relation.

Theorem 3.2 (Arizmendi-Hasebe [AHb]). *For $r, u > 0,\ 2 \ge \alpha > 0,\ s \in \mathbb{C}\backslash\{0\}$, we obtain*

$$F^{\alpha}_{s,r} \circ F^{\alpha}_{us,u} = F^{\alpha}_{us,ur}.$$

In the particular case $u = \frac{1}{r}$, this relation reads $(F^{\alpha}_{s,r})^{-1} = F^{\alpha}_{s/r,1/r}$.

This deformation is considered so that the above relation holds. A remarkable point is that the inverse function $(F^{\alpha}_{s,r})^{-1}$ is contained in the original family, so that the computation of ϕ_{μ} is possible.

We have deformed the Stieltjes transform of $\mathbf{b}_s$, but we have to show that the deformed family still corresponds to probability measures.

Theorem 3.3. *Let $1 \le r < \infty$, $0 < \alpha \le 2$. Assume one of the following conditions:*

(i) $0 < \alpha \le 1$, $(1-\alpha)\pi \le \arg s \le \pi$;

(ii) $1 < \alpha \le 2$, $0 \le \arg s \le (2-\alpha)\pi$.

Then $G^{\alpha}_{s,r}$ is the Stieltjes transform of a probability measure $\mu^{\alpha}_{s,r}$.

The measures $\mu^{\alpha}_{s,r}$ contain some well known distributions. When $(\alpha, s, r) = (2, s, 2)$, the measure $\mu^{2}_{s,2}$ is the symmetrized beta $\mathbf{b}_s$, and when $(\alpha, s, r) = (1, -1, \frac{1}{a})$, the beta distribution

$$\beta_{1-a,1+a}(dx) = \frac{\sin(\pi a)}{\pi a}x^{-a}(1-x)^{a}\,dx, \quad 0 < x < 1$$

appears. In the particular case $a = \frac{1}{2}$, the measure $\beta_{1/2,3/2}$ coincides with the free Poisson law π_1 up to scaling. Using the explicit Voiculescu transform $\phi_{\mu^{\alpha}_{s,r}}$ and Theorem 2.6, we can prove the following.

Theorem 3.4. *Assume that (α, s, r) satisfies the assumption of Theorem 3.3. Then*

(1) *$\mu^{\alpha}_{s,2}$ is FID.*

(2) *$\mu^{\alpha}_{s,r}$ is FID if $0 < \alpha \le 1$ and $1 \le r \le 2$.*

(3) *$\mu^{\alpha}_{s,r}$ is FID if $1 \le \alpha \le 2$ and $1 \le r \le \frac{2}{\alpha}$.*

(4) *$\mu^{1}_{s,3}$ is FID if and only if s is purely imaginary.*

(5) *If $\alpha > 1$, there exists $r_0 = r_0(\alpha, s) > 1$ such that $\mu^{\alpha}_{s,r}$ is not FID for $r > r_0$.*

3.2 Explicit probability density but implicit Voiculescu transform

Recent work has found many other examples of FID distributions. In most cases, the Voiculescu transform ϕ_μ cannot be expressed explicitly, so that the proofs become more technical.

Theorem 3.5. *The following probability distributions are FID.*

(1) *(Belinschi et al. [BBLS11]) The Gaussian*

$$\mathbf{g}(dx) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} 1_{\mathbf{R}}(x)\, dx.$$

(2) *(Anshelevich et al. [ABBL10]) The q-Gaussian distribution*

$$\mathbf{g}_q(dx) = \frac{\sqrt{1-q}}{\pi} \sin\theta(x) \prod_{n=1}^{\infty}(1-q^n)|1-q^n e^{2i\theta(x)}|^2 1_{\left[-\frac{2}{\sqrt{1-q}}, \frac{2}{\sqrt{1-q}}\right]}(x)\, dx$$

for $q \in [0,1)$, where $\theta(x)$ is the solution of $x = \frac{2}{\sqrt{1-q}}\cos\theta$, $\theta \in [0,\pi]$. When $q \to 1$, $\mathbf{g}_q$ converges weakly to $\mathbf{g}$, and $\mathbf{g}_0$ coincides with $\mathbf{w}$. For $q \in (0,1)$, the density function of $\mathbf{g}_q$ can be written as [LM95]

$$\frac{1}{2\pi} q^{-\frac{1}{8}} (1-q)^{\frac{1}{2}} \Theta_1\left(\frac{\theta(x)}{\pi}, \frac{1}{2\pi i}\log q\right),$$

where $\Theta_1(z,\tau) := 2\sum_{n=0}^{\infty}(-1)^n (e^{i\pi\tau})^{(n+\frac{1}{2})^2}\sin(2n+1)\pi z$ is a Jacobi theta function.

(3) *(Arizmendi-Belinschi [AB]) The ultraspherical distribution*

$$\frac{1}{16^n B(n+\frac{1}{2}, n+\frac{1}{2})}(4-x^2)^{n-\frac{1}{2}} 1_{[-2,2]}(x)\, dx$$

for $n = 1, 2, 3, \cdots$.

(4) *(Arizmendi-Hasebe-Sakuma [AHS]) Let X be a random variable following Wigner's semicircle law. Then X^4 also follows a FID law.*

(5) *(Arizmendi-Hasebe-Sakuma [AHS]) The chi-square distribution*

$$\frac{1}{\sqrt{\pi x}} e^{-x} 1_{[0,\infty)}(x)\, dx.$$

(6) *(Arizmendi-Hasebe [AHa]) The Boolean stable law $\mathbf{b}_\alpha^\rho$ is defined by*

(i) $F_{\mathbf{b}_\alpha^\rho}(z) = z + e^{i\pi\rho\alpha} z^{-\alpha+1}$ *for $\alpha \in (0,1)$ and $\rho \in [0,1]$;*

(ii) $F_{\mathbf{b}_\alpha^\rho}(z) = z + 2\rho i - \frac{2(2\rho-1)}{\pi}\log z$ *for $\alpha = 1$ and $\rho \in [0,1]$;*

(iii) $F_{\mathbf{b}_\alpha^\rho}(z) = z - e^{i(\alpha-2)\rho\pi} z^{-\alpha+1}$ *for $\alpha \in (1,2]$ and $\rho \in [0,1]$.*

$\mathbf{b}_\alpha^\rho$ *is FID if and only if: (a)* $0 < \alpha \le \frac{1}{2}$ *and* $\rho \in [0,1]$*; (b)* $\frac{1}{2} \le \alpha \le \frac{2}{3}$ *and* $2 - \frac{1}{\alpha} \le \rho \le \frac{1}{\alpha} - 1$*; (c)* $\alpha = 1$ *and* $\rho = \frac{1}{2}$*. For* $\alpha < 1$*, the probability density function can be written in the form*

$$\frac{d\mathbf{b}_\alpha^\rho}{dx} = \begin{cases} \dfrac{\sin(\pi\rho\alpha)}{\pi} \dfrac{x^{\alpha-1}}{x^{2\alpha} + 2x^\alpha \cos(\pi\rho\alpha) + 1}, & x > 0, \\ \dfrac{\sin(\pi(1-\rho)\alpha)}{\pi} \dfrac{|x|^{\alpha-1}}{|x|^{2\alpha} + 2|x|^\alpha \cos(\pi(1-\rho)\alpha) + 1}, & x < 0, \end{cases}$$

(7) *(Bożejko-Hasebe [BH]) The Meixner distribution*

$$\frac{4^t}{2\pi\Gamma(2t)} |\Gamma(t + ix)|^2 1_{\mathbb{R}}(x)\, dx$$

for $0 < t \le \frac{1}{2}$.

(8) *(Bożejko-Hasebe [BH]) The logistic distribution*

$$\frac{\pi}{2} \left(\frac{1}{\cosh \pi x} \right)^2 1_{\mathbb{R}}(x)\, dx.$$

(9) *(Hasebe [H]) The beta distribution*

$$\boldsymbol{\beta}_{p,q}(dx) = \frac{1}{B(p,q)} x^{p-1}(1-x)^{q-1} 1_{[0,1]}(x)\, dx$$

for $(p,q) \in D$*. The region* D *is shown in Fig. 5. This result extends (3).*

(10) *(Hasebe [H]) The beta prime distribution*

$$\frac{1}{B(p,q)} \frac{x^{p-1}}{(1+x)^{p+q}} 1_{[0,\infty)}(x)\, dx$$

for $(p,q) \in D'$*. The region* D' *is shown in Fig. 6.*

(11) *(Hasebe [H]) The t-distribution*

$$\frac{1}{B(\frac{1}{2}, q - \frac{1}{2})} \frac{1}{(1+x^2)^q} 1_{\mathbb{R}}(x)\, dx$$

for $q \in (\frac{1}{2}, 2] \cup [2 + \frac{1}{4}, 4] \cup [4 + \frac{1}{4}, 6] \cup \cdots$.

(12) *(Hasebe [H]) The gamma distribution*

$$\frac{1}{\Gamma(p)} x^{p-1} e^{-x} 1_{[0,\infty)}(x)\, dx$$

for $p \in (0, \frac{1}{2}] \cup [\frac{3}{2}, \frac{5}{2}] \cup [\frac{7}{2}, \frac{9}{2}] \cup \cdots$.

(13) *(Hasebe [H]) The inverse gamma distribution*

$$\frac{1}{\Gamma(p)} x^{-p-1} e^{-1/x} 1_{[0,\infty)}(x)\, dx$$

for $p \in (0, \frac{1}{2}] \cup [\frac{3}{2}, \frac{5}{2}] \cup [\frac{7}{2}, \frac{9}{2}] \cup \cdots$.

Remark 3.6. We mention some remarks on the above results.

(4) It is not known whether $|X|^q$ ($q \in \mathbb{R}$) is FID or not, except $q = 2, 4$. For $q = 2$, X^2 follows the free Poisson law $\boldsymbol{\pi}_1$ which is FID.

(6) The Boolean stable law is characterized by some stability, but not with respect to classical convolution, but Boolean convolution which appears as the sum of Boolean independent random variables [SW97].

(7) The Meixner distributions (for $t > 0$) are laws of a Lévy process, called a Meixner process [ST98], since they have the characteristic functions $\left(\frac{1}{\cosh(z/2)}\right)^{2t}$. When $t = \frac{1}{2}$, the Meixner distribution coincides with

$$\frac{1}{\cosh \pi x} 1_{\mathbb{R}}(x)\, dx,$$

which is called the hyperbolic secant distribution. It is known as the law of Lévy's stochastic area [L51].

It is unknown whether the Meixner distributions are FID for $t > \frac{1}{2}$ or not.

(9,10) The beta distributions contain the affine transformations of Wigner's semicircle law and the free Poisson law. The beta prime distributions contain the affine transformation of a free $\frac{1}{2}$-stable law [BP99].

Some parameters (p, q) outside the regions D, D' correspond to non FID distributions, but some still remain to be unknown whether they are FID or not.

(12,13) The gamma distributions and inverse gamma distributions are limits of beta and prime beta distributions, so that they are FID as consequences of Theorem 3.5(9), (10). The result (12) extends (5).

The probability measures above are ID too except (3), (4), (9) and part of (6).[6] The proofs can be found in Bondesson's book [B92].

In the same book, the class of GGCs (generalized gamma convolutions) is studied in details as a subclass of ID distributions. The main tool in the analysis of GGCs is Pick-Nevanlinna functions, the same tool as used in free probability. The author is now focusing on this similarity in two probabilities and hoping to discover a general theory behind them.

[6] The Boolean stable law is ID when positive, i.e. $\rho = 1$. If $\rho \neq 1$, the author does not know if the measure is ID or not.

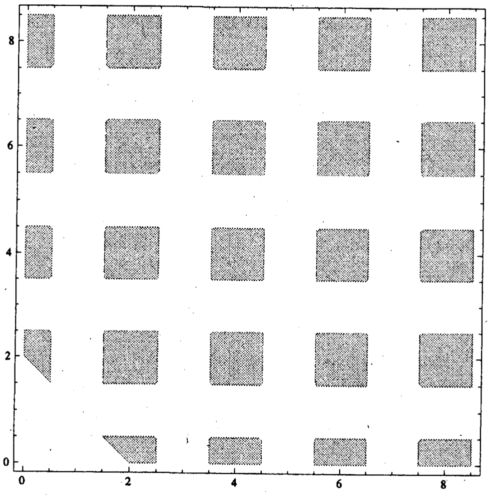

Figure 5: Region D

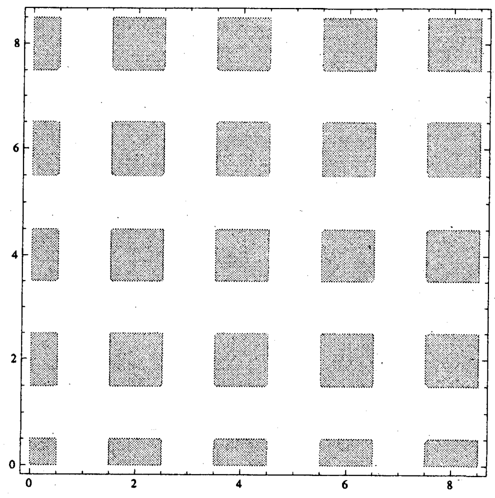

Figure 6: Region D'

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