数理解析研究所講究録
第 1859 巻 2013 年 115-126

Tunneling for spatially cut-off $P(\phi)_2$-Hamiltonians

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1 Introduction

Let U be a potential function on $\mathbb{R}^N$ such that

(1) $U(x)\geq 0$, $\liminf_{|x|\to\infty}U(x)>0$

(2) $\mathcal{Z}=\{x\mid U(x)=0\}$ is a finite set and the Hessians of U at $\mathcal{Z}$ are non-degenerate.

Let us consider a Schrödinger operator $-\Delta+\lambda^2U$ on $L^2(\mathbb{R}^N,dx)$. Here, λ is a parameter corresponding to the inverse of the Planck constant. Then for any $R>0$, the spectral subset $\sigma\left(\lambda^{-1}\left(-\Delta+\lambda^2U\right)\right)\cap[0,R]$ is discrete spectrum for large λ and the eigenvalues can be approximated by the eigenvalues of some harmonic oscillators. Moreover, If U is a symmetric double well type potential function, the gap between first two smallest eigenvalues are exponentially small when $\lambda\to\infty$. Also the exponential decay rate is given by the Agmon distance between zero points. Note that by the unitary map $f=f(x)\mapsto\lambda^{N/4}f(\lambda^{1/2}x)$, $\lambda^{-1}\left(-\Delta+\lambda^2U\right)$ is unitarily equivalent to $-\Delta+\lambda U\left(\cdot/\sqrt{\lambda}\right)$ on $L^2(\mathbb{R}^N,dx)$. In this sense, spatially cut-off $P(\phi)_2$-Hamiltonian is a self-adjoint operator $-L+V_\lambda$ on $L^2(\mathcal{S}'(\mathbb{R}),d\mu)$, where the probability measure μ is formally given by using the "Lebesgue measure" dw on $L^2(\mathbb{R},dx)$:

$$d\mu(w)=\det\left(\frac{\sqrt{m^2-\Delta}}{2\pi}\right)^{1/2}\exp\left(-\frac{1}{2}\left(\sqrt{m^2-\Delta}w,w\right)_{L^2(\mathbb{R})}\right)dw.$$

Hence $-L+V_\lambda$ is formally unitarily equivalent to $-\Delta_{L^2(\mathbb{R})}+\lambda U(w/\sqrt{\lambda})-\frac{1}{2}\mathrm{tr}(m^2-\Delta)^{1/2}$ on $L^2(L^2(\mathbb{R},dx),dw)$ where

$$U(w)=\frac{1}{4}\int_{\mathbb{R}}(w'(x)^2+m^2w(x)^2)dx+V(w),$$
$$V(w)=\int_{\mathbb{R}}:P(w(x)):g(x)dx$$

and $\Delta_{L^2(\mathbb{R}^n)}$ is the "Laplacian" on $L^2(\mathbb{R})$ and P is a polynomial bounded below. Therefore it is natural to expect that there are some relations between

(1) asymptotic behavior of low-lying spectrum of the operator $-L+V_\lambda$ as $\lambda\to\infty$

(2) global minimum points of U.

In this note, we discuss the asymptotic behavior of the first eigenvalue of $-L+V_\lambda$ and the gap of spectrum between the first and the second eigenvalue in terms of U based on [2] and [3]. The structure of this note is as follows. In Section 2, we recall semi-classical results for Schrödinger operators on $\mathbb{R}^N$. In Section 3, we give a definition of the spatially cut-off $P(\phi)_2$-Hamiltonian. In Section 4 and 5, we state our main results. In Section 6, we recall the proof of tunneling estimates for finite dimensional Schrödinger operators and give a rough sketch of the proof in our case. In Section 7, we explain basic properties of Agmon distance and the relation to instanton in our model.

2 Tunneling for Schrödinger operators on $\mathbb{R}^N$

Assume

(1) $U \in C^\infty(\mathbb{R}^N)$, $U(x) \geq 0$ for all $x \in \mathbb{R}^N$ and $\liminf_{|x|\to\infty} U(x) > 0$.

(2) $\{x \mid U(x) = 0\} = \{x_1, \ldots, x_n\}$.

(3) $Q_i = \frac{1}{2}D^2U(x_i) > 0$ for all i.

Then the first eigenvalue $E_1(\lambda)$ of $-\Delta + \lambda U(\cdot/\sqrt{\lambda})$ is simple and

$$\lim_{\lambda\to\infty} E_1(\lambda) = \min_{1\leq i\leq n} \inf \sigma(-\Delta + (Q_i x, x)).$$

In addition to the assumptions above, we assume the symmetry of U:

(4) $U(x) = U(-x)$,

(5) $\{x \mid U(x) = 0\} = \{-x_0, x_0\} \quad (x_0 \neq 0)$.

Let $E_2(\lambda)$ be the second eigenvalue. By Harrell, Jona-Lasinio, Martinelli and Scoppola, Simon, Helffer and Sjöstrand ([17, 21, 32, 33, 19, 20]) and others

$$\lim_{\lambda\to\infty} \frac{\log(E_2(\lambda) - E_1(\lambda))}{\lambda} = -d_U^{Ag}(-x_0, x_0),$$

where $d_U^{Ag}(-x_0, x_0)$ is the Agmon distance between $-x_0$ and x_0 ([1, 25]) and

$$d_U^{Ag}(-x_0, x_0) = \inf\left\{ \int_{-T}^{T} \sqrt{U(x(t))}|\dot{x}(t)|dt \;\middle|\; x \text{ is a smooth curve on } \mathbb{R}^N \text{ with } x(-T) = -x_0, x(T) = x_0 \right\}.$$

Carmona and Simon[6] gave another representation d_U^{CS} of d_U^{Ag} using an action integral:

$$d_U^{CS}(-x_0, x_0) = \inf\left\{ \int_{-\infty}^{\infty} \left(\frac{1}{4}|x'(t)|^2 + U(x(t))\right) dt \;\middle|\; \lim_{t\to-\infty} x(t) = -x_0, \lim_{t\to\infty} x(t) = x_0 \right\}.$$

Remark 1. *The classical Newton's equation corresponding to $-\Delta + U$ is*

$$x''(t) = -2(\nabla U)(x(t)).$$

The above action integral is euclidean action integral.
The minimizing path $x_E = x_E(t)$ $(-\infty < t < \infty)$ is called an instanton which satisfies

$$x''(t) = 2(\nabla U)(x(t)).$$

3 Definition of spatially cut-off $P(\phi)_2$-Hamiltonian

Let $m > 0$. Let μ be the Gaussian measure on the space of tempered distributions $\mathcal{S}'(\mathbb{R})$ such that

$$\int_W {}_{\mathcal{S}(\mathbb{R})}\langle \varphi, w\rangle^2_{\mathcal{S}'(\mathbb{R})} d\mu(w) = \left((m^2 - \Delta)^{-1/2}\varphi, \varphi\right)_{L^2}.$$

Let $\mathcal{E}$ be the Dirichlet form defined by

$$\mathcal{E}(f, f) = \int_W \|\nabla f(w)\|^2_{L^2(\mathbb{R},dx)} d\mu(w) \quad f \in \mathrm{D}(\mathcal{E}),$$

where $\nabla f(w)$ is the unique element in $L^2(\mathbb{R}, dx)$ such that

$$\lim_{\varepsilon \to 0} \frac{f(w + \varepsilon\varphi) - f(w)}{\varepsilon} = (\nabla f(w), \varphi)_{L^2(\mathbb{R},dx)}.$$

The generator $-L(\geq 0)$ of $\mathcal{E}$ is one of expressions of a free Hamiltonian. Let $P(x) = \sum_{k=0}^{2M} a_k x^k$, where $a_{2M} > 0$. Let $g \in C_0^\infty(\mathbb{R})$ with $g(x) \geq 0$ for all x and define for $h \in H^1(= H^1(\mathbb{R}))$,

$$V(h) = \int_{\mathbb{R}} P(h(x))g(x)dx$$
$$U(h) = \frac{1}{4}\int_{\mathbb{R}} \left(h'(x)^2 + m^2 h(x)^2\right) dx + V(h)$$

We want to consider an operator like

$$-L + \lambda V(w/\sqrt{\lambda}) \quad \text{on} \quad L^2(\mathcal{S}'(\mathbb{R}), d\mu).$$

The difficulty is in the definition of $w(x)^k$ because w is an element of the Schwartz distribution. Instead of $w(x)^k$, we use Wick power $: w(x)^k :$ which requires renormalizations for which we refer the readers to [12, 31, 34, 7]. For $P = P(x) = \sum_{k=0}^{2M} a_k x^k$ with $a_{2M} > 0$, define

$$\int_{\mathbb{R}} : P\left(\frac{w(x)}{\sqrt{\lambda}}\right) : g(x)dx = \sum_{k=0}^{2M} a_k \int_{\mathbb{R}} : \left(\frac{w(x)}{\sqrt{\lambda}}\right)^k : g(x)dx.$$

We write

$$: V\left(\frac{w}{\sqrt{\lambda}}\right) : = \int_{\mathbb{R}} : P\left(\frac{w(x)}{\sqrt{\lambda}}\right) : g(x)dx,$$
$$V_\lambda(w) = \lambda : V\left(\frac{w}{\sqrt{\lambda}}\right) : .$$

Definition 2. *The spatially cut-off $P(\phi)_2$-Hamiltonian $-L+V_\lambda$ is defined to be the unique self-adjoint extension operator of $(-L+V_\lambda, \mathfrak{F}C_b^\infty(\mathcal{S}'(\mathbb{R})))$.*

It is known that $-L+V_\lambda$ is bounded from below and the first eigenvalue $E_1(\lambda)$ is simple and the corresponding positive eigenfunction $\Omega_{1,\lambda}$ exists. See [12, 31, 34].

4 Semi-classical limit of the first eigenvalue

Assumption 3. (A1) *$U(h) \geq 0$ for all $h \in H^1$ and*

$$\mathcal{Z} = \{h \in H^1 \mid U(h) = 0\} = \{h_1, \ldots, h_n\}$$

is a finite set.
(A2) *The Hessian $\nabla^2 U(h_i)$ $(1 \leq i \leq n)$ is strictly positive.*

Remark 4. *Since for any $h \in H^1$,*

$$\nabla^2 U(h_i)(h,h) = \frac{1}{2}\int_{\mathbf{R}} h'(x)^2 dx + \int_{\mathbf{R}} \left(\frac{m^2}{2}h(x)^2 + P''(h_i(x))g(x)h(x)^2\right) dx,$$

the non-degeneracy is equivalent to

$$\inf \sigma(-\Delta + m^2 + 2P''(h_i(x))g(x)) > 0.$$

Theorem 5. *Assume* (A1) *and* (A2) *and let $E_1(\lambda) = \inf \sigma(-L+V_\lambda)$. Then*

$$\lim_{\lambda\to\infty} E_1(\lambda) = \min_{1\leq i\leq n} E_i,$$

where

$$E_i = \inf \sigma(-L+Q_i), \quad Q_i(w) = \frac{1}{2}\int_{\mathbf{R}} : w(x)^2 : P''(h_i(x))g(x)dx. \tag{4.1}$$

Let $H^s(\mathbb{R})$ be the Sobolev space with the norm:

$$\|\varphi\|_{H^s(\mathbb{R})} = \|(m^2-\Delta)^{s/2}\varphi\|_{L^2(\mathbb{R},dx)}.$$

Let $H = H^{1/2}(\mathbb{R})$. Then H is the Cameron-Martin subspace of μ and μ exists on $W \subset \mathcal{S}'(\mathbb{R})$:

$$W = \left\{w \in \mathcal{S}'(\mathbb{R}) \;\middle|\; \|w\|_W^2 = \int_{\mathbf{R}} |(1+|x|^2-\Delta)^{-1}w(x)|^2 dx < \infty\right\}.$$

The triple (W, H, μ) is an abstract Wiener space [15]. The proof of Theorem 5 is done by using

(1) IMS localization argument[32]

(2) Lower bound estimate for the bottom of the spectrum of $-L+V_\lambda$ which follows from logarithmic Sobolev ineqaulities [16]

(3) Large deviation and Laplace method for Wick polynomials (Wiener chaos)[5, 23, 24]

See [2, 3] for the detail of the proof.

5 Tunneling for spatially cut-off $P(\phi)_2$-Hamiltonians

Let

$$E_2(\lambda) = \inf\{\sigma(-L + V_\lambda) \setminus \{E_1(\lambda)\}\}.$$

It is known that $E_2(\lambda) > E_1(\lambda)$ (due to [34]). We prove that $E_2(\lambda) - E_1(\lambda)$ is exponentially small when $\lambda \to \infty$ in the case where the potential function is double well type.

Assumption 6. (A3) *For all x, $P(x) = P(-x)$ and $\mathcal{Z} = \{h_0, -h_0\}$, where $h_0 \neq 0$.*

Theorem 7. *Assume* (A1), (A2), (A3). *Then*

$$\limsup_{\lambda\to\infty} \frac{\log(E_2(\lambda) - E_1(\lambda))}{\lambda} \leq -d_U^{Ag}(h_0, -h_0).$$

It is still an open problem to obtain more precise asymptotics of the gap of the spectrum.

Example 8. *Fix $g \in C_0^\infty(\mathbb{R})$. Let $n \in \mathbb{N}$. For sufficiently large $a > 0$, the polynomial*

$$P(x) = a(x^2 - 1)^{2n} - C$$

satisfies (A1), (A2), (A3). *Here C is a positive constant which depends on a, g.*

We define the Agmon distance $d_U^{Ag}(-h_0, h_0)$.

Assumption 9. *In the definition below, we always assume $U(h) \geq 0$ for all h.*

Note that $h_0, -h_0 \in H^1(\mathbb{R})$. Hence it suffices to define the Agmon distance on $H^1(\mathbb{R})$. Let $0 < T < \infty$ and $h, k \in H^1(\mathbb{R})$. Let $AC_{T,h,k}(H^1(\mathbb{R}))$ be the set of all absolutely continuous paths $c : [0, T] \to H^1(\mathbb{R})$ satisfying $c(0) = h, c(T) = k$.

Definition 10. *We define the Agmon distance between h, k by*

$$d_U^{Ag}(h,k) = \inf\left\{\ell_U(c) \mid c \in AC_{T,h,k}(H^1(\mathbb{R}))\right\},$$

where

$$\ell_U(c) = \int_0^T \sqrt{U(c(t))}\|c'(t)\|_{L^2} dt.$$

Agmon metric is conformal to L^2-metric. However the function U is defined on H^1. So it is natural to consider on which space the Agmon distance is defined. The following classical result gives a suggestion for this problem:
For any $h, k \in H^{1/2}(\mathbb{R})$, there exists $u(= u(t,x)) \in H^1((0,T) \times \mathbb{R})$ such that

(1) $u(0,x) = h(x)$ and $u(T,x) = k(x)$,

(2) $\displaystyle\int_0^T \sqrt{U(u(t))}\|u'(t)\|_{L^2} dt < \infty$

Thus we extend the definition of the Agmon distance to the space $H^{1/2}$.

Definition 11 ([3]).
(1) *Let $h, k \in H^{1/2}$. Let $\mathcal{P}_{T,h,k,U}$ be all continuous paths $c = c(t)$ $(0 \le t \le T)$ on $H^{1/2}$ such that*

(i) $c \in AC_{T,h,k}(L^2(\mathbb{R}))$, $c(0) = h, c(T) = k$,

(ii) $c(t) \in H^1(\mathbb{R})$ *for* $\|c'(t)\|_{L^2}dt$ *-a.e.* $t \in [0,T]$ *and the length of c is finite:*

$$\ell_U(c) = \int_0^T \sqrt{U(c(t))}\|c'(t)\|_{L^2}dt < \infty.$$

(2) *Let $0 < T < \infty$. We define the Agmon distance between $h, k \in H^{1/2}(\mathbb{R})$ by*

$$d_U^{Ag}(h,k) = \inf\left\{\ell_U(c) \mid c \in \mathcal{P}_{T,h,k,U}\right\}.$$

It is not difficult to see the two definitions above of d_U^{Ag} coincides with each other on H^1.

Now let us recall some idea of the proof in [33] of the tunneling estimate in finite dimensional cases. Assume the assumptions (1), (2), (3), (4), (5) in Section 2. Then for the ground state $\Psi_{1,\lambda}$ of $-\Delta + \lambda^2 U$, we have

$$\lim_{\lambda\to\infty} \frac{1}{\lambda}\log\Psi_{1,\lambda}(x) = -\min\left(d_U^{Ag}(x,x_0), d_U^{Ag}(x,-x_0)\right).$$

This and estimates on the second eigenfunction implies

$$\lim_{\lambda\to\infty} \frac{\log\left(E_2(\lambda) - E_1(\lambda)\right)}{\lambda} = -d_U^{Ag}(x_0,-x_0).$$

Now let us consider the spatially cut-off $P(\phi)_2$-Hamiltonian as an infinite dimensional Schrödinger operator. Assume (A1), (A2), (A3). Let

$$d\mu_{\lambda,U} = \Omega_{1,\lambda}^2 d\mu, \quad \mu_U^\lambda = (S_\lambda)_*\mu_{\lambda,U}, \tag{5.1}$$

where $\Omega_{1,\lambda}$ is the ground state of $-L+V_\lambda$ and $S_\lambda w = \frac{w}{\sqrt{\lambda}}$. Formally $d\mu_U^\lambda(w) = \Psi_{1,\lambda}(w)^2dw$, where $\Psi_{1,\lambda}$ is the ground state for

$$-\Delta_{L^2(\mathbb{R})} + \lambda^2 U(w) - \frac{\lambda}{2}\mathrm{tr}(m^2-\Delta)^{1/2}.$$

It is natural to conjecture that μ_U^λ satisfies the large deviation principle with good rate function I_U:

$$I_U(h) = 2\min\left(d_U^{Ag}(h_0,h), d_U^{Ag}(-h_0,h)\right).$$

We prove a version of the upper bound estimate of this large deviation result which is sufficient for the proof of Theorem 7.

6 Proof of Theorem 7

Assume U satisfies (A1), (A2). Let $\mathcal{F}_U^W$ be the set of non-negative bounded globally Lipschitz continuous functions u on W such that

(i) $0 \le u(h) \le U(h)$ for all $h \in H^1$ and

$$\{h \in H^1 \mid U(h) - u(h) = 0\} = \{h_1, \dots, h_n\} = \{U = 0\}.$$

(ii) u is C^2 in $\cup_{i=1}^n B_{\delta_0}(h_i)$ for some $\delta_0 > 0$, where $B_\delta(h) = \{w \in W \mid \|w - h\|_W < \delta\}$.

(iii) The Hessians $\nabla^2 (U - u)(h_i)$ $(1 \le i \le n)$ are strictly positive.

Let $u \in \mathcal{F}_U^W$. For $w_1, w_2 \in W$, we define $\rho_u^W(w_1, w_2)$ by

(i) if $w_1 - w_2 \in L^2(\mathbb{R})$,

$$\rho_u^W(w_1, w_2) = \inf\left\{ \int_0^T \sqrt{u(w_1 + c(t))} \|c'(t)\|_{L^2} dt \;\middle|\; c \text{ is an absolutely continuous path on } L^2(\mathbb{R}) \text{ with } c(0) = 0,\, c(T) = w_2 - w_1 \right\}.$$

(ii) if $w_1 - w_2 \notin L^2(\mathbb{R})$, $\quad \rho_u^W(w_1, w_2) = \infty$.

Further define

$$\underline{\rho}_u^W(w_1, w_2) = \lim_{\varepsilon \to 0} \inf\left\{ \rho_u^W(w, \eta) \;\middle|\; w \in B_\varepsilon(w_1), \eta \in B_\varepsilon(w_2) \right\}.$$

In the case where $W = H = \mathbb{R}^N$, for any w_1, w_2, clearly,

$$\sup_{u \in \mathcal{F}_U^W} \underline{\rho}_u^W(w_1, w_2) = d_U^{Ag}(w_1, w_2).$$

Lemma 12. *Assume* (A1), (A2) *and $\mathcal{Z}$ consists two points $\{h, k\}$. Then*

$$d_U^{Ag}(h, k) = \sup_{u \in \mathcal{F}_U^W} \underline{\rho}_u^W(h, k).$$

We proved the above in the case of $h = h_0$, $k = -h_0$, where $\pm h_0$ are the zero points of U in [3]. But I think the equality holds for all points in $H^{1/2}$ under the assumptions (A1) and (A2).

Lemma 13. *Let $u \in \mathcal{F}_U^W$.*
(1) *Let O be a non-empty open subset of W and set $\rho_u^W(O, w) = \inf\{\rho_u^W(\phi, w) \mid \phi \in O\}$. Then*

$$\rho_u^W(O, \cdot) \in \mathrm{D}(\mathcal{E}),$$
$$|\nabla \rho_u^W(O, w)|_{L^2(\mathbb{R}, dx)} \le \sqrt{u(w)} \quad \mu\text{-a.s.}w.$$

(2) *Assume* (A1) *and* (A2). *Set $u_\lambda(w) = \lambda u(w/\sqrt{\lambda})$, $E_1(\lambda, u) = \inf \sigma(-L + V_\lambda - u_\lambda)$. Then $\lim_{\lambda \to \infty} E_1(\lambda, u)$ converges.*

Lemma 14. *Assume* (A1), (A2). *Let* $d\mu_{\lambda,U}(w) = \Omega_{1,\lambda}^2(w)d\mu$, *where* $\Omega_{1,\lambda}$ *is the ground state of* $-L + V_\lambda$. *Let* $r > \kappa$ *and* $0 < q < 1$. *Let* $B_\varepsilon(\mathcal{Z}) = \cup_{i=1}^n B_\varepsilon(h_i)$. *For large* λ,

$$\mu_{\lambda,U}\left(\left\{w \in W \;\middle|\; \rho_u^W\left(\frac{w}{\sqrt{\lambda}}, B_\varepsilon(\mathcal{Z})\right) \geq r\right\}\right) \leq \frac{C_1 e^{-2q\lambda(r-\kappa)}\|u\|_\infty}{\kappa^2(\lambda(1-q^2)\varepsilon^2 - C_2)}, \tag{6.1}$$

where C_i *are positive constants independent of* λ, r, κ.

Proof of Theorem 7. Note that

$$E_2(\lambda) - E_1(\lambda) = \inf\left\{ \frac{\int_W |\nabla f(w)|_{L^2}^2 d\mu_{\lambda,U}(w)}{\int_W f(w)^2 d\mu_{\lambda,U}(w)} \;\middle|\; f \in \mathrm{D}(\mathcal{E}) \cap L^\infty(W,\mu), f \not\equiv 0, f \perp 1 \text{ in } L^2(\mu_{\lambda,U}) \right\},$$

where $d\mu_{\lambda,U} = \Omega_{1,\lambda}^2 d\mu$ Note that the ground state measure $\mu_{\lambda,U}$ concentrate on $\{\pm\sqrt{\lambda}h_0\}$ by Lemma 14. So we introduce a function f such that $f = \pm 1$ in a neighborhood of $\pm\sqrt{\lambda}h_0$. Then $f \perp 1$ approximately in $L^2(\mu_{\lambda,U})$. This f can be constructed by using $\rho_u^W\left(w/\sqrt{\lambda}, B_\varepsilon(h_0)\right)$, $\rho_u^W\left(w/\sqrt{\lambda}, B_\varepsilon(-h_0)\right)$. Using the property

$$\|\nabla f\|_\infty < \infty, \quad \mathrm{supp}|\nabla f| \subset \left\{w \;\middle|\; \rho_u^W\left(\frac{w}{\sqrt{\lambda}}, B_\varepsilon(\mathcal{Z})\right) \approx \frac{\rho_u^W(h_0, -h_0)}{2}\right\}$$

and calculating the the ratio of the integrals of $|\nabla f|$ and f and applying Lemma 14, the proof is completed. □

7 Properties of Agmon distance and instanton

We already defined the Agmon distance d_U^{Ag} on $H^{1/2}$. Actually this is a continuous distance function on $H^{1/2}$ and the topology is the same as the one defined by the Sobolev norm. Also we can prove the existence of the geodesics between two zero points and the existence of instanton. The readers find these results in [3]. I do not prove the uniqueness of them yet.

Theorem 15 (Existence of geodesic). *Assume* (A1), (A2) *and* $\mathcal{Z}$ *consists of two points* $\{h, k\}$. *There exists a continuous curve* $c_\star$ *on* $H^{1/2}(\mathbb{R})$ *such that* $c_\star \in AC_{T,h,k}(L^2(\mathbb{R}))$ *and* $d_U^{Ag}(h,k) = \ell_U(c_\star)$. *Moreover* $c_\star$ *satisfies the following.*
(1) $c_\star(0) = h$, $c_\star(1) = k$ *and* $c_\star(t) \neq h, k$ *for* $0 < t < 1$.
(2) $c_\star = c_\star(t,x)$ *is a* C^∞ *function of* $(t,x) \in (0,1) \times \mathbb{R}$ *and* $c_\star \in H^1((\varepsilon, 1-\varepsilon) \times \mathbb{R})$ *for all* $0 < \varepsilon < 1$.
(3) *For almost every* t *in the Lebesgue measure, we have*

$$\sqrt{U(c_\star(t))}\|c_\star'(t)\|_{L^2} = d_U^{Ag}(h,k).$$

(4) $\int_0^\varepsilon \|c_\star'(t)\|_{L^2}^2 dt = \int_{1-\varepsilon}^1 \|c_\star'(t)\|_{L^2}^2 dt = +\infty$ *for all* $\varepsilon > 0$.

The instanton equation $\frac{\partial^2 u}{\partial t^2}(t,x) = 2(\nabla U)(u(t,x))$ reads

$$\frac{\partial^2 u}{\partial t^2}(t,x) + \frac{\partial^2 u}{\partial x^2}(t,x) = m^2 u(t,x) + 2P'(u(t,x))g(x). \tag{7.1}$$

Let $T > 0$ and define the action integral

$$\begin{aligned} I_{T,P}(u) &= \frac{1}{4}\iint_{(-T,T)\times\mathbb{R}} \left(\left|\frac{\partial u}{\partial t}(t,x)\right|^2 + \left|\frac{\partial u}{\partial x}(t,x)\right|^2 \right) dtdx \\ &\quad + \iint_{(-T,T)\times\mathbb{R}} \left(\frac{m^2}{4}u(t,x)^2 + P(u(t,x))g(x) \right) dtdx \end{aligned}$$

and

$$I_{\infty,P}(u) = \frac{1}{4}\int_{-\infty}^{\infty} \|\partial_t u(t)\|^2_{L^2(\mathbb{R})} dt + \int_{-\infty}^{\infty} U(u(t))dt.$$

Theorem 16 (Existence of instanton). *There exists a solution $u_\star = u_\star(t,x)$ $((t,x) \in \mathbb{R}^2)$ to the instanton equation which satisfies the following.*
(1) *For any $T > 0$, $u_\star|_{(-T,T)\times\mathbb{R}} \in H^1((-T,T)\times\mathbb{R}) \cap C^\infty((-T,T)\times\mathbb{R})$ and*

$$\lim_{t\to-\infty} \|u_\star(t) - h\|_{H^{1/2}} = 0, \quad \lim_{t\to\infty} \|u_\star(t) - k\|_{H^{1/2}} = 0.$$

(2) $I_{\infty,P}(u_\star) = d_U^{Ag}(h,k)$.

(3) *The function $u_\star$ is a minimizer of the functional $I_{\infty,P}$ in the set of functions u satisfying the following conditions:*

(i) $u|_{(-T,T)\times\mathbb{R}} \in H^1((-T,T),\mathbb{R})$ *for all* $T > 0$,

(ii) $\lim_{t\to-\infty} \|u(t) - h\|_{H^{1/2}} = 0, \quad \lim_{t\to\infty} \|u(t) - k\|_{H^{1/2}} = 0.$

Now we explain the relation between $c_\star$ and $u_\star$. Let

$$\begin{aligned} \rho(t) &= \frac{1}{2d_U^{Ag}(h,k)} \int_{1/2}^{t} \|c'(s)\|^2_{L^2} ds \quad 0 < t < 1, \\ \sigma(t) &= \frac{1}{2d_U^{Ag}(h,k)} \int_{-\infty}^{t} \|u'(s)\|^2_{L^2} ds \quad t \in \mathbb{R}. \end{aligned}$$

Then $\rho^{-1}(t) = \sigma(t)$ $(t \in \mathbb{R})$ and

$$\begin{aligned} u_\star(t,x) &= c_\star(\sigma(t),x) \quad t \in \mathbb{R}, \\ u_\star(\rho(t),x) &= c_\star(t,x) \quad 0 < t < 1. \end{aligned}$$

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