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Kernel functions and symbols of pseudodifferential operators of infinite order with an apparent parameter

By

Takashi AOKI*, Naofumi HONDA** and Susumu YAMAZAKI***

Abstract

We introduce a new representation of pseudodifferential operators of infinite order (or holomorphic microlocal operators) and symbol class.

Introduction

The aim of this article is to announce our recent results about a complete symbol theory for the sheaf $\mathscr{E}_X^{\mathbb{R}}$ of pseudodifferential operators in the complex analytic category. The foundation of the symbol theory of $\mathscr{E}_X^{\mathbb{R}}$ at the present stage (see [1], [3]) is quite unsatisfactory. There are two issues: first one is that, as Kamimoto and Kataoka have pointed out in their work [4], the space of the kernel functions which comes from standard Čech representation of cohomology groups is not closed under composition of kernel functions defined by naive integration employed in [1], [3]. Regarding this issue, [4], [5] give a possible solution by introducing the notion of formal kernels. Second issue is that the relation between the action of operators by integration of kernel functions and canonical action through cohomological definition was not clarified (see [6], [7]). Therefore, by using new isomorphism of cohomology groups (see Proposition 1.2), we establish a new symbol theory, and solve two issues above. Details in this article will be published in our forthcoming paper [2].

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*Department of Mathematics, Kinki University, Higashi-Osaka 577–8502, Japan.
**Department of Mathematics, Faculty of Science, Hokkaido University, Sapporo 060-0810, Japan.
***Department of General Education, College of Science and Technology, Nihon University, Chiba 274–8501, Japan.

§ 1. Local cohomology groups on a vector space

We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of integers, of real numbers and of complex numbers respectively. Further set $\mathbb{N} := \{m \in \mathbb{Z};\, m \geqslant 0\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and $\mathbb{C}^\times := \{c \in \mathbb{C};\, c \neq 0\}$. Let X be a finite dimensional $\mathbb{R}$-vector space, and define an open proper sector $S \subset \mathbb{C}$ by

$$S := \{\eta \in \mathbb{C};\, a < \arg\eta < b,\, 0 < |\eta| < r\}$$

for some $0 < b - a < \pi$ and $r > 0$. We set $\widehat{X} := X \times \mathbb{C}_\eta$ with coordinates (x, η), and let $\pi_\eta\colon \widehat{X} \ni (x, \eta) \mapsto x \in X$ be the canonical projection. Let $G \subset X$ be a closed subset (not necessarily convex) and $U \subset X$ an open neighborhood of the origin. In this section we give another representation of local cohomology groups $H^k_{G\cap U}(U; \mathcal{F})$ for a sheaf $\mathcal{F}$ on X. For this purpose, we need some preparations. Let Z be a closed subset in X and $\varphi\colon X \times [0,1] \to X$ a continuous deformation mapping which satisfies the following conditions:

(i) $\varphi(x, 1) = x$ for any $x \in X$ and $\varphi(z, s) = z$ for any $z \in Z$.
(ii) $\varphi(\varphi(x, s), 0) = \varphi(x, 0)$ for any $s \in [0, 1]$ and $x \in X$.
(iii) We set

$$\rho_\varphi(x, s) := |\varphi(x, s) - \varphi(x, 0)|.$$

Then $\rho_\varphi(x, s)$ is a strictly increasing function of s outside Z, i.e. if $s_1 < s_2$, we have $\rho_\varphi(x, s_1) < \rho_\varphi(x, s_2)$ for any $x \in X \setminus Z$.

We set, for short

$$\rho_\varphi(x) := \rho_\varphi(x, 1) = |\varphi(x, 1) - \varphi(x, 0)| = |x - \varphi(x, 0)|.$$

Here we remark

$$\rho_\varphi(\varphi(x, s)) = |\varphi(x, s) - \varphi(\varphi(x, s), 0)| = |\varphi(x, s) - \varphi(x, 0)| = \rho_\varphi(x, s).$$

1.1. Example. Let ζ be a unit vector in $X := \mathbb{C}^n$ and $Z = \{x \in X;\, \langle x, \zeta\rangle = 0\}$ with $\langle x, \zeta\rangle := \sum_{i=1}^{n} x_i\zeta_i$. Define the deformation mapping $\varphi\colon X \times [0,1] \to X$ by

$$\varphi(x, s) := x + (s - 1)\langle x, \zeta\rangle\overline{\zeta}.$$

Here $\overline{\zeta}$ denotes the complex conjugate of ζ. Note that $\varphi(x, 1) = x$ and $\varphi(x, 0)$ gives the orthogonal projection to the complex hyperplane Z with respect to the standard Hermitian metric $|x| = \langle x, \overline{x}\rangle^{1/2}$.

Let $\varrho > 0$ a positive constant. We define the subsets in $\widehat{X}$ by

$$\widehat{G} := \{(\varphi(x, s), \eta) \in \widehat{X};\, \rho_\varphi(x) \leqslant \varrho|\eta|,\, 0 \leqslant s \leqslant 1,\, x \in G\},$$
$$\widehat{U} := \{(x, \eta) \in U \times S;\, \rho_\varphi(x) < \varrho|\eta|\}.$$

Note that $\widehat{G} \cap \widehat{U}$ is a closed subset in $\widehat{U}$. Then we have the following proposition.

1.2. Proposition. *Let $\mathcal{F}$ be a complex of Abelian sheaves on X. Assume that U satisfies that $\sup_{x\in U}\rho_\varphi(x) < \varrho r$. Then there exists the following quasi-isomorphism:*

$$\boldsymbol{R}\Gamma_{G\cap U}(U;\mathcal{F}) \simeq \boldsymbol{R}\Gamma_{\widehat{G}\cap\widehat{U}}(\widehat{U};\pi_\eta^{-1}\mathcal{F}).$$

§ 2. Holomorphic microfunctions with an apparent parameter

Let X be a n-dimensional $\mathbb{C}$-vector space with the coordinates $z = (z_1,\dots,z_n)$, and Y the closed complex submanifold of X defined by $\{z' = 0\}$ where $z = (z', z'')$ with $z' := (z_1,\dots,z_d)$ for some $1 \leqslant d \leqslant n$. Set $\widehat{X} := X \times \mathbb{C}$, and let $\pi_\eta\colon \widehat{X} \ni (z,\eta) \mapsto z \in X$ be the canonical projections as in § 1. In what follows, we denote an object defined on the space $\widehat{X}$ by a symbol with $\widehat{\ }$ like $\widehat{U}_{\boldsymbol{\kappa}}$ etc. For any $z \in \mathbb{C}^n$, we set $\|z\| := \max_{1\leqslant i\leqslant n}\{|z_i|\}$. Let $\mathscr{O}_X$ be the sheaf of *holomorphic functions* on X, and $\mathscr{C}^{\mathbb{R}}_{Y|X}$ the sheaf of *real holomorphic microfunctions* along Y on the conormal bundle T^*_YX to Y (see [8], [9]). Let $z_0^* = (0;1,0,\dots,0) \in T^*_YX$ for simplicity. Let $\boldsymbol{\kappa} := (r, r', \varrho, \theta) \in \mathbb{R}^4$ be a 4-tuple of positive constants with

$$0 < \theta < \frac{\pi}{2}, \quad 0 < \varrho < 1, \quad 0 < r < \varrho r'. \tag{2.1}$$

Then we set

$$U_{\boldsymbol{\kappa}} := \bigcap_{i=2}^{n}\{z \in X;\ |z_1| < \varrho r,\ |z_i| < r'\},$$
$$G_{\boldsymbol{\kappa}} := \bigcap_{i=2}^{d}\{z \in X;\ |\arg z_1| \leqslant \frac{\pi}{2} - \theta,\ \varrho^2|z_i| \leqslant |z_1|\}.$$

By the definition of $\mathscr{C}^{\mathbb{R}}_{Y|X}$, we have

$$\mathscr{C}^{\mathbb{R}}_{Y|X,z_0^*} = \varinjlim_{\boldsymbol{\kappa}} H^d_{G_{\boldsymbol{\kappa}}\cap U_{\boldsymbol{\kappa}}}(U_{\boldsymbol{\kappa}};\mathscr{O}_X).$$

Now we apply the result in § 1 to this case. We set

$$S_{\boldsymbol{\kappa}} := S_{r,\theta/4} = \{\eta \in \mathbb{C};\ |\arg \eta| < \frac{\theta}{4},\ 0 < |\eta| < r\} \tag{2.2}$$
$$\widehat{U}_{\boldsymbol{\kappa}} := \bigcap_{i=2}^{n}\{(z,\eta) \in X \times S_{\boldsymbol{\kappa}};\ |z_1| < \varrho|\eta|,\ |z_i| < r'\},$$
$$\widehat{G}_{\boldsymbol{\kappa}} := \bigcap_{i=2}^{d}\{(z,\eta) \in \widehat{X};\ |\arg z_1| \leqslant \frac{\pi}{2} - \theta,\ \varrho|z_i| \leqslant |\eta|\}.$$

We adopt the deformation mapping given in Example 1.1 and assume that U is sufficiently small so that the assumption of Proportion 1.2 is satisfied (explicitly, $\varphi(z,s) =$

$(sz_1, z_2, \ldots, z_n))$. Thus, from the exact sequence $0 \to \pi_\eta^{-1}\mathscr{O}_X \to \mathscr{O}_{\widehat{X}} \xrightarrow{\partial_\eta} \mathscr{O}_{\widehat{X}} \to 0$, we obtain the following distinguished triangle:

$$\begin{array}{c} \boldsymbol{R}\varGamma_{\widehat{G}_{\boldsymbol{\kappa}}\cap\widehat{U}_{\boldsymbol{\kappa}}}(\widehat{U}_{\boldsymbol{\kappa}}; \pi_\eta^{-1}\mathscr{O}_X) \longrightarrow \boldsymbol{R}\varGamma_{\widehat{G}_{\boldsymbol{\kappa}}\cap\widehat{U}_{\boldsymbol{\kappa}}}(\widehat{U}_{\boldsymbol{\kappa}}; \mathscr{O}_{\widehat{X}}) \xrightarrow{\partial_\eta} \boldsymbol{R}\varGamma_{\widehat{G}_{\boldsymbol{\kappa}}\cap\widehat{U}_{\boldsymbol{\kappa}}}(\widehat{U}_{\boldsymbol{\kappa}}; \mathscr{O}_{\widehat{X}}) \xrightarrow{+1} . \\ \uparrow\wr \\ \boldsymbol{R}\varGamma_{G_{\boldsymbol{\kappa}}\cap U_{\boldsymbol{\kappa}}}(U_{\boldsymbol{\kappa}}; \mathscr{O}_X) \end{array}$$

Further we can prove:

2.1. Proposition. *If $k \neq d$, then*

$$H^k_{\widehat{G}_{\boldsymbol{\kappa}}\cap\widehat{U}_{\boldsymbol{\kappa}}}(\widehat{U}_{\boldsymbol{\kappa}}; \mathscr{O}_{\widehat{X}}) = H^k_{G_{\boldsymbol{\kappa}}\cap U_{\boldsymbol{\kappa}}}(U_{\boldsymbol{\kappa}}; \mathscr{O}_X) = 0.$$

2.2. Definition. By Proposition 2.1, we define

$$\begin{aligned} \widehat{C}^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa}) &:= H^d_{\widehat{G}_{\boldsymbol{\kappa}}\cap\widehat{U}_{\boldsymbol{\kappa}}}(\widehat{U}_{\boldsymbol{\kappa}}; \mathscr{O}_{\widehat{X}}), \\ C^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa}) &:= \operatorname{Ker}(\partial_\eta : \widehat{C}^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa}) \to \widehat{C}^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa})). \end{aligned}$$

Summing up, we obtain:

2.3. Theorem. *There exist isomorphisms*

$$\begin{array}{ccc} H^d_{G_{\boldsymbol{\kappa}}\cap U_{\boldsymbol{\kappa}}}(U_{\boldsymbol{\kappa}}; \mathscr{O}_X) & \xrightarrow{\sim} & C^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa}) \\ \downarrow & & \downarrow \\ \mathscr{C}^{\mathbb{R}}_{Y|X,z_0^*} & \xrightarrow{\sim} & \varinjlim_{\boldsymbol{\kappa}} C^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa}) \end{array}$$

We now consider a Čech representation of $C^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa})$. We set

$$\begin{aligned} V^{(1)}_{\boldsymbol{\kappa}} &:= \{z \in U_{\boldsymbol{\kappa}};\ \frac{\pi}{2} - \theta < \arg z_1 < \frac{3\pi}{2} + \theta\}, \\ V^{(i)}_{\boldsymbol{\kappa}} &:= \{z \in U_{\boldsymbol{\kappa}};\ \varrho^2|z_i| > |z_1|\} \qquad (2 \leqslant i \leqslant d), \\ \widehat{V}^{(1)}_{\boldsymbol{\kappa}} &:= \{(z,\eta) \in \widehat{U}_{\boldsymbol{\kappa}};\ \frac{\pi}{2} - \theta < \arg z_1 < \frac{3\pi}{2} + \theta\}, \\ \widehat{V}^{(i)}_{\boldsymbol{\kappa}} &:= \{(z,\eta) \in \widehat{U}_{\boldsymbol{\kappa}};\ \varrho|z_i| > |\eta|\} \qquad (2 \leqslant i \leqslant d). \end{aligned}$$

Let $\mathcal{P}_d$ be the set of all the subset of $\{1, \ldots, d\}$ and $\mathcal{P}_d^\vee \subset \mathcal{P}_d$ consisting of $\alpha \in \mathcal{P}_d$ with $\#\alpha = d - 1$ ($\#\alpha$ denotes the number of elements in α). For $\alpha \in \mathcal{P}_d$, we define

$$\widehat{V}^{(\alpha)}_{\boldsymbol{\kappa}} := \bigcap_{i\in\alpha} \widehat{V}^{(i)}_{\boldsymbol{\kappa}}, \qquad V^{(\alpha)}_{\boldsymbol{\kappa}} := \bigcap_{i\in\alpha} V^{(i)}_{\boldsymbol{\kappa}}, \qquad \widehat{V}^{(*)}_{\boldsymbol{\kappa}} := \widehat{V}^{(\{1,\ldots,d\})}_{\boldsymbol{\kappa}} = \bigcap_{i=1}^{d} \widehat{V}^{(i)}_{\boldsymbol{\kappa}}.$$

As each $\widehat{V}_{\boldsymbol{\kappa}}^{(\alpha)}$ (resp. $V_{\boldsymbol{\kappa}}^{(\alpha)}$) is pseudoconvex, we have

$$H^d_{G_{\boldsymbol{\kappa}}\cap U_{\boldsymbol{\kappa}}}(U_{\boldsymbol{\kappa}};\mathscr{O}_X)=\frac{\Gamma(V_{\boldsymbol{\kappa}}^{(*)};\mathscr{O}_X)}{\sum\limits_{\alpha\in\mathcal{P}_d^\vee}\Gamma(V_{\boldsymbol{\kappa}}^{(\alpha)};\mathscr{O}_X)}\xrightarrow{\sim}\{u\in\frac{\Gamma(\widehat{V}_{\boldsymbol{\kappa}}^{(*)};\mathscr{O}_{\widehat{X}})}{\sum\limits_{\alpha\in\mathcal{P}_d^\vee}\Gamma(\widehat{V}_{\boldsymbol{\kappa}}^{(\alpha)};\mathscr{O}_{\widehat{X}})};\,\partial_\eta u=0\}=C^{\mathbb{R}}_{Y|X}(\boldsymbol{\kappa}).$$

§ 3. Cohomological representation of $\mathscr{E}_X^{\mathbb{R}}$ with an apparent parameter

We inherit the same notation from the previous section. Set $X^2:=X\times X$ with the coordinates (z,w), and let (z,w,η) be coordinates of $\widehat{X}^2:=X^2\times\mathbb{C}$. Let $\Delta\subset X^2$ be the diagonal set. We identify X with Δ, and

$$T^*X=\{(z;\zeta)\}\simeq\{(z,z;\zeta,-\zeta)\}=T^*_\Delta X^2.$$

Let $\mathscr{E}_X^{\mathbb{R}}$ denote the sheaf of pseudodifferential operators on the cotangent bundle T^*X of X. Let Ω_X be the sheaf of holomorphic n-forms on X, and set $p_2\colon X^2\ni(z,w)\mapsto w\in X$. Then by the definition, $\mathscr{E}_X^{\mathbb{R}}=\mathscr{C}^{\mathbb{R}}_{X|X^2}\underset{p_2^{-1}\mathscr{O}_X}{\otimes}p_2^{-1}\Omega_X$. We fix $z_0^*=(z_0;\zeta_0):=(0;1,0,\dots,0)\in T^*X$. Let $\boldsymbol{\kappa}=(r,r',\varrho,\theta)\in\mathbb{R}^4$ be parameters satisfying the conditions (2.1). We set

$$\begin{aligned}
\widehat{U}_{\Delta,\boldsymbol{\kappa}}&:=\bigcap_{i=2}^{n}\{(z,w,\eta)\in\widehat{X}^2;\,\|z\|<r',\,\eta\in S_{\boldsymbol{\kappa}},\,|z_1-w_1|<\varrho|\eta|,\,|z_i-w_i|<r'\},\\
\widehat{V}^{(1)}_{\Delta,\boldsymbol{\kappa}}&:=\{(z,w,\eta)\in\widehat{U}_{\Delta,\boldsymbol{\kappa}};\,\frac{\pi}{2}-\theta<\arg(z_1-w_1)<\frac{3\pi}{2}+\theta\},\\
\widehat{V}^{(i)}_{\Delta,\boldsymbol{\kappa}}&:=\{(z,w,\eta)\in\widehat{U}_{\Delta,\boldsymbol{\kappa}};\,\varrho|z_i-w_i|>|\eta|\}\quad(2\leqslant i\leqslant n),\\
U_{\Delta,\boldsymbol{\kappa}}&:=\bigcap_{i=2}^{n}\{(z,w)\in X^2;\,\|z\|<r',\,|z_1-w_1|<\varrho r,\,|z_i-w_i|<r'\},\\
V^{(1)}_{\Delta,\boldsymbol{\kappa}}&:=\{(z,w)\in U_{\Delta,\boldsymbol{\kappa}};\,\frac{\pi}{2}-\theta<\arg(z_1-w_1)<\frac{3\pi}{2}+\theta\},\\
V^{(i)}_{\Delta,\boldsymbol{\kappa}}&:=\{(z,w)\in U_{\Delta,\boldsymbol{\kappa}};\,\varrho^2|z_i-w_i|>|z_1-w_1|\}\quad(2\leqslant i\leqslant n).
\end{aligned}$$

We also set

$$E_X^{\mathbb{R}}(\boldsymbol{\kappa}):=C^{\mathbb{R}}_{X|X^2}(\boldsymbol{\kappa})\underset{p_2^{-1}\mathscr{O}_X}{\otimes}p_2^{-1}\Omega_X\,.$$

Then by Theorem 2.3, we obtain

$$\begin{array}{ccc}
H^n_{G_{\Delta,\boldsymbol{\kappa}}\cap U_{\Delta,\boldsymbol{\kappa}}}(U_{\Delta,\boldsymbol{\kappa}};\mathscr{O}^{(0,n)}_{X^2}) & \xrightarrow{\quad\sim\quad} & E_X^{\mathbb{R}}(\boldsymbol{\kappa})\\
\| & & \|\\
\dfrac{\Gamma(V^{(*)}_{\Delta,\boldsymbol{\kappa}};\mathscr{O}^{(0,n)}_{X^2})}{\sum\limits_{\alpha\in\mathcal{P}_n^\vee}\Gamma(V^{(\alpha)}_{\Delta,\boldsymbol{\kappa}};\mathscr{O}^{(0,n)}_{X^2})} & \xrightarrow{\sim} & \{K\in\dfrac{\Gamma(\widehat{V}^{(*)}_{\Delta,\boldsymbol{\kappa}};\mathscr{O}^{(0,n,0)}_{\widehat{X}^2})}{\sum\limits_{\alpha\in\mathcal{P}_n^\vee}\Gamma(\widehat{V}^{(\alpha)}_{\Delta,\boldsymbol{\kappa}};\mathscr{O}^{(0,n,0)}_{\widehat{X}^2})};\,\partial_\eta K=0\}\\
\downarrow & & \downarrow\\
\mathscr{E}^{\mathbb{R}}_{X,z_0^*} & \xrightarrow{\quad\sim\quad} & \varinjlim\limits_{\boldsymbol{\kappa}}E_X^{\mathbb{R}}(\boldsymbol{\kappa})
\end{array}$$

Here $\mathscr{O}_{X^2}^{(0,n,0)}$ is the sheaf of holomorphic n-forms with respect to $dw_1, \dots, dw_n$, and $\mathscr{O}_{X^2}^{(0,n)}$ is the sheaf of holomorphic n-forms with respect to $dw_1, \dots, dw_n$. Let $(z,\eta) \in \widehat{X}$. Set $\beta_0 := \dfrac{\varrho}{2} e^{-\sqrt{-1}(\pi+\theta)/2}$ and $\beta_1 := \dfrac{\varrho}{2} e^{\sqrt{-1}(\pi+\theta)/2}$, and we define, for a sufficiently small $\varepsilon > 0$, the path $\gamma_i(z,\eta;\varrho,\theta)$ in $\mathbb{C}_{w_i}$ by

$$\begin{aligned}\gamma_1(z,\eta;\varrho,\theta) &:= \{w_1 = z_1 + t\beta_0\eta;\ 1 \geqslant t \geqslant \varepsilon\} \vee \{w_1 = z_1 + \varepsilon\beta_0\eta\, e^{\sqrt{-1}(\pi+\theta)t};\ 0 \leqslant t \leqslant 1\} \\ &\quad \vee \{w_1 = z_1 + t\beta_1\eta;\ \varepsilon \leqslant t \leqslant 1\}, \\ \gamma_i(z,\eta;\varrho) &:= \{w_i = z_i + \Big(\frac{|\eta|}{\varrho} + \varepsilon\Big)e^{2\pi\sqrt{-1}t};\ 0 \leqslant t \leqslant 1\} \quad (2 \leqslant i \leqslant n).\end{aligned}$$

Note that $\gamma_1(z,\eta;\varrho,\theta)$ joins the two points $z_1 + \beta_0\eta$ and $z_1 + \beta_1\eta$, which depend on the variables z_1 and η holomorphically. Define the real n-dimensional chain in X made from these paths by

$$\gamma(z,\eta;\varrho,\theta) := \gamma_1(z,\eta;\varrho,\theta) \times \gamma_2(z,\eta;\varrho) \times \cdots \times \gamma_n(z,\eta;\varrho) \subset X.$$

3.1. Theorem. *The bi-linear morphism*

$$\begin{array}{ccc} \mu\colon E_X^{\mathbb{R}}(\boldsymbol{\kappa}) \underset{\mathbb{C}}{\otimes} C_{Y|X}^{\mathbb{R}}(\boldsymbol{\kappa}) & \longrightarrow & C_{Y|X}^{\mathbb{R}}(\widetilde{\boldsymbol{\kappa}}) \\ \Downarrow & & \Downarrow \\ [\psi(z,w,\eta)\,dw] \otimes [u(z,\eta)] & \longmapsto & [\displaystyle\int_{\gamma(z,\eta;\varrho,\theta)} \psi(z,w,\eta)\,u(w,\eta)\,dw] \end{array}$$

is well defined. Here $\widetilde{\boldsymbol{\kappa}} = (\tilde{r}, \tilde{r}', \tilde{\varrho}, \tilde{\theta})$ is a 4-tuple of positive constants satisfying

$$0 < \tilde{r} < r, \quad 0 < \tilde{r}' < \frac{r'}{2}, \quad 0 < \tilde{\theta} < \frac{\theta}{4}, \quad 0 < \tilde{\varrho} < \frac{\varrho}{2}\sin\frac{\theta}{4}.$$

Moreover,

$$\mathscr{E}_{X,z_0^*}^{\mathbb{R}} \underset{\mathbb{C}}{\otimes} \mathscr{C}_{Y|X,z_0^*}^{\mathbb{R}} = \varinjlim_{\boldsymbol{\kappa}} (E_X^{\mathbb{R}}(\boldsymbol{\kappa}) \underset{\mathbb{C}}{\otimes} C_{Y|X}^{\mathbb{R}}(\boldsymbol{\kappa})) \xrightarrow{\mu} \varinjlim_{\boldsymbol{\kappa}} C_{Y|X}^{\mathbb{R}}(\boldsymbol{\kappa}) = \mathscr{C}_{Y|X,z_0^*}^{\mathbb{R}}$$

coincides with the cohomological action of $\mathscr{E}_{X,z_0^*}^{\mathbb{R}}$ on $\mathscr{C}_{Y|X,z_0^*}^{\mathbb{R}}$.

As a corollary of the theorem, we have the result on the composition on $E_X^{\mathbb{R}}(\boldsymbol{\kappa})$:

3.2. Corollary. *Let $\widetilde{\boldsymbol{\kappa}} = (\tilde{r}, \tilde{r}', \tilde{\varrho}, \tilde{\theta}) \in \mathbb{R}^4$ satisfying*

$$0 < \tilde{r} < r, \quad 0 < \tilde{r}' < \frac{r'}{8}, \quad 0 < \tilde{\theta} < \frac{\theta}{4}, \quad 0 < \tilde{\varrho} < \frac{\varrho}{2}\sin\frac{\theta}{4},$$

and the corresponding conditions to (2.1)*. Then there exists the bi-linear morphism*

$$\begin{array}{ccc} \mu\colon E_X^{\mathbb{R}}(\boldsymbol{\kappa}) \underset{\mathbb{C}}{\otimes} E_X^{\mathbb{R}}(\boldsymbol{\kappa}) & \longrightarrow & E_X^{\mathbb{R}}(\widetilde{\boldsymbol{\kappa}}) \\ \Downarrow & & \Downarrow \\ [\psi_1(z,w,\eta)\,dw] \otimes [\psi_2(z,w,\eta)\,dw] & \longmapsto & \Big[\Big(\displaystyle\int_{\gamma(z,\eta;\varrho,\theta)} \psi_1(z,\widetilde{w},\eta)\,\psi_2(\widetilde{w},w,\eta)\,d\widetilde{w}\Big)dw\Big]. \end{array}$$

Moreover the multiplication of the ring $\mathscr{E}^{\mathbb{R}}_{X,z_0^*}$ *coincides with*

$$\mathscr{E}^{\mathbb{R}}_{X,z_0^*} \underset{\mathbb{C}}{\otimes} \mathscr{E}^{\mathbb{R}}_{X,z_0^*} = \varinjlim_{\kappa}(E^{\mathbb{R}}_X(\kappa) \underset{\mathbb{C}}{\otimes} E^{\mathbb{R}}_X(\kappa)) \xrightarrow{\mu} \varinjlim_{\kappa} E^{\mathbb{R}}_X(\kappa) = \mathscr{E}^{\mathbb{R}}_{X,z_0^*}.$$

§ 4. Symbols with an apparent parameter

Let $X := \mathbb{C}^n$ and consider $T^*X \simeq X \times \mathbb{C}^n = \{(z;\zeta)\}$. Let $\pi\colon T^*X \to X$ be a canonical projection. If $V \subset T^*X$ is a conic set and $d > 0$, we set

$$V[d] := \{(z,\zeta) \in V;\ \|\zeta\| \geqslant d\}.$$

For any open conic subset $\Omega \subset T^*X$ and $\rho \geqslant 0$, we set

$$\Omega_\rho := \mathrm{Cl}\Big[\bigcup_{(z,\zeta)\in\Omega}\{(z+z';\zeta+\zeta') \in \mathbb{C}^{2n};\ \|z'\| \leqslant \rho,\ \|\zeta'\| \leqslant \rho\|\zeta\|\}\Big].$$

Here Cl means the closure. In particular, $\Omega_0 = \mathrm{Cl}\,\Omega$. For any $d > 0$ and $\rho \in [0,1[$, we set $d_\rho := d(1-\rho)$ for short. Let U, V be conic subsets of T^*X. Then we write $V \underset{\text{conic}}{\Subset} U$ if V is generated by a compact subset of U. We recall definitions of symbols and formal symbols of $\mathscr{E}^{\mathbb{R}}_X$:

4.1. Definition. Let $\Omega \underset{\text{conic}}{\Subset} T^*X$ be an open conic subset.

(1) We call $P(z,\zeta)$ a *symbol* on Ω if there exist $d > 0$ and $\rho \in \,]0,1[$ such that $P(z,\zeta) \in \Gamma(\Omega_\rho[d_\rho];\mathscr{O}_{T^*X})$, and for any $h > 0$ there exists $C_h > 0$ such that

$$|P(z,\zeta)| \leqslant C_h e^{h\|\zeta\|} \quad ((z;\zeta) \in \Omega_\rho[d_\rho]).$$

We denote by $\mathscr{S}(\Omega)$ the set of symbols on Ω.

(2) We call $P(z,\zeta)$ a *null-symbol* on Ω if there exist $d > 0$ and $\rho \in \,]0,1[$ such that $P(z,\zeta) \in \Gamma(\Omega_\rho[d_\rho];\mathscr{O}_{T^*X})$, and there exist C, $\delta > 0$ such that

$$|P(z,\zeta)| \leqslant Ce^{-\delta\|\zeta\|} \quad ((z;\zeta) \in \Omega_\rho[d_\rho]).$$

We denote by $\mathscr{N}(\Omega)$ the set of null-symbols on Ω.

(3) For any $z_0^* \in T^*X$, we set

$$\mathscr{S}_{z_0^*} := \varinjlim_{\Omega\ni z_0^*}\mathscr{S}(\Omega) \supset \mathscr{N}_{z_0^*} := \varinjlim_{\Omega\ni z_0^*}\mathscr{N}(\Omega)$$

where $\Omega \underset{\text{conic}}{\Subset} T^*X$ ranges though open conic neighborhoods of z_0^*. Then it is known that $\mathscr{E}^{\mathbb{R}}_{X,z_0^*} \xrightarrow{\sim} \mathscr{S}_{z_0^*}/\mathscr{N}_{z_0^*}$ (see [1], [3]).

Next, set for short

$$S := S_{\kappa} \tag{4.1}$$

for some $r,\ \theta \in \left]0, \frac{1}{2}\right[$ (recall (2.2)). In particular we always assume that $|\eta| < \frac{1}{2}$ for any $\eta \in S$. For $Z \Subset S$, we set $m_Z := \min_{\eta \in Z} |\eta| > 0$.

4.2. Definition. We define a set $\mathfrak{N}(\Omega; S)$ as follows: $P(z,\zeta,\eta) \in \mathfrak{N}(\Omega; S)$ if

(i) $P(z,\zeta,\eta) \in \Gamma(\Omega_\rho[d_\rho] \times S; \mathscr{O}_{T^*X\times\mathbb{C}})$ for some $d > 0$ and $\rho \in]0,1[$,

(ii) there exists $\delta > 0$ so that for any $Z \Subset S$, there exists a constant $C_Z > 0$ satisfying

$$|P(z,\zeta,\eta)| \leqslant C_Z e^{-\delta\|\eta\zeta\|} \quad ((z;\zeta,\eta) \in \Omega_\rho[d_\rho] \times Z).$$

4.3. Lemma. *If $P(z,\zeta,\eta) \in \mathfrak{N}(\Omega; S)$, it follows that $\partial_\eta P(z,\zeta,\eta) \in \mathfrak{N}(\Omega; S)$.*

4.4. Proposition. *Assume that $P(z,\zeta,\eta) \in \Gamma(\Omega_\rho[d_\rho] \times S; \mathscr{O}_{T^*X\times\mathbb{C}})$ satisfies that $\partial_\eta P(z,\zeta,\eta) \in \mathfrak{N}(\Omega; S)$.*

(1) *The following conditions are equivalent:*

(i) *there exists a constant $v > 0$ satisfying the following: for any $Z \Subset S$ there exists a constant $C_Z > 0$ such that*

$$|P(z,\zeta,\eta)| \leqslant C_Z e^{v\|\eta\zeta\|} \quad ((z;\zeta,\eta) \in \Omega_\rho[d_\rho] \times Z).$$

(ii) *for any $h > 0$ and $Z \Subset S$ there exists constant $C_{h,Z} > 0$ such that*

$$|P(z,\zeta,\eta)| \leqslant C_{h,Z}\, e^{h\|\zeta\|} \quad ((z;\zeta,\eta) \in \Omega_\rho[d_\rho] \times Z).$$

(2) *Assume that $P(z,\zeta,\eta)$ satisfies the equivalent conditions of* (1) (*resp. $P(z,\zeta,\eta) \in \mathfrak{N}(\Omega; S)$*). *Then for any $\eta_0 \in S$, it follows that $P(z,\zeta,\eta_0) \in \mathscr{S}(\Omega)$ (resp. $P(z,\zeta,\eta_0) \in \mathscr{N}(\Omega)$) and further $P(z,\zeta,\eta) - P(z,\zeta,\eta_0) \in \mathfrak{N}(\Omega; S)$.*

4.5. Definition. (1) We define a set $\mathfrak{S}(\Omega; S)$ as follows: $P(z,\zeta,\eta) \in \mathfrak{S}(\Omega; S)$ if

(i) $P(z,\zeta,\eta) \in \Gamma(\Omega_\rho[d_\rho] \times S; \mathscr{O}_{T^*X\times\mathbb{C}})$ for some $d > 0$ and $\rho \in]0,1[$,

(ii) $\partial_\eta P(z,\zeta,\eta) \in \mathfrak{N}(\Omega; S)$,

(iii) $P(z,\zeta,\eta)$ satisfies the equivalent conditions of Proposition 4.4.

Note that $\mathfrak{N}(\Omega; S) \subset \mathfrak{S}(\Omega; S)$ holds by Lemma 4.3.

(2) For $z_0^* \in \dot{T}^*X$, we set

$$\mathfrak{S}_{z_0^*} := \varinjlim_{\Omega, S} \mathfrak{S}(\Omega; S) \supset \mathfrak{N}_{z_0^*} := \varinjlim_{\Omega, S} \mathfrak{N}(\Omega; S).$$

Here $\Omega \underset{\text{conic}}{\Subset} T^*X$ ranges though open conic neighborhoods of z_0^*, and the inductive limits with respect to S are taken by $r,\ \theta \to 0$ in (4.1).

We call each element of $\mathfrak{S}(\Omega; S)$ (resp. $\mathfrak{N}(\Omega; S)$) a *symbol* (resp. *null-symbol*) *on* Ω *with an apparent parameter in* S. It is easy to see that $\mathfrak{S}(\Omega; S)$ is a $\mathbb{C}$-algebra under the ordinary operations of functions, and $\mathfrak{N}(\Omega; S)$ is a subalgebra. By definition, we can regard that

$$\mathscr{S}(\Omega) = \{P(z,\zeta,\eta) \in \mathfrak{S}(\Omega; S);\ \partial_\eta P(z,\zeta,\eta) = 0\} \subset \mathfrak{S}(\Omega; S),$$
$$\mathscr{N}(\Omega) = \mathscr{S}(\Omega) \cap \mathfrak{N}(\Omega; S) \subset \mathfrak{N}(\Omega; S).$$

Hence we have an injective mapping $\mathscr{S}(\Omega)/\mathscr{N}(\Omega) \hookrightarrow \mathfrak{S}(\Omega; S)/\mathfrak{N}(\Omega; S)$. Moreover

4.6. Proposition. *There exists the following isomorphism:*

$$\mathscr{S}(\Omega)/\mathscr{N}(\Omega) \xrightarrow{\sim} \mathfrak{S}(\Omega; S)/\mathfrak{N}(\Omega; S).$$

Take any $K(z,w,\eta)\,dw = [\psi(z,w,\eta)\,dw] \in \varinjlim_{\kappa} E_X^{\mathbb{R}}(\kappa)$. Then we set

$$\sigma(\psi)(z,\zeta,\eta) := \int_{\gamma(0,\eta;\varrho,\theta)} \psi(z, z+w, \eta)\, e^{\langle w,\zeta\rangle}\, dw.$$

Then we can prove the following theorem:

4.7. Theorem. *The mapping* σ *induces an isomorphism* $\mathscr{E}^{\mathbb{R}}_{X,z_0^*} \xrightarrow{\sim} \mathfrak{S}_{z_0^*}/\mathfrak{N}_{z_0^*}$.

§ 5. Classical formal symbols with an apparent parameter

5.1. Definition. Let t be an indeterminate.

(1) $P(t;z,\zeta) = \sum_{\nu=0}^{\infty} t^\nu P_\nu(z,\zeta) \in \widehat{\mathscr{S}_{\mathrm{cl}}}(\Omega)$ if $P(t;z,\zeta) \in \Gamma(\Omega_\rho[d_\rho]; \mathscr{O}_{T^*X})[\![t]\!]$ for some $d > 0$ and $\rho \in \,]0,1[$, and there exists a constant $A > 0$ satisfying the following: for any $h > 0$ there exists a constant $C_h > 0$ such that

$$|P_\nu(z,\zeta)| \leqslant \frac{C_h A^\nu \nu!\, e^{h\|\zeta\|}}{\|\zeta\|^\nu} \quad (\nu \in \mathbb{N}_0,\ (z;\zeta) \in \Omega_\rho[d_\rho]).$$

(2) $P(t;z,\zeta) = \sum_{\nu=0}^{\infty} t^\nu P_\nu(z,\zeta,\eta) \in \widehat{\mathscr{S}_{\mathrm{cl}}}(\Omega)$ is an element of $\widehat{\mathscr{N}_{\mathrm{cl}}}(\Omega)$ if there exists a constant $A > 0$ satisfying the following: for any $h > 0$ there exists a constant $C_h > 0$ such that

$$\Big|\sum_{\nu=0}^{m-1} P_\nu(z,\zeta)\Big| \leqslant \frac{C_h\, A^m m!\, e^{h\|\zeta\|}}{\|\zeta\|^m} \quad (m \in \mathbb{N},\ (z;\zeta) \in \Omega_\rho[d_\rho]).$$

(3) We set

$$\widehat{\mathscr{S}}_{\mathrm{cl},z_0^*} := \varinjlim_{\Omega} \widehat{\mathscr{S}_{\mathrm{cl}}}(\Omega) \supset \widehat{\mathscr{N}}_{\mathrm{cl},z_0^*} := \varinjlim_{\Omega} \widehat{\mathscr{N}_{\mathrm{cl}}}(\Omega).$$

We call each element of $\widehat{\mathscr{S}_{\mathrm{cl}}}(\Omega)$ (resp. $\widehat{\mathscr{N}_{\mathrm{cl}}}(\Omega)$) a *classical formal symbol* (resp. *classical formal null-symbol*) *on* Ω.

5.2. Definition. Let t be an indeterminate. Then we define a set $\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)$ as follows: $P(t;z,\zeta,\eta)=\sum_{\nu=0}^{\infty}t^{\nu}P_{\nu}(z,\zeta,\eta)\in\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)$ if

(i) $P(t;z,\zeta,\eta)\in\Gamma(\Omega_{\rho}[d_{\rho}]\times S;\mathscr{O}_{T^*X\times\mathbb{C}})[\![t]\!]$ for some $d>0$ and $\rho\in\,]0,1[$,

(ii) there exists a constant $A>0$, and for any $Z\Subset S$, $h>0$, there exists $C_{h,Z}>0$ such that

$$\Big|\sum_{\nu=0}^{m-1}P_{\nu}(z,\zeta,\eta)\Big|\leqslant\frac{C_{h,Z}\,A^{m}\,m!\,e^{h\|\zeta\|}}{\|\eta\zeta\|^{m}}\qquad(m\in\mathbb{N},\ (z;\zeta,\eta)\in\Omega_{\rho}[d_{\rho}]\times Z).$$

5.3. Definition. We say that $P(t;z,\zeta,\eta)=\sum_{\nu=0}^{\infty}t^{\nu}P_{\nu}(z,\zeta,\eta)\in\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S)$ if

(i) $P(t;z,\zeta,\eta)\in\Gamma(\Omega_{\rho}[d_{\rho}]\times S;\mathscr{O}_{T^*X\times\mathbb{C}})[\![t]\!]$ for some $d>0$ and $\rho\in\,]0,1[$,

(ii) there exists a constant $A>0$, and for any $Z\Subset S$, $h>0$ there exists $C_{h,Z}>0$ such that

$$|P_{\nu}(z,\zeta,\eta)|\leqslant\frac{C_{h,Z}\,A^{\nu}\nu!\,e^{h\|\zeta\|}}{\|\eta\zeta\|^{\nu}}\qquad(\nu\in\mathbb{N}_0,\ (z;\zeta,\eta)\in\Omega_{\rho}[d_{\rho}]\times Z).$$

(iii) $\partial_{\eta}P(t;z,\zeta,\eta)\in\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)$.

We call each element of $\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S)$ (resp. $\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)$) a *classical formal symbol* (resp. *classical formal null-symbol*) *on* Ω *with an apparent parameter in* S. We remark that $\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)\subset\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S)$. For any $z_0^*\in\dot{T}^*X$, we set

$$\widehat{\mathfrak{S}}_{\mathrm{cl},z_0^*}:=\varinjlim_{\Omega,S}\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S)\supset\widehat{\mathfrak{N}}_{\mathrm{cl},z_0^*}:=\varinjlim_{\Omega,S}\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S).$$

5.4. Proposition. *Let* $P(t;z,\zeta,\eta)\in\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S)$. *Then for any* $\eta_0\in S$, *it follows that* $P(t;z,\zeta,\eta_0)\in\widehat{\mathscr{S}_{\mathrm{cl}}}(\Omega)$ *and* $P(t;z,\zeta,\eta)-P(t;z,\zeta,\eta_0)\in\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)$.

We can regard that

$$\widehat{\mathscr{S}_{\mathrm{cl}}}(\Omega)=\{P(t;z,\zeta,\eta)\in\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S);\ \partial_{\eta}P(t;z,\zeta,\eta)=0\}\subset\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S),$$
$$\widehat{\mathscr{N}_{\mathrm{cl}}}(\Omega)=\widehat{\mathscr{S}_{\mathrm{cl}}}(\Omega)\cap\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)\subset\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S).$$

Hence we have an injective mapping $\widehat{\mathscr{S}_{\mathrm{cl}}}(\Omega)/\widehat{\mathscr{N}_{\mathrm{cl}}}(\Omega)\hookrightarrow\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S)/\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)$. Moreover

5.5. Proposition. $\widehat{\mathscr{S}_{\mathrm{cl}}}(\Omega)/\widehat{\mathscr{N}_{\mathrm{cl}}}(\Omega)\simeq\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S)/\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)$.

5.6. Theorem. *Let* $\Omega\underset{\text{conic}}{\Subset}T^*X$ *be any sufficiently small neighborhood of* $z_0^*\in\dot{T}^*X$. *Then* $\mathfrak{S}(\Omega;S)/\mathfrak{N}(\Omega;S)\simeq\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S)/\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)$.

5.7. Definition. As in the case of $\mathfrak{S}(\Omega; S)$, for any $P(t; z, \zeta, \eta) \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega; S)$ we set

$$:P(t; z, \zeta, \eta): := P(t; z, \zeta, \eta) \bmod \widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega; S) \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega; S)/\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega; S).$$

Take $\Omega \underset{\text{conic}}{\Subset} T^*\mathbb{C}^n$. Let $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ be local coordinates on a neighborhood of $\mathrm{Cl}\,\pi(\Omega) \subset X$, and $(z; \zeta)$, $(w; \lambda)$ corresponding local coordinates on a neighborhood of $\mathrm{Cl}\,\Omega$. Let $z = \Phi(w)$ be the coordinate transformation. We define $J^*_\Phi(z', z)$ by the relation $\Phi^{-1}(z') - \Phi^{-1}(z) = J^*_\Phi(z', z)(z' - z)$. Then ${}^{\mathrm{t}}J^*_\Phi(z, z)\lambda = {}^{\mathrm{t}}[\frac{\partial w}{\partial z}(z)]\lambda = \zeta$. Let $P(t; z, \zeta, \eta) \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega; S)$ with respect to $(z; \zeta)$. Then we set $\Phi^* P(t; w, \lambda, \eta) = \sum_{\nu=0}^{\infty} t^\nu \Phi^* P_\nu(w, \lambda, \eta)$ by

$$\Phi^* P(t; w, \lambda, \eta) := e^{t\langle \partial_{\zeta'}, \partial_{z'} \rangle} P(t; \Phi(w), \zeta' + {}^{\mathrm{t}}J^*_\Phi(\Phi(w) + z', \Phi(w))\lambda, \eta)\big|_{\substack{z'=0 \\ \zeta'=0}},$$

i.e.

$$\Phi^* P_\nu(w, \lambda, \eta) = \sum_{k+|\alpha|=\nu} \frac{1}{\alpha!} \partial_{\zeta'}^{\alpha} \partial_{z'}^{\alpha} P_k(\Phi(w), \zeta' + {}^{\mathrm{t}}J^*_\Phi(\Phi(w) + z', \Phi(w))\lambda, \eta)\big|_{\substack{z'=0 \\ \zeta'=0}}.$$

5.8. Theorem. (1) *$\Phi^* P(t; w, \lambda, \eta)$ defines an element of $\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega; S)$ with respect to $(w; \lambda)$. Moreover if $P(t; z, \zeta, \eta) \in \widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega; S)$, it follows that $\Phi^* P(t; w, \lambda, \eta) \in \widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega; S)$.*

(2) *$\mathbb{1}^*$ is the identity, and if $z = \Phi(w)$ and $w = \Psi(v)$ are complex coordinate transformations, $\Psi^*\Phi^* = (\Phi\Psi)^*$ holds.*

5.9. Definition. Under the notation above, we define a coordinate transformation Φ^* associated with Φ by

$$\Phi^*(:P:)(t; w, \lambda, \eta) := :\Phi^* P(t; w, \lambda, \eta):.$$

5.10. Theorem. *For any $P(t; z, \zeta)$, $Q(t; z, \zeta) \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega; S)$, set*

$$\begin{aligned} Q \circ P(t; z, \zeta, \eta) &:= e^{t\langle \partial_{\zeta'}, \partial_{z'} \rangle} Q(t; z, \zeta', \eta)\, P(t; z', \zeta, \eta)\big|_{\substack{z'=z \\ \zeta'=\zeta}} \\ &= e^{t\langle \partial_{\zeta'}, \partial_{z'} \rangle} Q(t; z, \zeta + \zeta', \eta)\, P(t; z + z', \lambda, \eta)\big|_{\substack{z'=0 \\ \zeta'=0}}. \end{aligned}$$

(1) *$Q \circ P(t; z, \zeta, \eta) \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega; S)$. Moreover if either $P(t; z, \zeta, \eta)$ or $Q(t; z, \zeta, \eta)$ is an element of $\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega; S)$, it follows that $Q \circ P(t; z, \zeta, \eta) \in \widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega; S)$.*

(2) *$R \circ (Q \circ P) = (R \circ Q) \circ P$ holds.*

(3) *Let $\Phi(w) = z$ be a holomorphic coordinate transformation. Then*

$$\Phi^* Q \circ \Phi^* P(t; w, \lambda, \eta) = \Phi^*(Q \circ P)(t; w, \lambda, \eta).$$

5.11. Definition. For any $:P(t; z, \zeta, \eta):$, $:Q(t; z, \zeta, \eta): \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega; S)/\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega; S)$, we define the product by:

$$:Q(t; z, \zeta, \eta): :P(t; z, \zeta, \eta): := :Q \circ P(t; z, \zeta, \eta):.$$

5.12. Theorem. *For any* $P(t;z,\zeta,\eta) \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S)$, *set*

$$P^*(t;z,\zeta,\eta) := e^{t\langle\partial_\zeta,\partial_z\rangle}P(t;z,-\zeta).$$

(1) $P^*(t;z,\zeta,\eta) \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega^a;S)$, *where* $\Omega^a := \{(z;\zeta);\,(z;-\zeta) \in \Omega\}$, *and* $P^{**} = P$. *Moreover if* $P(t;z,\zeta) \in \widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)$, *it follows that* $P^*(t;z,\zeta) \in \widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega^a;S)$.

(2)

$$(Q \circ P)^*(t;z,\zeta,\eta) = P^*(t;z,\zeta) \circ Q^*(t;z,\zeta,\eta).$$

(3) *For any holomorphic coordinate transformation* $\Phi(w) = z$, *on* $\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega^a;S) \underset{\mathscr{O}_X}{\otimes} \Omega_X$ *it follows that*

$$\Phi(P^*)(t;w,\lambda,\eta) \otimes dw = \Phi(P)^*(t;w,\lambda,\eta) \otimes dw.$$

5.13. Definition. For any $:P(t;z,\zeta,\eta): \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S)/\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S)$, we define the formal adjoint by

$$(:P(t;z,\zeta,\eta):)^* := :e^{t\langle\partial_\zeta,\partial_z\rangle}P(t;z,-\zeta,\eta): \in \widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega^a;S)/\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega^a;S).$$

5.14. Theorem. *Let* $[\psi(z,w,\eta)\,dw]$, $[\varphi(z,w,\eta)\,dw] \in \mathscr{E}^{\mathbb{R}}_{X,z_0^*}$. *Then the following hold:*

(1) $\sum_\alpha \frac{1}{\alpha!}\partial_\zeta^\alpha\sigma(\psi)(z,\zeta,\eta)\,\partial_z^\alpha\sigma(\varphi)(z,\zeta,\eta) \in \mathfrak{S}_{z_0^*}$.

(2) $\sum_\alpha \frac{1}{\alpha!}\partial_\zeta^\alpha\sigma(\psi)(z,\zeta,\eta)\,\partial_z^\alpha\sigma(\varphi)(z,\zeta,\eta) - \sigma(\psi)\circ\sigma(\varphi)(z,\zeta,\eta) \in \widehat{\mathfrak{N}}_{\mathrm{cl},z_0}$.

(3) $\sigma(\mu(\psi\otimes\varphi))(z,\zeta,\eta) - \sum_\alpha \frac{1}{\alpha!}\partial_\zeta^\alpha\sigma(\psi)(z,\zeta,\eta)\,\partial_z^\alpha\sigma(\varphi)(z,\zeta,\eta) \in \mathfrak{N}_{z_0^*}$.

5.15. Remark. Let $[\psi(z,w,\eta)\,dw] \in \mathscr{E}^{\mathbb{R}}_{X,z_0^*}$. Then we can also prove the following:

(1) We have

$$:P^*(t;z,\zeta,\eta): = :\int_{\gamma(0,\eta;\varrho,\theta)} \psi(z-w,z,\eta)\,e^{-\langle w,\zeta\rangle}\,dw:.$$

(2) Let $z = \Phi(w)$ be a complex coordinate transformation. Then

$$:\Phi^*P(t;w,\lambda,\eta): = :\int_{\gamma(z,\eta;\varrho,\theta)} \psi(z,z',\eta)\,e^{\langle\Phi^{-1}(z')-\Phi^{-1}(z),\lambda\rangle}dz':.$$

§ 6. Formal symbols with an apparent parameter

6.1. Definition. Let t be an indeterminate.

(1) $P(t;z,\zeta) = \sum_{\nu=0}^{\infty} t^\nu P_\nu(z,\zeta) \in \widehat{\mathscr{S}}(\Omega)$ if $P_\nu(z,\zeta) \in \Gamma(\Omega_\rho[(\nu+1)d_\rho];\mathscr{O}_{T^*X})$ for some $d>0$ and $\rho \in \,]0,1[$, and there exists a constant $A \in \,]0,1[$ satisfying the following: for any $h>0$ there exists a constant $C_h>0$ such that

$$|P_\nu(z,\zeta)| \leqslant C_h A^\nu e^{h\|\zeta\|} \quad (\nu \in \mathbb{N}_0,\ (z;\zeta) \in \Omega_\rho[(\nu+1)d_\rho]).$$

(2) Let $P(t;z,\zeta) = \sum_{\nu=0}^{\infty} t^\nu P_\nu(z,\zeta) \in \widehat{\mathscr{S}}(\Omega)$. Then $P(t;z,\zeta)$ is an element of $\widehat{\mathscr{N}}(\Omega)$ if there exists a constant $A \in \left]0,1\right[$ satisfying the following: for any $h > 0$ there exists a constant $C_h > 0$ such that

$$\left|\sum_{\nu=0}^{m-1} P_\nu(z,\zeta)\right| \leqslant C_h A^m e^{h\|\zeta\|} \quad (m \in \mathbb{N},\ (z;\zeta) \in \Omega_\rho[md_\rho]).$$

(3) For $z_0^* \in \dot{T}^*X$, we set

$$\widehat{\mathscr{S}}_{z_0^*} := \varinjlim_{\Omega} \widehat{\mathscr{S}}(\Omega) \supset \widehat{\mathscr{N}}_{z_0^*} := \varinjlim_{\Omega} \widehat{\mathscr{N}}(\Omega).$$

We call each element of $\widehat{\mathscr{S}}(\Omega)$ (resp. $\widehat{\mathscr{N}}(\Omega)$) a *formal symbol* (resp. *formal null-symbol*) *on* Ω.

For $U \subset S$ and $m \in \mathbb{N}$, we set

$$(\Omega_\rho * U)[md_\rho] := \{(z;\zeta,\eta) \in \Omega_\rho \times Z;\ \|\eta\zeta\| \geqslant md_\rho\} \subset \Omega_\rho[md_\rho] \times S.$$

6.2. Definition. Let t be an indeterminate. Then, we say that $P(t;z,\zeta,\eta) = \sum_{\nu=0}^{\infty} t^\nu P_\nu(z,\zeta,\eta)$ is an element of $\widehat{\mathfrak{N}}(\Omega;S)$ if

(i) $P_\nu(z,\zeta,\eta) \in \Gamma((\Omega_\rho * S)[(\nu+1)d_\rho]; \mathscr{O}_{T^*X\times\mathbb{C}})$ for some $d > 0$ and $\rho \in \left]0,1\right[$,

(ii) there exists a constant $A \in \left]0,1\right[$, and for any $Z \Subset S$, $h > 0$ there exists $C_{h,Z} > 0$ such that

$$\left|\sum_{\nu=0}^{m-1} P_\nu(z,\zeta,\eta)\right| \leqslant C_{h,Z} A^m e^{h\|\zeta\|} \quad (m \in \mathbb{N},\ (z;\zeta,\eta) \in (\Omega_\rho * Z)[md_\rho]).$$

6.3. Definition. (1) We say that $P(t;z,\zeta,\eta) = \sum_{\nu=0}^{\infty} t^\nu P_\nu(z,\zeta,\eta)$ is an element of $\widehat{\mathfrak{S}}(\Omega;S)$ if

(i) $P_\nu(z,\zeta,\eta) \in \Gamma((\Omega_\rho * S)[(\nu+1)d_\rho]; \mathscr{O}_{T^*X\times\mathbb{C}})$ for some $d > 0$ and $\rho \in \left]0,1\right[$,

(ii) there exists a constant $A \in \left]0,1\right[$, and for any $Z \Subset S$, $h > 0$, there exists $C_{h,Z} > 0$ such that

$$|P_\nu(z,\zeta,\eta)| \leqslant C_{h,Z} A^\nu e^{h\|\zeta\|} \quad (\nu \in \mathbb{N}_0,\ (z;\zeta,\eta) \in (\Omega_\rho * Z)[(\nu+1)d_\rho]).$$

(iii) $\partial_\eta P(t;z,\zeta,\eta) \in \widehat{\mathfrak{N}}(\Omega;S)$.

We call each element of $\widehat{\mathfrak{S}}(\Omega;S)$ (resp. $\widehat{\mathfrak{N}}(\Omega;S)$) a *formal symbol* (resp. *formal null-symbol*) *on* Ω *with an apparent parameter in* S.

We set

$$\widehat{\mathfrak{S}}_{z_0^*} := \varinjlim_{\Omega,S} \widehat{\mathfrak{S}}(\Omega;S) \supset \widehat{\mathfrak{N}}_{z_0^*} := \varinjlim_{\Omega,S} \widehat{\mathfrak{N}}(\Omega;S).$$

6.4. Proposition. (1) *Let* $P(t;z,\zeta,\eta) \in \widehat{\mathfrak{S}}(\Omega;S)$. *Then for any* $\eta_0 \in S$, *it follows that* $P(t;z,\zeta,\eta_0) \in \widehat{\mathscr{S}}(\Omega)$ *and* $P(t;z,\zeta,\eta) - P(t;z,\zeta,\eta_0) \in \widehat{\mathfrak{N}}(\Omega;S)$.

(2) *There exists the following isomorphism:*

$$\widehat{\mathscr{S}}(\Omega)/\widehat{\mathscr{N}}(\Omega) \simeq \widehat{\mathfrak{S}}(\Omega;S)/\widehat{\mathfrak{N}}(\Omega;S).$$

6.5. Theorem. (1) $\widehat{\mathfrak{S}}_{\mathrm{cl}}(\Omega;S) \subset \widehat{\mathfrak{S}}(\Omega;S)$ *and* $\widehat{\mathfrak{N}}_{\mathrm{cl}}(\Omega;S) \subset \widehat{\mathfrak{N}}(\Omega;S)$.

(2) *For any* $z_0^* \in \dot{T}^*X$, *the inclusions* $\mathfrak{S}_{z_0^*} \subset \widehat{\mathfrak{S}}_{\mathrm{cl},z_0^*} \subset \widehat{\mathfrak{S}}_{z_0^*}$ *and* $\mathfrak{N}_{z_0^*} \subset \widehat{\mathfrak{N}}_{\mathrm{cl},z_0^*} \subset \widehat{\mathfrak{N}}_{z_0^*}$ *induce*

$$\begin{array}{ccccccc}
\mathscr{E}^{\mathbb{R}}_{X,z_0^*} & \xrightarrow{\sim} & \mathscr{S}_{z_0^*}/\mathscr{N}_{z_0^*} & \xrightarrow{\sim} & \widehat{\mathscr{S}}_{\mathrm{cl},z_0^*}/\widehat{\mathscr{N}}_{\mathrm{cl},z_0^*} & \xrightarrow{\sim} & \widehat{\mathscr{S}}_{z_0^*}/\widehat{\mathscr{N}}_{z_0^*} \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\
\varinjlim_{\kappa} E^{\mathbb{R}}_X(\kappa) & \xrightarrow{\sim} & \mathfrak{S}_{z_0^*}/\mathfrak{N}_{z_0^*} & \xrightarrow{\sim} & \widehat{\mathfrak{S}}_{\mathrm{cl},z_0^*}/\widehat{\mathfrak{N}}_{\mathrm{cl},z_0^*} & \xrightarrow{\sim} & \widehat{\mathfrak{S}}_{z_0^*}/\widehat{\mathfrak{N}}_{z_0^*}
\end{array}$$

Note that $\mathscr{E}^{\mathbb{R}}_{X,z_0^*} \simeq \mathscr{S}_{z_0^*}/\mathscr{N}_{z_0^*}$ is induced by

$$\psi(z,w)\,dw \mapsto \int_{\gamma(0,\eta_0;\varrho,\theta)} \psi(z,z+w)\,e^{\langle w,\zeta\rangle}dw$$

for any fixed $\eta_0 \in S$ by the constructions of other isomorphisms (see [1], [3]).

We use the notation of Theorem 5.8. For any $P(t;z,\zeta,\eta) \in \widehat{\mathfrak{S}}(\Omega;S)$, we also set

$$\Phi^*P(t;w,\lambda,\eta) := e^{t\langle\partial_{\zeta'},\partial_{z'}\rangle}P(t;\Phi(w),\zeta' + {}^{\mathrm{t}}J_{\Phi}^*(\Phi(w)+z',\Phi(w))\lambda,\eta)\big|_{\substack{z'=0\\ \zeta'=0}}.$$

6.6. Theorem. (1) $\Phi^*P(t;w,\lambda,\eta) \in \widehat{\mathfrak{S}}(\Omega;S)$ *with respect to coordinate system* $(w;\lambda)$. *Further if* $P(t;z,\zeta,\eta) \in \widehat{\mathfrak{S}}(\Omega;S)$, *it follows that* $\Phi^*P(t;w,\lambda,\eta) \in \widehat{\mathfrak{S}}(\Omega;S)$.

(2) $\mathbb{1}^*$ *is the identity, and for complex coordinate transformations* $z = \Phi(w)$ *and* $w = \Psi(v)$, *it follow that* $\Psi^*\Phi^*P(t;v,\xi) - (\Phi\Psi)^*P(t;v,\xi) \in \widehat{\mathfrak{N}}_{(v;\xi)}$.

6.7. Theorem. *For any* $P(t;z,\zeta)$, $Q(t;z,\zeta) \in \widehat{\mathfrak{S}}(\Omega;S)$, *set*

$$\begin{aligned}
Q\circ P(t;z,\zeta,\eta) &:= e^{t\langle\partial_{\zeta'},\partial_{z'}\rangle}Q(t;z,\zeta',\eta)\,P(t;z',\zeta,\eta)\big|_{\substack{z'=z\\ \zeta'=\zeta}} \\
&= e^{t\langle\partial_{\zeta'},\partial_{z'}\rangle}Q(t;z,\zeta+\zeta',\eta)\,P(t;z+z',\lambda,\eta)\big|_{\substack{z'=0\\ \zeta'=0}}.
\end{aligned}$$

(1) $Q\circ P(t;z,\zeta,\eta) \in \widehat{\mathfrak{S}}(\Omega;S)$. *Moreover if either* $P(t;z,\zeta,\eta)$ *or* $Q(t;z,\zeta,\eta)$ *is an element of* $\widehat{\mathfrak{N}}(\Omega;S)$, *it follows that* $Q\circ P(t;z,\zeta,\eta) \in \widehat{\mathfrak{N}}(\Omega;S)$.

(2) $R\circ(Q\circ P) = (R\circ Q)\circ P$ *holds.*

(3) *Let* $\Phi(w) = z$ *be a holomorphic coordinate transformation. Then*

$$\Phi^*Q\circ\Phi^*P(t;w,\lambda,\eta) - \Phi^*(Q\circ P)(t;w,\lambda,\eta) \in \widehat{\mathfrak{N}}_{(v;\xi)}.$$

6.8. Theorem. *For any* $P(t;z,\zeta,\eta)\in\widehat{\mathfrak{S}}(\Omega;S)$ *set*

$$P^*(t;z,\zeta,\eta):=e^{t\langle\partial_\zeta,\partial_z\rangle}P(t;z,-\zeta,\eta).$$

(1) $P^*(t;z,\zeta,\eta)\in\widehat{\mathfrak{S}}(\Omega^a;S)$ *and* $P^{**}=P$. *Moreover if* $P(t;z,\zeta)\in\widehat{\mathfrak{N}}(\Omega;S)$, *it follows that* $P^*(t;z,\zeta)\in\widehat{\mathfrak{N}}(\Omega^a;S)$.

(2) $(Q\circ P)^*(t;z,\zeta,\eta)=P^*(t;z,\zeta)\circ Q^*(t;z,\zeta,\eta)$.

(3) *For any holomorphic coordinate transformation* $\Phi(w)=z$, *on* $\widehat{\mathfrak{S}}(\Omega^a;S)\underset{\mathcal{O}_X}{\otimes}\Omega_X$ *it follows that*

$$\Phi(P^*)(t;w,\lambda,\eta)\otimes dw=\Phi(P)^*(t;w,\lambda,\eta)\otimes dw.$$

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