数理解析研究所講究録
第 1859 巻 2013 年 127-137

Gauge Invariance, Gauge Fixing, and Gauge Independence*

Izumi OJIMA (RIMS, Kyoto University)

1 Why a fixed law of dynamics for each system?

One of the implicit but standard preconceptions in physical sciences seems to be that

"a physical theory should describe a specific physical system ***with a fixed law of dynamics***".

Sticking to this belief causes the following difficulties:

i) difficulty caused by "***singular constrained dynamics***" with a degenerate symplectic form (typically in a gauge theory to be quantized), and

ii) (mathematical) difficulty in treating "***explicitly broken symmetris***" (e.g., broken scale invariance and approximate flavour symmetry of hadrons).

If we are free from the above prejudice, we can easily relativize the situation concerning ***overlap*** vs. ***separation*** between responses of a physical system against two sorts of actions on its internal ($\Rightarrow$ symmetries) and external ($\Rightarrow$ dynamics) degrees of freedom.

A geometric analogy: resolution of singularity

Between i) & ii) with no inevitable relations seen at first, some aspect of mutual duality starts to show up. To understand it, the analogy to geometric configurations of two diagrams (like an arc and a surface) ***with or without*** intersections will be helpful:

by shifting the diagrams along the axis orthogonal to the intersections, we can freely change their contact relations ***with or without*** intersections, which is a simple-minded picture of the blowing-up method for singularity resolutions.

Or, even if we do not touch on given two separated diagrams, they can be ***viewed*** as intersecting by choosing suitable angles of axis of our sight.

The former is in the active version of deformations, while the latter in the passive one.

*於 数研研究集会 『量子場の数理とその周辺』 (2012.11)

Duality between gauge constraints and explicit breaking

Going back from this analogy to our physical context of symmetry and dynamics, we can interpret, respectively,
· ***constrained*** dynamics with overlap of internal- & space-time symmetry
$\longleftrightarrow$ two diagrams with intersection as ***singularity***, and
· ***explicitly broken*** symmetry $\leftrightarrow$ ***blowing-up*** of intersection singularity.

Then, the two questions must be asked about ***local gauge invariance***:

a) Why is such an awkward detour inevitable that the beautiful ***gauge symmetry*** should be ***explicitly violated by hand*** via "gauge fixing conditions" which make the 1st-class ***constraints*** into the 2nd-class ones? And,

b) Is the vital essence of ***local gauge invariance*** really recovered after the ***explicit breaking***? What are the mutual relations among ***different theories*** quantized with different gauge fixing conditions?

2 To relativize dynamics of a system

In special or general theory of relativity, ***standard reference systems are relativized and pluralized for one and the same physical system***, whose mutual relations are controlled, deformed and compared via Lorentz or general coordinate transformations, respectively.

Generalizing this excellent idea of relativity, we can naturally and legitimately ***relativize and pluralize dynamics*** of a physical system whose ***mutual relations are controlled, deformed and compared;***
the ***freedom*** attained by this extension is expected to liberate us from the stereotyped spell of "a physical system *with a fixed law of dynamics*". Then, it will enable us to examine a theoretical framework of theories to describe "a physical systems with ***a family of dynamical laws*** exhibited in an array, where their mutual relations are systematically examined from the viewpoints of deformation and evolutionary theories": in [1], the essence of this line of thought has been proposed under the name of "***Theory Bundle***", bundles of theories patched together by the "***method of variation of natural constants***".

2.1 Relevance to/in controls and evolutions

While someone dubious may take this idea as a groundless "fairy tale" or "science fiction", its essence may have alreay been embodied (partially) in the control theory to treat physical systems from the viewpoint of controls; such a natural way of thinking can be treated as a heterodoxical idea only within the traditional stance of theoretical and/or mathematical physics (if it sticks to the precise description of *objective nature* existing outside of us).

Along the above line of thought, the rational way of ontogenetic descriptions of historical evolution processes of physical nature can be envisaged;

the goal will, perhaps, be attained in parallel with the scientific explanations of the biological evolutions to be implemented through the bi-directional passage between the ontogenesis of species with fixed law of repeatable dynamics and the phylogenesis consisting of evolution of laws governing species-speficic laws.

3 Framework for multiple laws of dynamics and applications

Now the required mathematical bases for this purpose can be consolidated by connecting the essence of multiple laws of dynamics based on the "***groupoid dynamical systems***" with the notions of "***sectors***" and of "***sector space***" in the framework of "***quadrality scheme***" based on "***Micro-Macro duality***" [2].

After brief explanation of groupoids and groupoid dynamical systems, an application of the framework will be discussed in the case of local gauge invariance: on the basis of the duality in gauge theories between

gauge constraints $\rightleftarrows$ gauge fixing conditions,

we aim at providing the answer to the afore-mentioned questions in terms of gauge sectors and of gauge equivalence.

The same method can be applied also to the problems of renormalization and of unification of four interactions. In the former, the duality plays important roles between "cutoffs" (or, regularizations of ultra-violet divergences) to circumvent Haag's no-go theorem and renormalization conditions and, in the latter, the evolution of dynamical laws can be discussed in "historical evolution of physical nature".

3.1 Groupoids and groupoid dynamical systems

First we need to explain the notion of "***groupoid***" and the "***groupoid dynamical system***".

In a word, "groupoid" is a family of invertible transformations from an initial point to a final one, which can be thought of as a family of groups scattered over spacetime. In this sense, it provides not only a generalization of the notion of groups in close relation with the basic ideas of local gauge invariance and of general relativity, but also an algebraic and generalized formulation of "equivalence relations" ubiquitously found at the basis of any kind of mathematical descriptions.

Definition of a groupoid Γ

A groupoid Γ is defined on a set $\Gamma^{(0)}$ (called unit space) and two maps $s, t : \Gamma \to \Gamma^{(0)}$ satisfying the following three properites R1), R2), R3). When $t(\gamma) = x, s(\gamma) = y$, we write $x \overset{\gamma}{\leftarrow} y$ or $\gamma : x \leftarrow y$, where γ is called an arrow from y to x and for which we have:

R1) For any $x \in \Gamma^{(0)}$, there is an arrow $x \overset{1_x}{\leftarrow} x$ from x to x called a unit arrow :

R2) when $x \overset{\gamma_1}{\leftarrow} y$ and $y \overset{\gamma_2}{\leftarrow} z$, there exists a composition $x \overset{\gamma_1\gamma_2}{\leftarrow} z$ of arrows γ_1 and of γ_2 from z to x.

R3) when γ is an arrow $x \overset{\gamma}{\leftarrow} y$ from y to x, there exists the inverse $\gamma^{-1} \in \Gamma$ from x to y in the sense of $\gamma\gamma^{-1} = 1_x : x \leftarrow x$ and of $\gamma^{-1}\gamma = 1_y : y \leftarrow y$.

If we define a relation R on $\Gamma^{(0)}$ by $R(x, y) = (\exists\gamma \in \Gamma$ such that $x \overset{\gamma}{\leftarrow} y)$, then R1), R2), R3) are equivalent to laws of symmetry, transitivity, and reflexivity, respectively. In this way, a groupoid is an algebraic generalization of an equivalence relation. While the equivalence relation $R(x, y)$ is symmetric in x, y owing to R3), we retain the direction of arrows $x \overset{\gamma}{\leftarrow} y$ for the purpose of unified treatment of such relations with preferred directions as order relations or arrows of time. The totality of the arrows γ is called a groupoid Γ and the set $\Gamma^{(0)}$ of x, y, etc., connected by the arrows $\gamma \in \Gamma$ in such a way as $x \overset{\gamma}{\leftarrow} y$ is called the "unit space" of the groupoid Γ. The element $y \in \Gamma^{(0)}$ in $x \overset{\gamma}{\leftarrow} y$ is called the source of γ and denoted by $s(\gamma) = y$, and, in this situation, $x \in \Gamma^{(0)}$ is called the target of γ and denoted by $t(\gamma) = y$.

In this context, a groupoid Γ can be viewed as a special sort of categories, all of the arrows of which are invertible. Then, the unit space $\Gamma^{(0)}$ is nothing but the set of objects of the category Γ, where

R1) means the assignment of the identity arrow 1_x corresponding to an object $x \in \Gamma^{(0)}$,

R2) explains the relation among the source, target and the composition of arrows in the category Γ,

R3) means the invertibility of all the arrows in Γ.

It can be easily understood that a groupoid is a generalization of the concept of a group and that a group is a special case of a groupoid: for this purpose, we equip a group G with a (virtual) object $*$ which is regarded as connected by any group element $g \in G$ to itself: $* \overset{g}{\leftarrow} *$. In this way, a group G can be viewed as a groupoid G whose unit space is given by $\{*\}$.

The important difference between a general groupoid and a group can be found in that any pair $(g_1, g_2) \in G \times G$ of group elements can be composable: $(g_1, g_2) \longmapsto g_1 g_2 \in G$,

whereas the product $\gamma_1\gamma_2$ of a pair $(\gamma_1, \gamma_2) \in \Gamma \times \Gamma$ can be defined only when the condition $s(\gamma_1) = t(\gamma_2)$ is satisfied: $\gamma_1\gamma_2 = [t(\gamma_1) \leftarrow s(\gamma_1) = t(\gamma_2) \leftarrow s(\gamma_2)] = [t(\gamma_1) \overset{\gamma_1\gamma_2}{\leftarrow} s(\gamma_2)]$.

The set of all the composable pairs (γ_1, γ_2) is denoted by $\Gamma^{(2)}$, which can be identified with the fiber product : $\Gamma^{(2)} := \{(\gamma_1, \gamma_2) \in \Gamma \times \Gamma; s(\gamma_1) = t(\gamma_2)\} = \Gamma \underset{\Gamma^{(0)}}{\times} \Gamma$ characterized by a commutative diagram:

$$\begin{array}{ccc} \Gamma & \overset{pr_1}{\leftarrow} & \Gamma \underset{\Gamma^{(0)}}{\times} \Gamma \\ s \downarrow & \circlearrowleft & \downarrow pr_2 \\ \Gamma^{(0)} & \underset{t}{\leftarrow} & \Gamma \end{array}$$

For $x, y \in \Gamma^{(0)}$ we denote $\Gamma^x_y := \{\gamma \in \Gamma; t(\gamma) = x, s(\gamma) = y\} = \Gamma(x \leftarrow y)$. Since $\Gamma^x_x \subset \Gamma^{(2)}$, any pair of elements in the subgroupoid Γ^x_x are composable, and hence it is a group $\Gamma^x_x \subset \Gamma$, among many such contained in Γ.

Transformation groupoid:

In contrast to a group G as a groupoid with a trivial unit space $\Gamma^{(0)} = \{*\}$, a typical example of a groupoid with non-trivial unit space $\Gamma^{(0)}$ can be found in a "transformation groupoid" associated with an action of a group G as follows:

An action of a group G on a space M is specified by a map $\alpha : G \times M \to M$ satisfying the following two properties:

$$\alpha(e, x) = x,$$
$$\alpha(g_1, \alpha(g_2, x)) = \alpha(g_1 g_2, x).$$

If we write $\alpha(g, x) = \alpha_g(x) = g \cdot x$, this means

$$\alpha_e = id_M, \alpha_{g_1} \circ \alpha_{g_2} = \alpha_{g_1 g_2},$$

or,

$$e \cdot x = x, g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x.$$

Namely, $G \ni g \longmapsto \alpha_g \in Aut(M)$ gives a representation α on M, where $Aut(M)$ denotes the totality of automorphisms transforming the space M onto itself with leaving the structure of M unchanged.

In this situation, a groupoid $\Gamma := G \times M$ consisting of the unit space $\Gamma^{(0)} = M$ which is acted on by arrows $\gamma = (x, g) := (g \cdot x \leftarrow x)$ (or $\gamma = (x, g) := (x \leftarrow g^{-1} \cdot x)$) is called a ***transformation groupoid***.

In a word, a transformation groupoid $\Gamma := G \times M$ consists of the pairs (g, x) of group elements $g \in G$ and points $x \in M$ which specify the motion $(g \cdot x \leftarrow x)$ of a point $x \in M$ under the action of $g \in G$. Or, it can also be viewed as a trivial G-principal bundle over a base space M with a fiber G, specified by an exact sequence $G \hookrightarrow \Gamma = G \times M \twoheadrightarrow M = \Gamma^{(0)} = \Gamma/G$. It can also be viewed as the graph $\{(x, g) = ((g \cdot x \leftarrow x)); x \in M, g \in G\}$ of G-action on M.

Sector decomposition by central measure

To classify symmetry breaking patterns in a universal way, we need Tomita decomposition theorem on sector decompositions:

Theorem 1 (Tomita decomposition theorem) *For a state ω of a C^*-algebra $\mathcal{X}$, a unique measure μ_ω called a central measure, (pseudo-)supported by factor states $\in F_{\mathcal{X}}$, exists with its barycenter $b(\mu_\omega) := \int_{E_{\mathcal{X}}} \rho d\mu_\omega(\rho) = \omega$ such that*
(0) $\left(\int_\Delta \rho \, d\mu_\omega(\rho)\right) \overset{|}{\circ} \left(\int_{E_{\mathcal{X}} \setminus \Delta} \rho \, d\mu_\omega(\rho)\right)$ for Borel set $\Delta \subset E_{\mathcal{X}}$,
(1) $\exists$unique projection $P = [\mathfrak{Z}_{\pi_\omega}(\mathcal{X})\Omega_\omega]$ on GNS space $\mathfrak{H}_\omega$ s.t. $P\Omega_\omega =$

Ω_ω, $P\pi_\omega(\mathcal{X})P \subseteq \{P\pi_\omega(\mathcal{X})P\}'$ *with center* $\mathfrak{Z}_{\pi_\omega}(\mathcal{X}) = \pi_\omega(\mathcal{X})'' \cap \pi_\omega(\mathcal{X})'$;
(2) $\mathfrak{Z}_{\pi_\omega}(\mathcal{X}) = \{\pi_\omega(\mathcal{X}) \cup P\}'$;
(3) $\mu_\omega(\gamma(A_1)_1 \cdots \gamma(A_n)) = \langle \Omega_\omega, \pi_\omega(A_1)P \cdots P\pi_\omega(A_n)\Omega_\omega\rangle$;
(4) $\mathfrak{Z}_{\pi_\omega}(\mathcal{X})$ *is *-isomorphic to the range of map* $L^\infty(E_\mathcal{X}, \mu_\omega) \ni f \mapsto \kappa_\omega(f) \in \pi_\omega(\mathcal{X})'$ *defined by*

$$\langle \Omega_\omega, \kappa_\omega(f)\pi_\omega(A)\Omega_\omega\rangle = \int_{E_\mathcal{X}} f(\rho)[\gamma(A)](\rho)d\mu_\omega(\rho)$$

and for $A, B \in \mathcal{X}$

$$\kappa_\omega(\gamma(A))\pi_\omega(B)\Omega_\omega = \pi_\omega(B)P\pi_\omega(A)\Omega_\omega,$$

where $\gamma(A) := (E_\mathcal{X} \ni \rho \longmapsto \rho(A)) \in L^\infty(E_\mathcal{X}, \mu_\omega)$.

κ_ω as *-algebraic embedding defines a projection-valued measure κ_ω : $(\mathcal{B}(\mathrm{supp}\ \mu_\omega) \ni \Delta \mapsto \kappa_\omega(\Delta) := \kappa_\omega(\chi_\Delta) \in Proj(\mathfrak{Z}_\omega(\mathcal{X})))$ on Borel subsets $\Delta \in \mathcal{B}(\mathrm{supp}\ \mu_\omega)$ of $E_\mathcal{X}$, satisfying

$$\langle \Omega_\omega, \kappa_\omega(\Delta)\Omega_\omega\rangle = \mu_\omega(\Delta).$$

Hilbert C*-module associated with sector structure

For a C*-dynamical system $G \underset{\tau}{\curvearrowright} \mathcal{X}$ with a G-action τ on $\mathcal{X}$, a Hilbert C*-bimodule $\widetilde{\mathcal{X}} := C(E_\mathcal{X}) \otimes \mathcal{X}$ can be defined with left $C(E_\mathcal{X})$ action and right $\mathcal{X}$ action together with $C(E_\mathcal{X})$-valued left inner product and $\mathcal{X}$-valued right inner product for $\hat{F}_1, \hat{F}_2 \in \widetilde{\mathcal{X}}$ as follows:

$$|\hat{F}_1\rangle\langle\hat{F}_2|_l := \Lambda(\hat{F}_1 \cdot \hat{F}_2^*) \in C(E_\mathcal{X});$$
$$\langle\hat{F}_1|\hat{F}_2\rangle_r := \widehat{\mu_\omega}(\hat{F}_1^* \cdot \hat{F}_2) \in \mathcal{X},$$

where $\Lambda : \widetilde{\mathcal{X}} \ni \hat{F} \longmapsto (E_\mathcal{X} \ni \rho \longmapsto \rho(\hat{F}(\rho))) \in C(E_\mathcal{X})$ and $\widehat{\mu_\omega} : \widetilde{\mathcal{X}} \ni \hat{F} \longmapsto \mu_\omega(\hat{F}) \in \mathcal{X}$ are conditional expectations.

$$\begin{array}{ccccc} C(E_\mathcal{X}) & \underset{\Lambda}{\overset{\kappa}{\underset{\twoheadleftarrow}{\hookrightarrow}}} & \widetilde{\mathcal{X}} = C(E_\mathcal{X}) \otimes \mathcal{X} & \underset{\iota}{\overset{\widehat{\mu_\omega}}{\underset{\hookleftarrow}{\twoheadrightarrow}}} & \mathcal{X} \\ \mu_\omega, \kappa_\omega & & \kappa_\omega \ltimes \pi_\omega & & \pi_\omega \end{array}$$

G-equivariance relation between ω and μ_ω

Using Λ and representation $\kappa_\omega \ltimes \pi_\omega$ of $f \otimes A \in \widetilde{\mathcal{X}}$, we rewrite the equation, $\langle \Omega_\omega, \kappa_\omega(f)\pi_\omega(A)\Omega_\omega\rangle = \int_{E_\mathcal{X}} f(\rho)[\gamma(A)](\rho)d\mu_\omega(\rho)$, for defining κ_ω as follows:

$$\begin{aligned}\langle \Omega_\omega, (\kappa_\omega \ltimes \pi_\omega)(f \otimes A)\Omega_\omega\rangle &= \langle \Omega_\omega, \kappa_\omega(f)\pi_\omega(A)\Omega_\omega\rangle \\ &= \mu_\omega(f \cdot \gamma(A)) = \mu_\omega(\Lambda(f \otimes A)) = [\Lambda^*(\mu_\omega)](f \otimes A)).\end{aligned}$$

Central measure $\mu_\omega \in M^1(F_\mathcal{X})$: uniquely determined by $\omega \in E_\mathcal{X} \Longrightarrow$ From $\omega \circ \tau_g = b(\mu_\omega) \circ \tau_g = b(\mu_{\omega\circ\tau_g})$ we have

$$d\mu_{\omega\circ\tau_g}(\rho) = d\mu_\omega(\rho \circ \tau_g^{-1}) \quad \text{or} \quad \mu_{\omega\circ\tau_g}(f) = \mu_\omega(f \circ \tau_g^*), \qquad \text{(a)}$$

for $f \in C(E_\mathcal{X}), g \in G, \rho \in E_\mathcal{X}$, where

$$(f \circ \tau_g^*)(\rho) = f(\rho \circ \tau_g) = ((\tau_g)_* f)(\rho).$$

Then, G-action on $\hat{F} \in \widetilde{\mathcal{X}}$ given by $[\hat{\tau}_g(\hat{F})](\rho) = \tau_g(\hat{F}(\rho \circ \tau_g))$ implies the G-equivariance relation between ω and μ_ω:

$$\langle \Omega_\omega \mid (\kappa_\omega \ltimes \pi_\omega)(\hat{\tau}_g(\hat{F}))\Omega_\omega \rangle = \langle \Omega_{\omega\circ\tau_g} \mid (\kappa_\omega \ltimes \pi_{\omega\circ\tau_g})(\hat{F})\Omega_{\omega\circ\tau_g} \rangle \quad \text{(b)}$$

Operational meaning of G-equivariance relation

Combining Eqs. (a) and (b), we see the microscopic G-action $\hat{F} \to \hat{\tau}_g(\hat{F})$ can be transformed into the state change $\omega \to \omega \circ \tau_g$, which can further be transformed into the change in macroscopic observable $f \to f \circ \tau_g^*$.

In this way, G-equivariance relation (b) between ω and μ_ω plays such an important role as making microscopic effects of G visible at the macroscopic level.

Once a sector structure emerges, this equivariance relation (b) always holds, irrespective of whether group G is written in terms of a unitary representation or not. For the purpose of distinguishing kinematical and dynamical symmetries, we consider next the problem as to whether a symmetry is unitarily implemented or not.

Transformation groupoid associated with G-quasi-invariant ω

A state $\omega \in E_\mathcal{X}$ is called G-quasi-invariant if its corresponding central measure $\mu_\omega \in M^1(E_\mathcal{X})$ is a G-quasi-invariant measure on $E_\mathcal{X}$ in the sense that μ_ω and $\mu_\omega \circ (\tau_g)_* = \mu_{\omega\circ\tau_g}$ are equivalent measures, namely, both are absolutely continuous w.r.t. the other: $\mu_\omega \ll \mu_{\omega\circ\tau_g}$ and $\mu_{\omega\circ\tau_g} \ll \mu_\omega$.

On the basis of this G-quasi-invariance, unitary representation U_ω of G can be given in $L^2(E_\mathcal{X}; \mu_\omega) \otimes \mathfrak{H}_\omega$ by

$$[U_\omega(g)\xi](\rho) := \sqrt{\frac{d(\mu_{\omega\circ\tau_g})}{d\mu_\omega}}\,\xi(\rho \circ \tau_g)$$

for $g \in G, \rho \in E_\mathcal{X}, \xi \in L^2(E_\mathcal{X}; \mu_\omega) \otimes \mathfrak{H}_\omega$.

Kinematics vs. dynamics

Under the assumption of transitivity, this action can be identified with transformation groupoid $\Gamma = G \times \Gamma^{(0)}$ with the G-transitive unit space $\Gamma^{(0)} := \operatorname{supp} \mu_\omega$ embedded in $Spec(\mathfrak{Z}_\pi(\mathcal{X})) \subset F_\mathcal{X}$.

Because of the unitary representation U_ω, the G-quasi-invariant action on classifying unit space $\Gamma^{(0)}$ can be viewed as ***kinematical*** and a G-action violating quasi-invariance of ω is to be viewed as ***dynamical*** since it does not leave the unit space $\Gamma^{(0)}$ invariant.

Some remarks on transitivity
$\Gamma^{(0)} = G/H$: transitivity + symmetric space $\subset$ ergodicity = measure-theoretical transitivity
$\Gamma^{(0)} = \amalg(G/H_i)$ $\because$ orbit decomposition, ergodic decomposition

Symmetry breaking patterns classified by unit space $\Gamma^{(0)}$

Then, in terms of the unit space $\Gamma^{(0)} \subset F_{\mathcal{X}}$, breaking patterns of the symmetry described by $G \underset{\tau}{\curvearrowright} \mathcal{X}$ can be classified into ***unbroken***, ***spontaneously broken***, ***explicitly broken*** ones as follows:

(i) ***unbroken***: $\Gamma^{(0)}_{unbroken}$ = one-point set (or, disjoint union of such sets)

(ii) ***spontanesously broken***: $\Gamma^{(0)}_{SSB}$ = sector bundle $G \underset{H}{\times} \hat{H}$ of a theory with a ***fixed dynamics***, whose base space G/H consists of degenerate vacua and whose fibers consist of sectors $\hat{H}$ of unbroken symmetry H

(iii) ***explicitly broken***: $\Gamma^{(0)}_{\text{explicit br.}}$ = double-layer bundle of sectors, whose base space consists of physical constants to parametrize ***different dynamics***, upon each point of which we have a sector bundle $\Gamma^{(0)}_{SSB}$ of SSB corresponding to a fixed dynamics

4 Local Gauge Invariance

In the case of local gauge invariance, a "gauge sector" is specified by a gauge-fixing condition, the totality of which defines the unit space $\Gamma^{(0)} = \mathcal{G}/G_{BRS}$ of the transformation groupoid $\mathcal{G}\times\Gamma^{(0)}$ of local gauge transformations $\mathcal{G}$ (where G_{BRS} is the BRS cohomology representing unbroken gauge symmetry in each gauge sector).

Each point of $\Gamma^{(0)}$ is a "gauge sector" parametrized by a "name" (f, α) of the gauge-fixing condition $f(A)+\alpha B \approx 0$ specified in terms of the Nakanishi-Lautrup B-field, each of which is transferred to another ***gauge sector*** by the ***action of broken gauge transformations in*** $\mathcal{G}$ ($\because$ transitivity of $\Gamma^{(0)} = \mathcal{G}/G_{BRS}$).

Geometry of gauge configuration space What is most important here is the role played by the Nakanishi-Lautrup B-field as the Lie derivative $\mathcal{L}$ on the configuration space of gauge field $\because$

$$\begin{aligned} \mathcal{L} &: [Lie(\mathcal{G}) \ni X \longmapsto -i\mathcal{L}_X \\ &= -i(d_B \circ i_X + i_X \circ d_B) = -i\{d_B, i_X\}] \\ &= -i\delta_{BRS}(\bar{c}) = B. \end{aligned}$$

Namely, the differential operator $\mathcal{L} + if(A)/\alpha$ determined by the quantum field B acts as a ***covariant derivative*** on configuration space of gauge field, whose connection coefficient is provided by the Lie-algebra-valued function $\Gamma^{(0)} \ni (f, \alpha) \longmapsto if(A)/\alpha \in Lie(\mathcal{G})$ on $\Gamma^{(0)}$.

Gauge fixing= parallel transport & BRS cochains Then, the gauge-fixing condition $f(A) + \alpha B \approx 0$ to determine a gauge sector $(f, \alpha) \in \Gamma^{(0)}$ plays the role of ***parallel transport on configuration space of gauge field*** :

$$(\pounds + if(A)/\alpha)\psi = 0$$

for state vectors $\psi \in$ gauge sector (f, α),

which specifies the geometric meaning of a gauge sector (f, α).

Inside of each gauge sector (f, α), we can find
FP ghost $c \in \wedge Lie(\mathcal{G})^*$: Maurer-Cartan form on $Lie(\mathcal{G})$,
anti-FP ghost $\bar{c} = (Lie(\mathcal{G}) \ni X \longmapsto i_X$: interior product) $\in \wedge Lie(\mathcal{G})$,
B-field $B = (Lie(\mathcal{G}) \ni X \longmapsto -i\pounds_X) \in \vee Lie(\mathcal{G})$

BRS cohomology which determine the BRS cohomology,

$$\delta_{BRS} A_\mu = D_\mu c, \quad \delta_{BRS} D_\mu c = 0,$$
$$\delta_{BRS} c = -\frac{1}{2} gc \wedge c,$$
$$\delta_{BRS} \bar{c} = iB, \quad \delta_{BRS} B = 0,$$

acting on the gauge sector (f, α) as the ***unbroken remaining symmetry***. The equation of FP ghost c in the gauge sector (f, α):

$$\frac{\delta f(A)}{\delta A_\mu} D_\mu c = \frac{\delta f(A)}{\delta A_\mu} \delta_{BRS} A_\mu = \delta_{BRS}(f(A))$$
$$= \delta_{BRS}(f(A) + \alpha B) = 0$$

guarantees the consistency between the gauge-fixing condition $f(A) + \alpha B \approx 0$ and the action of the BRS coboundary δ_{BRS} in the sense that the action of BRS coboundary δ_{BRS} is restricted to the inside of each sector without leakage to other sectors (: *unbroken symmetry*!).

4.1 Answers to questions a) & b)

In this way, the basic structures found in the quantum gauge theory can be consistently understood as the sector structure associated with the explicitly broken symmetry under the group of local gauge transformations which answers the above first question:

a) Why is such an awkward detour inevitable that the beautiful ***gauge symmetry*** should be ***explicitly violated by hand*** via "gauge fixing conditions" which make the 1st-class ***constraints*** into the 2nd-class ones?

The role of the explicit breaking via gauge-fixing is just to disentangle or "blow up" the overlaps between the dynamical and the internal-symmetry transformations (as local gauge transformations) taking the form of a first class constraint. The essence of local gauge invariance is ***unfolded*** by this procedure into the coexistence of many gauge sectors (f, α), the totality of which constitute the unit space $\Gamma^{(0)}$ of the transformation groupoid $\Gamma = \mathcal{G}\times\Gamma^{(0)}$ with the group $\mathcal{G}$ of local gauge transformations as a broken symmetry.

As ***parallel transports***, each of gauge fixing conditions represents a specific direction in ***the configuration space of gauge field***, the totality of which are mutually transformed by the local gauge transformations $\mathcal{G}$. Therefore, the gauge invariance of the theory containing all the gauge sectors can naturally be understood by its construction. This is the answer to the question:

b) Is the vital essence of ***local gauge invariance*** really recovered after the ***explicit breaking***? What are the mutual relations among ***different theories*** quantized with different gauge fixing conditions?

What remains at the end is such a natural question as to whether it should be possible to judge in a simpler way the meaning of gauge invariance without checking all the gauge sectors. While I have not encountered the detailed discussions on the meaning of ***gauge equivalence***, it can be interpreted here that "the contents of ***BRS-invariant sector*** are all the same over all gauge sectors".

4.2 Gauge equivalence

If the positivity is guaranteed of the inner products in the BRS-invariant sectors contained in each of gauge sectors, the action of local gauge transformations as a broken symmetry connecting different sectors can consistently be restricted to BRS-invariant sectors. Since the transitive action of $\mathcal{G}$ over all gauge sectors $\Gamma^{(0)}$ connects different gauge sectors by conjugacy transformations $g(-)g^{-1}$, the required conclusion is easily seen to hold by $g\iota g^{-1} = \iota$, in such a form as the triviality $Ind^{\mathcal{G}}_{G_{BRS}}(\iota) = id_{\mathcal{G}/G_{BRS}} \otimes \iota$ of the induced representation from the trivial representation of BRS transformation. In the relativistic context, therefore, this picture is just the expected natural realization of the ***gauge equivalence.***

Gauge-dependent classical observables

In the low energy regimes, however, we cannot deny the possible emergence of macroscopic classical fields like the Coulomb tails or Cooper pairs, as ***gauge-dependent physical modes*** due to the condensation effects of soft photons.[1]

In this case, we have to be prepared for such a possibility that different gauge sectors are not equivalent and that the different choices of gauges may

[1] This is in sharp contrast to the wrong naïve claim of "group invariance" of measurable quantities, as is seen, e.g., in S. Tanimura, arXiv 1112.5701.

result in different ways of realization of physical phenomenta.

The possible "emergence of physical but ***gauge-dependent classical modes***" can be interpreted as the result of ***spontaneous symmetry breaking (SSB) of BRS invariance in each gauge sector*** arising from the explicit breaking by gauge fixing condition. These aspects seem to provide useful viewpoints for the systematic analysis of phase transitions and infrared photon-like modes (in progress).

References

[1] Ojima, I., Lévy Process and Innovation Theory in the context of Micro-Macro Duality, 15 December 2006 at The 5th Lévy Seminar in Nagoya, Japan.

[2] Ojima, I., A unified scheme for generalized sectors based on selection criteria –Order parameters of symmetries and of thermality and physical meanings of adjunctions–, Open Systems and Information Dynamics, **10**, 235-279 (2003) (math-ph/0303009); Micro-macro duality in quantum physics, 143-161, Proc. Intern. Conf. "Stochastic Analysis: Classical and Quantum", World Sci., 2005, arXiv:math-ph/0502038.