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On the Structure of Hyperfunctions and Ultradistributions

By

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Abstract

In this article, we study global representation of ultradistributions and hyperfunctions. For non-quasi-analytic ultradistributions, we need to assume suitable global growth conditions for the global representation to hold. On the other hand, it holds for any quasi-analytic ultradistribution and hyperfunction.

§ 1. Introduction

In this paper, we discuss the structure of generalized functions. It is well known that any distribution f is locally represented as $f=P(D)g$, where $P(D)$ is a finite order differential operator with constant coefficients and g is a continuous function, which is the structure theorem for distributions. The structure theorems for non quasi-analytic ultradistributions ([1], [6]), quasi-analytic ultradistributions ([10], [11]) and hyperfunctions ([3]) are also known, among which, it is only the structure of hyperfunctions that is proved to hold globally. In this paper, we shall give global structure theorem of distributions and non quasi-analytic ultradistributions, by assuming suitable global decay conditions. The main purpose of this article is to give the global structure theorem for all quasi-analytic ultradistributions.

§ 2. Ultradistributions

In this section, we review the definition of ultradistributions. Let $\Omega \subset \mathbb{R}^n$ be an open subset and M_p, $p=0,1,\ldots,$ be a sequence of positive numbers. For non-quasi-analytic classes, we impose the following conditions on M_p.

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$(M.0)$ (normalization)

$$M_0 = M_1 = 1.$$

$(M.1)$ (logarithmic convexity)

$$M_p^2 \le M_{p-1} M_{p+1} \; (p = 1, 2, \dots).$$

$(M.2)$ (stability under ultradifferential operators)

there exist G, $H > 0$ such that $M_p \le GH^p \min_{0 \le q \le p} M_q M_{p-q}$ $(p = 1, 2, \dots)$.

$(M.3)$ (strong non quasi-analyticity)

there exists $G > 0$ such that $\displaystyle\sum_{q=p+1}^{\infty} \frac{M_{q-1}}{M_q} \le Gp \frac{M_p}{M_{p+1}}$ $(p = 1, 2, \dots)$.

$(M.2)$ and $(M.3)$ are often replaced by the following weaker conditions respectively;

$(M.2)'$ (stability under differential operators)

there exist G, $H > 0$ such that $M_{p+1} \le GH^p M_p$ $(p = 0, 1, \cdots)$.

$(M.3)'$ (non-quasi-analyticity)

$$\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty.$$

For two sequences M_p and N_p of positive numbers we define their *orders*.

Definition 2.1. Let M_p and N_p be the sequences of positive numbers.

(i) $M_p \subset N_p$ if there exist constants $L > 0$ and $C > 0$ such that $M_p \le CL^p N_p$ for any p.

(ii) $M_p \prec N_p$ if for any $L > 0$ there exists a constant $C > 0$ such that $M_p \le CL^p N_p$ for any p.

In order to define quasi-analytic classes, we impose the following conditions, (QA) and (NA), instead of $(M.3)$ or $(M.3)'$.

(QA) (quasi-analyticity)

$$p! \subset M_p, \quad \sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} = \infty.$$

Let M_p be a sequence of positive numbers satisfying (QA). If

$$\liminf_{p \to \infty} \sqrt[p]{\frac{p!}{M_p}} > 0$$

then $\mathcal{E}^{\{M_p\}}$ is the class of analytic functions. We impose the condition that $\{M_p\}$ would not define the analytic class, namely,

(NA) (non-analyticity)

$$\lim_{p \to \infty} \sqrt[p]{\frac{p!}{M_p}} = 0.$$

Definition 2.2. Let M_p be a sequence of positive numbers and $\Omega \subset \mathbb{R}^n$ be an open subset. A function $f \in \mathcal{E}(\Omega) = C^\infty(\Omega)$ is called an *ultradifferentiable function* of the class (M_p) (resp. $\{M_p\}$) if and only if for any compact subset $K \subset \Omega$ and for any $h > 0$ there exists a constant C (resp. for any compact subset $K \subset \Omega$ there exist constants h and C) such that

$$\sup_{x\in K} |D^\alpha \varphi(x)| \le C h^{|\alpha|} M_{|\alpha|} \quad \text{for all } \alpha \tag{2.1}$$

holds. Denote the set of the ultradifferentiable functions of the class (M_p) (resp. $\{M_p\}$) on Ω by $\mathcal{E}^{(M_p)}(\Omega)$ (resp. $\mathcal{E}^{\{M_p\}}(\Omega)$) and denote by $\mathcal{D}^*(\Omega)$ the set of all functions in $\mathcal{E}^*(\Omega)$ with their supports compact in Ω, where $* = (M_p)$ or $\{M_p\}$.

Let $K \subset \mathbb{R}^n$ be a compact set, and assume that M_p satisfy $(M.1)$ and (NA). Denote by $\mathcal{E}^*[K]$ the set of the ultradifferentiable functions of the class $* = (M_p)$ or $\{M_p\}$ defined on some neighborhood of K. We define $\varphi \in \mathcal{E}^{\{M_p\},h}[K]$ by $\varphi \in \mathcal{E}^{\{M_p\}}[K]$ and (2.1) holds for given $h > 0$.

For M_p satisfying $(M.3)'$ and a compact subset $K \subset \Omega$, set

$$\mathcal{D}^*_K = \{\varphi \in \mathcal{D}^*(\mathbb{R}^n);\ \mathrm{supp} f \subset K\}, \tag{2.2}$$

where $* = (M_p)$ or $\{M_p\}$ and we define

$$\mathcal{D}^{\{M_p\},h}_K = \bigcup_{C>0} \{\varphi \in \mathcal{D}^{\{M_p\}}_K;\ \sup_{x\in K} |D^\alpha \varphi(x)| \le C h^{|\alpha|} M_{|\alpha|}\}. \tag{2.3}$$

Let M_p satisfy $(M.1)$ and $(M.3)'$. We define $\mathcal{D}^{*\prime}(\Omega)$ as the strong dual of $\mathcal{D}^*(\Omega)$ for any open set Ω and call it *the set of ultradistributions of the class* $*$ defined on Ω. These spaces are endowed with natural structure of locally convex spaces.

For non quasi-analytic ultradifferentiable functions and non quasi-analytic ultradistributions confer [6] and [7].

Definition 2.3. Let $K \subset \mathbb{R}^n$ be a compact set, M_p satisfy $(M.1)$ and (NA). For $f \in \mathcal{E}^{\{M_p\},h}[K]$ we define its norm by

$$\|f\|_{\mathcal{E}^{\{M_p\},h}[K]} := \sup_{x\in K,\alpha} \frac{|D^\alpha f(x)|}{h^{|\alpha|} M_{|\alpha|}}. \tag{2.4}$$

Let Ω be an open set and K be a compact set. Topologies of the spaces of ultradifferentiable functions are defined as follows.

$$\begin{aligned} \mathcal{E}^{\{M_p\}}[K] &= \varinjlim_{h\to\infty} \mathcal{E}^{\{M_p\},h}[K], \\ \mathcal{E}^{\{M_p\}}(\Omega) &= \varprojlim_{K\Subset\Omega} \mathcal{E}^{\{M_p\}}[K], \end{aligned} \tag{2.5}$$

$$\mathcal{E}^{(M_p)}[K] = \varprojlim_{h \to 0} \mathcal{E}^{\{M_p\},h}[K],$$

$$\mathcal{E}^{(M_p)}(\Omega) = \varprojlim_{K \Subset \Omega} \mathcal{E}^{(M_p)}[K].$$

We define $\mathcal{E}^{*}_{K}{}'$ as the strong dual of $\mathcal{E}^*[K]$ and call it *the set of ultradistributions of the class $*$ supported by K*. We also define $\mathcal{E}^{*\prime}(\Omega) := \bigcup_{K \subset \Omega} \mathcal{E}^{*}_{K}{}'$.

Let us define the sheaf of ultradistributions.

Definition 2.4. Let a sequence M_p of positive numbers satisfy $(M.0)$, $(M.1)$, $(M.2)'$ and

$$\limsup_{p \to \infty} \sqrt[p]{\frac{p!}{M_p}} < \infty. \tag{2.6}$$

For an open bounded set Ω, we define

$$Db^*(\Omega) := \mathcal{E}^{*\prime}(\mathbb{R}^n)/\mathcal{E}^{*\prime}(\mathbb{R}^n \setminus \Omega), \tag{2.7}$$

where $* = (M_p)$ or $\{M_p\}$. We abuse the notation and by Db^* we define the presheaf induced by (2.7). We denote the associated sheaf by $\mathcal{D}b^*$. If M_p satisfies $(M.3)'$ then $\mathcal{D}b^* = \mathcal{D}^{*\prime}$. If M_p satisfies (QA) and (NA), then we call $\mathcal{D}b^*$ *the sheaf of the quasi-analytic ultradistributions of class $*$*.

The ultradistributions are represented as the boundary values of holomorphic functions (cf. [6], [8]).

Definition 2.5. For a positive sequence M_p satisfying (NA), define its *associated functions* by

$$M(t) := \sup_p \frac{t^p}{M_p}, \quad M^*(t) := \sup_k \frac{t^k k!}{M_k}, \tag{2.8}$$

for $t > 0$.

Proposition 2.6. *Let Ω be an open set in $\mathbb{R}^n$ and Γ_j $(j = 1, \ldots, N)$ open cones in $\mathbb{R}^n$. The following two conditions are equivalent.*

(i) $f(x) \in \mathcal{D}b^{(M_p)}$ *(resp. $\mathcal{D}b^{\{M_p\}}$).*

(ii) *The function $f(x)$ is represented as a hyperfunction*

$$f(x) = \sum_{j=1}^{N} F_j(x + i\Gamma_j 0),$$

where the functions F_j *are holomorphic in*

$$\{z \in \mathbb{C}^n;\ z \in \Omega + i\Gamma_j,\ |\mathrm{Im}\, z| < \varepsilon \ \text{ for some } \varepsilon > 0\}$$

and for any compact set $K \subset \Omega$ *there exist constants* L *and* C *(resp. for any* $L > 0$ *there exists* C*) such that*

$$\sup_{x \in K} |F_j(x + iy)| \leq CM^*(L/|y|),$$

where $i := \sqrt{-1}$.

Definition 2.7. For two classes $*$ and $\dagger$ we define their inclusion relations.

$$\begin{aligned} \dagger \leq * &\iff \mathcal{E}^\dagger \subset \mathcal{E}^*, \\ \dagger < * &\iff \mathcal{E}^\dagger \subsetneq \mathcal{E}^*. \end{aligned} \tag{2.9}$$

Definition 2.8. A function $\varepsilon(t) > 0$ defined for $t > 0$ is said to be *subordinate* if it is continuous, monotonously increasing and $\varepsilon(t)/t$ is monotonously decreasing to zero as $t \to \infty$, in particular

$$\lim_{t \to \infty} \frac{\varepsilon(t)}{t} = 0. \tag{2.10}$$

Proposition 2.9 (cf. Lemma 3.10 in [6]). *For positive sequences* M_p *and* N_p *satisfying* $(M.1)$*, the following conditions are equivalent.*

(i) $M_p \prec N_p$.

(ii) *For any* $L > 0$*, there exists a constant* $C > 0$ *such that*

$$N(t) \leq CM(Lt), \quad \text{for } 0 < t < \infty.$$

(iii) *There exists a subordinate function* $\varepsilon(t)$ *such that*

$$N(t) \equiv M(\varepsilon(t)).$$

Definition 2.10. A differential operator $P(D) = \sum_\alpha a_\alpha D^\alpha$ of infinite order is defined to belong to the class (M_p) (resp. $\{M_p\}$), if there exist such constants L and C (resp. for any $L > 0$ there exists such a constant C) that $|a_\alpha| \leq (CL^{|\alpha|})/M_{|\alpha|}$ holds for any α. We call this operator *an ultradifferential operator* of the class (M_p) (resp. $\{M_p\}$).

§ 3. Known Structure Theorems

In this section, we review the known results on the structure theorems. The structure theorem for the distributions was proved by L. Schwartz.

Theorem 3.1 (cf. [9]). *Any distribution f is locally represented as*

$$f = P(D)g, \tag{3.1}$$

where $P(D)$ is a differential operator of the finite order with constant coefficients and g is a continuous function.

H. Komatsu [6] proved the structure theorem for the strong non quasi-analytic ultradistributions.

Theorem 3.2 (cf. [6]). *Let the sequence M_p satisfy the conditions $(M.1)$, $(M.2)$ and $(M.3)$. Then $f \in \mathcal{D}b^*$, where $*$ is (M_p) or $\{M_p\}$, is locally represented in the form* (3.1), *where $P(D)$ is an ultradifferential operator of class $*$ with constant coefficients and g is a continuous function.*

This theorem was extended by R. W. Braun [1] for the non quasi-analytic ultradistributions.

Theorem 3.3 (cf. [1]). *Let the sequence M_p satisfy the conditions $(M.1)$, $(M.2)$ and $(M.3)'$. Then for $f \in \mathcal{D}b^*$, where $*$ is (M_p) or $\{M_p\}$, and for any class $\dagger$ satisfying $* < \dagger$, there exist an ultradifferential operator $P(D)$ of class $*$ with constant coefficients and an ultradifferentiable function g of class $\dagger$ such that the representation* (3.1) *locally holds.*

In [3], [4], A. Kaneko proved the structure theorem for the hyperfunctions.

Theorem 3.4 (cf. [3], [4]). *Any hyperfunction f is globally represented as*

$$f = J(D)g, \tag{3.2}$$

where $J(D)$ is a local operator with constant coefficients, that is, $J(D)$ is an infinite order differential operator $J(D) = \sum_\alpha a_\alpha D^\alpha$ with the coefficients satisfying

$$\lim_{|\alpha| \to \infty} \sqrt[|\alpha|]{|a_\alpha| \alpha!} = 0,$$

and g is an infinitely differentiable function.

Theorem 3.4 was the first result to give the structure of generalized functions more singular than the distributions. After this work, the structure of generalized functions was well studied by other mathematicians, for example, Theorem 3.2 by H. Komatsu, Theorem 3.3 by R. W. Braun etc. A. Kaneko also applied the local operators to give a new characterization of analytic functions (cf. [3]). We note that the structure theorems for the distributions (Theorem 3.1) and for the non quasi-analytic ultradistributions (Theorems 3.2 and 3.3) hold only locally, whereas the representation in (3.2) is global on any open set by virtue of the properties of the (Fourier) hyperfunctions.

The structure theorem for the quasi-analytic ultradistributions had been left open, which was proved by the author [10].

Theorem 3.5 (cf. [10]). *Let M_p satisfy $(M.0)$, $(M.1)$, $(M.2)$, (QA) and (NA). Assume that $f \in \mathcal{D}b^*$, where $* = (M_p)$ or $\{M_p\}$. Then for any class $\dagger$ satisfying $* < \dagger$ there exist $g \in \mathcal{E}^\dagger$ and an ultradifferential operator $P(D)$ of class $*$ such that the representation*

$$f = P(D)g \tag{3.3}$$

locally holds.

The main purpose of this paper is to extend Theorem 3.5 in order that the structure theorem of quasi-analytic ultradistributions holds globally, which shall be discussed in the next section.

§ 4. Main Theorems

It is our main purpose in this article to give the global structure theorem for distributions and non-quasi-analytic ultradistributions with suitable global growth conditions and prove the global structure theorem for the all quasi-analytic ultradistributions, which shall be discussed in this section. We first define the ultradistributions with growth conditions.

Definition 4.1. Let M_p be a sequence of positive numbers. then a function $f \in \mathcal{E}^{(M_p)}(\mathbb{R}^n)$ (resp. $f \in \mathcal{E}^{\{M_p\}}(\mathbb{R}^n)$) belongs to $\mathcal{P}^{(M_p)}$ (resp. $\mathcal{P}^{\{M_p\}}$) if for any $h > 0$ there exists a constant $C = C_h > 0$ (resp. there exists a constants $h > 0,\ C > 0$) such that

$$\sup_{x \in \mathbb{R}^n} \frac{|D^\alpha f(x)|}{M(h|x|)} \leq C|h|^{|\alpha|} M_{|\alpha|}, \tag{4.1}$$

for any multi-index α. Let us define

$$\mathcal{P}^{\{M_p\},h} = \bigcup_{C>0} \{f \in \mathcal{P}^{\{M_p\}};\ \sup_{x \in \mathbb{R}^n} \frac{|D^\alpha f(x)|}{M(h|x|)} \leq C|h|^{|\alpha|} M_{|\alpha|} \text{ (for all } \alpha)\}.$$

For $f \in \mathcal{P}^{\{M_p\},h}$, we define its norm by

$$\|f\|_{\mathcal{P}^{\{M_p\},h}} := \sup_{x \in \mathbb{R}^n, \alpha, k} \frac{|D^\alpha f(x)|}{h^{|\alpha|} M_{|\alpha|} M(h|x|)}. \tag{4.2}$$

For topologies of ultradifferentiable classes, the following relations hold.

$$\mathcal{P}^{\{M_p\}} = \varinjlim_{h\to\infty} \mathcal{P}^{\{M_p\},h}, \quad \mathcal{P}^{(M_p)} = \varprojlim_{h\to 0} \mathcal{P}^{\{M_p\},h}. \tag{4.3}$$

The set $\mathcal{Q}^* := \mathcal{P}^{*\prime}$ is defined as the strong dual of $\mathcal{P}^*$, where $* = (M_p)$ or $\{M_p\}$, and is called as *the space of the Fourier ultradistributions of class* $*$.

Proposition 4.2. *Assume that a sequence M_p $(p = 0, 1, 2, \dots)$ of positive numbers satisfies the conditions $(M.0)$, $(M.1)$, $(M.2)$ and (NA). Then the following conditions are equivalent.*

(i) *The function $\widehat{f}$ is the Fourier-Laplace transform of $f \in \mathcal{P}^{(M_p)}$ (resp. $f \in \mathcal{P}^{\{M_p\}}$).*

(ii) *For $h > 0$ there exists a constant $C = C_h > 0$ (resp. there exist constants h, $C > 0$) such that*

$$|P_1(D)(P_2(\xi)\widehat{f}(\xi))| \le \frac{C}{M(h|\xi|)}, \quad \text{for } \xi \in \mathbb{R}^n, \tag{4.4}$$

for any ultradifferential operator $P_1(D)$ and $P_2(D)$ of the same class.

Theorem 4.3 (The Paley-Wiener Theorem for NA Ultradistributions).
Let M_p satisfy $(M.0)$, $(M.1)$, $(M.2)'$ and (NA). For any compact convex set $K \subset \mathbb{R}^n$, the following conditions are equivalent.

(i) *$\widehat{f}$ is the Fourier-Laplace transform of $f \in \mathcal{D}b_K^{(M_p)}$ (resp. $f \in \mathcal{D}b_K^{\{M_p\}}$).*

(ii) *$\widehat{f}(\zeta)$ is an entire function of $\zeta \in \mathbb{C}^n$ which satisfies the following: there exist L, $C > 0$ (resp. for any $L > 0$, there exists $C > 0$) such that for any $\xi \in \mathbb{R}^n$*

$$|\widehat{f}(\xi)| \le CM(L|\xi|), \tag{4.5}$$

and for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for any $\zeta \in \mathbb{C}^n$,

$$|\widehat{f}(\zeta)| \le C_\varepsilon \exp(H_K(\operatorname{Im}\zeta) + \varepsilon|\zeta|), \tag{4.6}$$

where $H_K(y) := \sup_{x\in K}\{x \cdot y\}$ $(y \in \mathbb{R}^n)$ is the supporting function of K.

(iii) *$\widehat{f}(\zeta)$ is an entire function of $\zeta \in \mathbb{C}^n$ which satisfies the following: there exist L, $C > 0$ (resp. for any $L > 0$, there exists $C > 0$) such that for any $\zeta \in \mathbb{C}^n$,*

$$|\widehat{f}(\zeta)| \le CM(L|\zeta|)e^{H_K(\operatorname{Im}\zeta)}. \tag{4.7}$$

Definition 4.4. Let $\mathbb{D}^n := \mathbb{R}^n \sqcup S^{n-1}$ be the directional compactification of $\mathbb{R}^n$. For a compact subset $K \subset \mathbb{D}^n$, we define the space of ultradifferentiable test functions as follows:

$$\mathcal{P}^{\{M_p\}}(K) = \{\varphi(x) \in C^\infty(K \cap \mathbb{R}^n); \text{there exist } C,\ h > 0 \text{ such that for any } x \in K \text{ and } \alpha,\ |D^\alpha\varphi(x)| \le Ch^{|\alpha|}M_{|\alpha|}M(h|x|)^{-1}\}.$$

We define $\mathcal{P}^{(M_p)}(K)$ in the same way. The growth condition is meaningful only if K contains points at infinity. Notice that compact subsets of $\mathbb{D}^n$ restricted to $\mathbb{R}^n$ not necessarily bounded in the usual sense. By the same way as Definition 2.4, $\mathcal{Q}^*$ itself is defined as a sheaf of Fourier ultradistributions of class $*$ on $\mathbb{D}^n$ whose restriction to $\mathbb{R}^n$ agrees with the usual sheaf $\mathcal{D}b^*$ of ultradistributions, since $\mathcal{Q}^*$ is defined on the directional compactification of $\mathbb{R}^n$. $\mathcal{P}^*$ being invariant under the Fourier transformation by virtue of Proposition 4.2, $\mathcal{Q}^*$ is also invariant under the Fourier transformation.

Theorem 4.5. *Let M_p satisfy $(M.0)$, $(M.1)$, $(M.2)'$ and (NA). The following conditions are equivalent.*

(i) $f \in \mathcal{Q}^{(M_p)}$ *(resp.* $f \in \mathcal{Q}^{\{M_p\}}$*)* .

(ii) $f \in \mathcal{D}b^{(M_p)}$ *(resp.* $f \in \mathcal{D}b^{\{M_p\}}$*) and there exist constants $L > 0$ and $C > 0$ (resp. for any $L > 0$ there exists a constant $C > 0$) such that for any $\xi \in \mathbb{R}^n$*

$$|\widehat{f}(\xi)| \leq CM(L|\xi|). \tag{4.8}$$

Theorem 4.6. *If the class $*$ is quasi-analytic, then*

(i) *$\mathcal{Q}^*$ is flabby.*

(ii) *The restriction $\mathcal{Q}^*(\mathbb{D}^n) \to \mathcal{D}b^*(\mathbb{R}^n)$ is surjective.*

This theorem is proved by L. Hörmander ([2]) for the $\{M_p\}$ classes, the idea of which can be extended for the (M_p) classes.

Now we study the global structure theorems. The following theorem is well known.

Theorem 4.7. *Any tempered distribution $f \in \mathcal{S}'$ is globally represented as*

$$f = P(D)g, \tag{4.9}$$

where $P(D)$ is a differential operator of finite order with constant coefficients and g is a continuous function.

It is essential for this theorem to hold that the Fourier-Laplace transformation is an isomorphism on $\mathcal{S}'$.

Let us prove our main theorem in this article.

Theorem 4.8. *Let the sequence M_p satisfy the conditions, $(M.1)$, $(M.2)$ and $p! \subset M_p$. Assume that $f \in \mathcal{Q}^*(\mathbb{D}^n)$, where $*$ is (M_p) or $\{M_p\}$. Then for any class $\dagger$ satisfying $* < \dagger$ there exist $g \in \mathcal{P}^\dagger(\mathbb{R}^n)$ and an ultradifferential operator $P(D)$ of class $*$ such that the representation*

$$f = P(D)g, \tag{4.10}$$

holds. If $M_p = p!$, we only consider the $\{M_p\}$ class, which yields that $\mathcal{Q}^{\{M_p\}}(\mathbb{D}^n) = \mathcal{Q}(\mathbb{D}^n)$ is the space of Fourier hyperfunctions.

Before proving this theorem, let us prepare a lemma.

Lemma 4.9. *Let a sequence M_p satisfy $(M.0)$, $(M.1)$, $(M.2)'$ and $p! \subset M_p$. If for any $h > 0$ there exists a constant $C = C_h > 0$ (resp. there exist constants $h > 0$ and $C > 0$) such that*

$$|f(x)| \leq \frac{C}{M(h|x|)}, \tag{4.11}$$

then $\widehat{f} \in \mathcal{E}^{(M_p)}$ (resp. $\widehat{f} \in \mathcal{E}^{\{M_p\}}$). If $M_p = p!$, we only consider the $\{M_p\}$ class.

Proof of Theorem 4.8.

I. *The proof for the (M_p) class.*

By Theorem 4.5, there exist $L > 0$ and $C > 0$ such that (4.8) holds. Define the ultradifferential operator of class (M_p) by

$$P(D) := \sum_{p=0}^{\infty} \frac{(-C\Delta)^p}{M_{2p}}. \tag{4.12}$$

for some suitable constant C such that $\left|\frac{\widehat{f}(\xi)}{P(\xi)}\right|$ is bounded. By lemma 4.9

$$g := \mathcal{F}^{-1}\Big(\frac{\widehat{f}(\xi)}{P(\xi)^2}\Big) \in \mathcal{E}^{\{M_p\}}, \tag{4.13}$$

where $\mathcal{F}^{-1}$ is the inverse Fourier-Laplace transformation operator. We have

$$f(x) = P(D)^2 g(x). \tag{4.14}$$

By virtue of $(M.2)$, we see that $P(D)^2$ is an ultradifferential operator of class (M_p).

II. *The proof for $\{M_p\}$ class.*

Let $\{M_p\} < \dagger = (N_p)$ or $\{N_p\}$. $L_p := \sqrt{M_p N_p}$ yields $M_p \prec L_p \prec N_p$. There exists a subordinate function ε_1 such that $L(t) = M(\varepsilon_1(t))$, hence there exist such a positive decreasing sequence $l_p^{(1)}$ with $\lim_{p\to\infty} l_p^{(1)} = 0$ and a constant $A_1 > 0$ that

$$P_1(\xi) := \sum_{p=0}^{\infty} \frac{(l_{2p}^{(1)}|\xi|)^{2p}}{M_{2p}} \geq A_1 M(\varepsilon_1(|\xi|)), \tag{4.15}$$

for any $\xi \in \mathbb{R}^n$. By virtue of Theorem 4.3, there exists a subordinate function ε_2 such that

$$|\widehat{f}(\xi)| \leq M(\varepsilon_2(|\xi|)), \tag{4.16}$$

for any $\xi \in \mathbb{R}^n$. There exist a positive decreasing sequence $l_p^{(2)}$ satisfying $\lim_{p\to\infty} l_p^{(2)} = 0$ and a constant $A_2 > 0$ such that

$$P_2(\xi) := \sum_{p=0}^{\infty} \frac{(l_{2p}^{(2)}|\xi|)^{2p}}{M_{2p}} \geq A_2 M(\varepsilon_2(|\xi|)), \tag{4.17}$$

for any $\xi \in \mathbb{R}^n$. Let us define

$$g := \mathcal{F}^{-1}\Big(\frac{\widehat{f}(\xi)}{P_1(\xi)P_2(\xi)}\Big), \tag{4.18}$$

then it is proved that $g \in \mathcal{E}^{\{L_p\}} \subset \mathcal{E}^{\dagger}$. We have

$$P_1(D)P_2(D)g(x) = f(x). \tag{4.19}$$

By the condition $(M.2)$, the ultradifferential operator$P_1(D)P_2(D)$ belongs to the $\{M_p\}$ class. □

By the proof of this theorem, we obtain the following global structure theorem for non quasi-analytic ultradistributions.

Theorem 4.10. *Let M_p satisfy the conditions $(M.1)$, $(M.2)$ and $(M.3)'$. Assume that $f \in \mathcal{D}b^*(\mathbb{D}^n)$, where $\Omega \subset \mathbb{R}^n$ is open and $*$ is (M_p) or $\{M_p\}$, satisfying the condition that there exist constants $L > 0$ and $C > 0$ (resp. for any $L > 0$ there exists a constant $C > 0$) such that the estimate (4.8) holds. Then for any class $\dagger$ satisfying $* < \dagger$ there exist $g \in \mathcal{P}^{\dagger}(\mathbb{R}^n)$ and an ultradifferential operator $P(D)$ of class $*$ such that the representation (4.10) globally holds.*

By virtue of Theorems 4.6 and 4.8, we obtain the global representation of any quasi-analytic ultradistribution.

Theorem 4.11. *Assume that $f \in \mathcal{D}b^*(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is open and $*$ is a quasi-analytic class satisfying the condition $(M.2)$. For any class $\dagger$ satisfying $* < \dagger$ there exist $g \in \mathcal{E}^{\dagger}$ and an ultradifferential operator $P(D)$ of class $*$ such that the representation*

$$f = P(D)g, \tag{4.20}$$

globally holds (on Ω).

§ 5. Conclusion

We have proved a global representation theorem for any quasi-analytic ultradistribution (Theorem 4.11), hyperfunctional counterpart of which has been proved A. Kaneko

(Theorem 3.4). The proofs of these two global representations essentially depends on the flabbiness of the sheaves of the quasi-analytic ultradistributions and the hyperfunctions (Theorem 4.6).

On the other hand, in order to obtain the global representation of the distributions and the non quasi-analytic ultradistributions, we had to restrict their growth toward infinity (Theorems 4.7 and 4.10). It may be interesting to study whether the assumptions in Theorems 4.7 and 4.10 are optimal for the global representations to hold.

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