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CONFORMALLY FLAT LORENTZ PARABOLIC MANIFOLD

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ABSTRACT. The purpose of this note is to introduce *Lorentz parabolic structure* on smooth manifolds. First we revisit (G,X)-*structure* on manifolds. Secondly we study *Lorentz similarity structure* and *Fefferman-Lorentz parabolic* structure.

1. INTODUCTION

In the first part of this paper we review (G,X)-*structure* introduced by Thurston, Kulkarni et al. Many results are known when (G,X) is a homogeneous Riemannian geometry. In 1980-90s non-Riemannian homogeneous geometries have been studied intensively. Specifically conformally flat geometry, spherical CR-geometry and flat quaternionic CR-geometry. Those geometries are obtained on the projective limit of the isometric actions of hyperbolic spaces. Similarly, another kind of non-Riemannian homogeneous geometry is obtained as the the boundary behavior of the isometric actions on *pseudo-hyperbolic spaces.* The typical example is *conformally flat Lorentz geometry.* In the second part of this paper, we introduce *conformally flat Lorentz parabolic* geometry. A Lorentz parabolic structure contains Lorentz similarity structure and Fefferman-Lorentz structure. It is explained that the fundamental group of a compact complete Lorentz similarity manifold M is virtually polycyclic. It turns out that a finite cover of M admits a Lorentz parabolic structute. We discuss *Fefferman-Lorentz parabolic* geometry. The conformally flat Lorentz geometry $(\mathrm{O}(2n+2,2), S^1\times S^{2n+1})$ contains this as a subgeometry $(\mathrm{U}(n+1,1), S^1\times S^{2n+1})$. Let Γ be a discrete subgroup of $\mathrm{U}(n+1,1)$ acting properly discontinuously on a domain $\#$ of $S^1\times S^{2n+1}$. We present a classification of compact conformally flat Fefferman-Lorentz parabolic manifolds $\#/\Gamma$ admitting a 1-parameter group $\mathsf{H}\leq \mathrm{Conf}(\#/\Gamma)$. This

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class contains $S^1 \times \mathcal{N}/\Delta$ where $\mathcal{N}$ is a 3-dimensional Heisenberg nilmanifold. Finally we discuss the deformation space of conformally flat Fefferma-Lorentz parabolic structures on the proudct $S^1 \times \mathcal{N}/\Delta$.

2. (G, x)-STRUCTURE

Our geometry is a pair (G, X) where G is a finite dimensional Lie group with finitely many components and X is an n dimensional homogeneous space of G. A geometric structure ((G, X)-structure) on a smooth n dimensional manifold M is a maximal collection of charts $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in\Lambda}$ whose coordinate changes belong to G. More precisely, $M = \bigcup_{\alpha\in\Lambda} U_\alpha$, $\phi_\alpha : U_\alpha \to X$ is a diffeomorphism onto its image. If $U_\alpha \cap U_\beta \neq \emptyset$ then it satisfies that there exists a unique element $g_{\alpha\beta} \in G$ such that $g_{\alpha\beta} \cdot \phi_\alpha = \phi_\beta$ on $U_\alpha \cap U_\beta$. We say that M is uniformized over X with respect to G (or simply, M is locally modelled on (G, X)). An n-manifold M is called a (G, X)-manifold if M is uniformized over X with respect to G. Using a collection of charts $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in\Lambda}$ we can construct a geometric invariant (ρ, dev) called a developing pair of M. (See [5].)

Lemma 2.1. *Given a (G, X)-structure on a smooth n-manifold M, there exists a pair $(\rho, \mathrm{dev}) : (\pi_1(M), \tilde{M}) \to (G, X)$ unique up to conjugation of elements of G, where* dev *is a (G, X)-structure preserving immersion and ρ is a homomorphism such that the diagram is commutative for each element $\gamma \in \pi_1(M)$;*

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\mathrm{dev}} & X \\ \gamma\downarrow & & \downarrow\rho(\gamma) \\ \tilde{M} & \xrightarrow{\mathrm{dev}} & X. \end{array} \tag{2.1}$$

Proof. Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in\Lambda}$ be a geometric structure on M. In the union $\bigcup_{\alpha\in\Lambda} (U_\alpha \times X)$, we define the following equivalence relation; for $(p, x) \in U_\alpha \times X$, $(q, y) \in U_\beta \times X$, then

$$(p, x) \sim (q, y) \text{ if and only if } p = q \in U_\alpha \cap U_\beta,\ g_{\alpha\beta}x = y, \\ (^\exists\, g_{\alpha\beta} \in G). \tag{2.2}$$

Put $E = \bigcup_\alpha (U_\alpha \times X)/\sim$. Let $\pi : E \to M$ be the map defined by $\pi([p, x]) = p$ if $p \in U_\alpha$. Then it is easy to see that $E \xrightarrow{\pi} M$ is a fiber bundle with fiber X. Recall that E is determined by the transitive functions $\{g_{\alpha\beta}\}$. Since $g_{\alpha\alpha} = 1$ and $g_{\alpha\beta} \cdot g_{\beta\gamma} = g_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$, $\{g_{\alpha\beta}\}$ is a 1-cocycle in the first cohomology $H^1(M; G)$. Here G is

viewed as the sheaf of germs of G-valued functions. Since $H^1(M;G) \approx \mathrm{Hom}(\pi_1(M), G)$, $\{g_{\alpha\beta}\}$ determines a homomorphism $\rho : \pi_1(M) \to G$. More precisely it follows that $E \approx \tilde{M} \underset{\rho}{\times} X$ in which each element $\gamma \in \pi_1(M)$ acts on $\tilde{M} \times X$ by $(\gamma, (b, x)) = (\gamma b, \rho(\gamma)x)$.
We construct a developing map. Let $s : M \to E$ be a section defined by $s(p) = [p, \phi_\alpha(p)]$ if $p \in U_\alpha$. Consider the pull back of the bundle:

$$(2.3)\qquad \begin{array}{ccccc} \pi_1(M) & \to & P^*E & \to & E \\ & & \downarrow & & \downarrow \\ \pi_1(M) & \to & \tilde{M} & \to & M. \end{array}$$

As before the bundle P^*E is determined by a lift $\{\tilde{g}_{\alpha\beta}\}$ of $\{g_{\alpha\beta}\}$. Since $H^1(\tilde{M}; G) = \{1\}$, the bundle P^*E is trivial. Choose a trivialization $\Psi : P^*E \to \tilde{M} \times X$. The section s extends to a section $\tilde{s} : \tilde{M} \to P^*E$. Put dev $= Pr_2 \cdot \Psi \cdot \tilde{s} : \tilde{M} \to X$. It is an immersion and preserves the (G, X)-structure. The map dev depends on the choice of sections and trivializations, however dev is unique up to elements of G.

On the other hand, we note that for $(\tilde{p}, x) \in \tilde{U}_\alpha \times X$, $(\tilde{q}, y) \in \tilde{U}_\beta \times X$ in $P^*E = \bigcup_\alpha (\tilde{U}_\alpha \times X)$, it follows that $(\tilde{p}, x) \sim (\tilde{q}, y)$ iff $\gamma\tilde{p} = \tilde{q}$, $\rho(\gamma)y = g_{\alpha\beta}x$ and $p = q \in U_\alpha \cap U_\beta$ for some $\gamma \in \pi_1(M)$ and $g_{\alpha\beta} \in G$. It is easy to see that

$$\mathrm{dev} \cdot \gamma = \rho(\gamma) \cdot \mathrm{dev} \ \text{ for every } \gamma \in \pi_1(M).$$

□

If $\mathrm{Aut}(\tilde{M})$ is the group of all (G, X)-structure preserving diffeomorphisms on $\tilde{M}$. Then note that $\pi_1(M) \leq \mathrm{Aut}(\tilde{M})$ and ρ extends naturally to a continuous homomorphism $\rho : \mathrm{Aut}(\tilde{M}) \to G$.

Definition 2.2. The map dev is called a developing map for a (G, X)-manifold M and the map ρ is called a holonomy homomorphism of M.

Let $\hat{\#}(M)$ be the space consisting of all possible developing pairs (ρ, dev). A topology on $\hat{\#}(M)$ is given by the following subbasis.

- $\mathcal{N}(U) = \{U\}$ where U is an open subset of $\mathrm{Map}(\tilde{M}, X)$ in the compact open topology of $\mathrm{Map}(\tilde{M}, X)$.
- $\mathcal{N}(K) = \{\mathrm{dev} \in \hat{\#}(M) \mid \mathrm{dev}|K \text{ is embedding}\}$ for a compact subset $K \subset \tilde{M}$.

(Compare [1].) Recall that the deformation space $\mathcal{T}(M)$ is a space of (G, X)-structures on marked manifolds homeomorphic to M. $\mathcal{T}(M)$ consists of equivalence classes of diffeomorphisms $f : M \to M'$ from

M to a (G,X)-manifolds M'. Two such diffeomorphisms $f_i : M \to M_i$ $(i = 1,2)$ are equivalent if and only if there is an isomorphism (i.e. a (G,X)-structure preserving diffeomorphism) $h : M_1 \to M_2$ such that $h \circ f_1$ is *isotopic* to f_2.

$$(2.4) \qquad \begin{array}{ccc} M & \xrightarrow{f_1} & M_1 \\ & {}_{f_2}\searrow \quad \simeq & \downarrow h \\ & & M_2 \end{array}$$

Denote by $\mathrm{Diff}^0(M)$ the subgroup of $\mathrm{Diff}(M)$ whose elements are *isotopic* to the identity map. Put $\pi = \pi_1(M)$. Consider the following exact sequences of the diffeomorphism groups, where $N_{\mathrm{Diff}(\tilde{M})}(\pi)$ (resp. $C_{\mathrm{Diff}(\tilde{M})}(\pi)$) is the normalizer (resp. centralizer) of π in $\mathrm{Diff}(\tilde{M})$

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi & \longrightarrow & N_{\mathrm{Diff}(\tilde{M})}(\pi) & \xrightarrow{\eta} & \mathrm{Diff}(M) & \longrightarrow & 1 \\ & & & & \uparrow & & \uparrow & & \\ & & & & C_{\mathrm{Diff}(\tilde{M})}(\pi) & \longrightarrow & \mathrm{Diff}^0(M) & & \end{array}$$

Put $\widehat{\mathrm{Diff}}(M) = \eta^{-1}(\mathrm{Diff}(M))$ and let $\widehat{\mathrm{Diff}}^0(M)$ be the identity component. Then $\eta(\widehat{\mathrm{Diff}}(M)) = \mathrm{Diff}^0(M)$ and $\widehat{\mathrm{Diff}}^0(M) \leq C_{\mathrm{Diff}(\tilde{M})}(\pi)$. The natural right action of $\widehat{\mathrm{Diff}}(M)$ and the left action of G on $\hat{\#}(M)$ are given by

$$(2.5) \qquad \begin{aligned} (\rho, \mathrm{dev}) \circ \tilde{f} &= (\rho \circ \mu(\tilde{f}), \mathrm{dev} \circ \tilde{f}), \\ g \circ (\rho, \mathrm{dev}) &= (g \circ \rho \circ g^{-1}, g \circ \mathrm{dev}), \end{aligned}$$

where $\mu(\tilde{f}) : \pi \to \pi$ is an isomorphism defined by $\mu(\tilde{f})(\gamma) = \tilde{f} \circ \gamma \circ \tilde{f}^{-1}$. Obviously both actions commute.

It is noted that two developing pairs (ρ_i, dev_i) $(i = 1,2)$ represent the same structure on M if and only if there exists an element $g \in G$ such that $g \circ \mathrm{dev}_1 = \mathrm{dev}_2$. Put

$$\#(M) = \hat{\#}(M)/\widehat{\mathrm{Diff}}^0(M).$$

The action of G induces an action of $\#(M)$. Then it is easy to show that

Lemma 2.3. *The elements of $\mathcal{T}(M)$ are in one-to-one correspondence with the orbits of $G \setminus \#(M)$.*

If $f : M \to M'$ is a representative element of $\mathcal{T}(H, M)$ then there is a developing pair $(\rho, \mathrm{dev}) : (\pi_1(M'), \tilde{M}') \to (G, X)$. We have the holonomy representation $\rho \circ f_\sharp : \pi \to G$ up to conjugate by an element of G. We then obtain a map $hol : \mathcal{T}(M) \to \mathrm{Hom}(\pi, G)/G$ which assigns to a marked structure its holonomy representation. By the definition hol lifts to a map $\widehat{hol} : \#(M) \to \mathrm{Hom}(\pi, G)$ which makes the following diagram commute.

$$\begin{array}{ccc} \#(M) & \xrightarrow{\widehat{hol}} & \mathrm{Hom}(\pi, G) \\ \downarrow & & \downarrow \\ \mathcal{T}(M) & \xrightarrow{hol} & \mathrm{Hom}(\pi, G)/G. \end{array}$$

Thurston has shown the following. (See [Lo],[J-M],[Th] for the proof.)

Theorem 2.4 (Holonomy Theorem). $\widehat{hol} : \#(M) \to \mathrm{Hom}(\pi, G)$ *is a local homeomorphism.*

3. Examples of non-Riemannian homogeneous geometry

3.1. **Homogeneous Riemannian geometry.** Let $X = G_x \backslash G$ be the simply connected homogeneous space ($x \in X$). If G_x is compact, then (G, X) is called *homogeneous Riemannian geometry.* If M is a compact manifold which admits a (G, X)-structure, then it follows that $M = X/\rho(\pi)$ where $\rho : \pi = \pi_1(M) \to G$ is a discrete faithful representation. This is obtained by the following lemma.

Lemma 3.1. *If $f : M \to N$ is a Riemannian immersion and M is complete, then f is a covering map.*

A Riemannian manifold is complete if every Cauchy sequence converges relative to the Riemannian metric. Specifically a compact Riemanniam manifold is complete.

Thus the deformation space $\mathcal{T}(M)$ is identified with the set of equivalence classes of discrete faithful representations $\mathrm{R}(\pi, G)/G$. For example, when we take $G = \mathrm{Isom}(\mathbb{H}^n_{\mathbb{K}})$ the full isometry group of the $\mathbb{K}$-hyperbolic space where $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$. The Mostow rigidity theorem says that $\mathrm{R}(\pi, G)/G$ is a single point. By Margulis-Mostow rigidity, the same result holds for a noncompact semisimple Lie group G of $\mathbb{R}$-rank ≥ 2. If $X = K \backslash G$, then X/Γ is a compact nonpositively curved Riemannian manifold. On the other hand, if M is noncompact, there occurs a remarkably distinct feature, one is Thurston bending while the other is Margulis super rigidity. After Thurston's hyperbolization theory several non-Riemannian homogeneous geometry surrounding hyperbolic geometry came to our interest in 1980s$\sim$1990s. The $\mathbb{K}$-hyperbolic space $\mathbb{H}^{n+1}_{\mathbb{K}}$ has the projective compactification $\partial\mathbb{H}^{n+1}_{\mathbb{K}}$ which is diffeomorphic to the sphere $S^{|\mathbb{K}|(n+1)-1}$. It is well known that the isometric action $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{K}})$ extends to a smooth action on $S^{|\mathbb{K}|(n+1)-1}$. This phenomenon occurs also for Hadamard manifolds (complete simply connected Riemannian manifold of nonpositive curvature). In general, an extended action on the boundary sphere is *topological*. But the above actions on $\partial\mathbb{H}^{n+1}_{\mathbb{K}}$ are known to be analytic. Denote $\mathrm{Aut}(S^{|\mathbb{K}|(n+1)-1})$ the (extended) action of $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{K}})$ on $S^{|\mathbb{K}|(n+1)-1}$. It is known that $\mathrm{Aut}(S^{|\mathbb{K}|(n+1)-1})$ acts transitively on $S^{|\mathbb{K}|(n+1)-1}$ with *noncompact* stabilizer $\mathrm{Aut}(S^{|\mathbb{K}|(n+1)-1})_\infty$ such that

$$S^{|\mathbb{K}|(n+1)-1} = \mathrm{Aut}(S^{|\mathbb{K}|(n+1)-1})_\infty\backslash\mathrm{Aut}(S^{|\mathbb{K}|(n+1)-1})$$

where $\infty \in S^{|\mathbb{K}|(n+1)-1}$. Hence we have a non-Riemannian homogeneous geometry $(\mathrm{Aut}(S^{|\mathbb{K}|(n+1)-1}), S^{|\mathbb{K}|(n+1)-1})$. According to whether $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, it is said to be

$$\begin{array}{lcc} & \text{Conformally flat geometry} & (\mathrm{PO}(n+1,1), S^n) \\ (3.1) & \text{Spherical } CR\text{-geometry} & (\mathrm{PU}(n+1,1), S^{2n+1}) \\ & \text{Quaternionic flat } CR\text{-geometry} & (\mathrm{PSp}(n+1,1), S^{4n+3}) \end{array}$$

It is an excellent result by Gromov-Lawson-Yau that a nontrivial S^1-bundle M^3 over a closed surface Σ_g of genus $g > 1$ admits a conformally flat structure. It is trivial that the product $S^1 \times \Sigma_g$ is a conformally flat manifold. On the other hand, in spherical CR-geometry $(\mathrm{PU}(2,1), S^3)$, the complement of geometric circle $S^3 - S^1$ has an invariant subgroup $\mathrm{U}(1,1) = \mathrm{P}(\mathrm{U}(1,1) \times \mathrm{U}(1))$. Choosing a discrete cocompact subgroup $\Gamma \leq \mathrm{U}(1,1)$, we get a spherical CR-manifold $S^3 - S^1/\Gamma$ which is a nontrivial S^1-bundle: $S^1 \to \mathrm{U}(1)\backslash\mathrm{U}(1,1)/\Gamma \to \mathrm{U}(1)\backslash\mathrm{PU}(1,1)/P(\Gamma)$. Here $\mathrm{U}(1)\backslash\mathrm{PU}(1,1)/P(\Gamma) = \mathbb{H}^1_{\mathbb{C}}/P(\Gamma) = \Sigma_g$. However, to our knowledge, the following problem hasn't been yet proved rigorously.

Problem. Does the product $S^1 \times \Sigma_g$ admit a spherical CR-structure?

4. Conformally flat Lorentz geometry

It is natural to consider how the isometry group of the pseudo-hyperbolic space acts on the compactification. Put $V^{m+2,2}_- = \{x \in \mathbb{R}^{m+4} \mid \mathcal{B}(x,x) = x_1^2 + \cdots + x_{m+2}^2 - x_{m+3}^2 - x_{m+4}^2 < 0\}$. If $P_{\mathbb{R}} : \mathbb{R}^{m+4} - \{0\} \to \mathbb{RP}^{m+3}$ is the canonical projection, then the real pseudo-hyperbolic space $\mathbb{H}^{m+2,1}_{\mathbb{R}}$ is defined to be $P_{\mathbb{R}}(V^{m+2,2}_-)$. For this reason, the $m+3$-dimensional quadrics $V^{m+2,2}_{-1} = \{x \in \mathbb{R}^{m+4} \mid x_1^2 + \cdots + x_{m+2}^2 - x_{m+3}^2 - x_{m+4}^2 = -1\}$ with Lorentz metric g is the complete pseudo-Riemannian manifold of signature $(m+1,1)$ and of constant curvature -1 such that $P_{\mathbb{R}}(V^{m+2,2}_-) = P_{\mathbb{R}}(V^{m+2,2}_{-1})$. Since $P_{\mathbb{R}} : V^{m+2,2}_{-1} \to \mathbb{H}^{m+2,1}_{\mathbb{R}}$

is a two-fold covering, so $\mathbb{H}_{\mathbb{R}}^{m+2,1}$ is a complete pseudo-hyperbolic space form. The action $O(m+2,2)$ on $V_-^{m+2,2}$ induces an action on $\mathbb{H}_{\mathbb{R}}^{m+2,1}$. The kernel of this action is the center $\mathbb{Z}/2 = \{\pm 1\}$ whose quotient is called *real pseudo-hyperbolic group* $PO(m+2,2)$. The projective compactification of $\mathbb{H}_{\mathbb{R}}^{m+2,1}$ is obtained by taking the closure $\overline{\mathbb{H}_{\mathbb{R}}^{m+2,1}}$ in $\mathbb{RP}^{m+3}$. Consider the commutative diagram:

$$\begin{array}{ccc}(\mathrm{GL}(m+4,\mathbb{R}), \mathbb{R}^{m+4}-\{0\}) & \xrightarrow{P} & (\mathrm{PGL}(m+4,\mathbb{R}), \mathbb{RP}^{m+3}) \\ \cup & & \cup \\ (\mathrm{O}(m+2,2), V_-^{m+2,2}\cup V_0) & \xrightarrow{P} & (\mathrm{PO}(m+2,2), \mathbb{H}_{\mathbb{R}}^{m+2,1}\cup S^{m+1,1})\end{array}$$

Here $V_0 = V_0^{m+2,1} = \{x \in \mathbb{R}^{m+4} \mid x_1^2 + \cdots + x_{m+2}^2 - x_{m+3}^2 - x_{m+4}^2 = 0\}$. It follows that

$$\overline{\mathbb{H}_{\mathbb{R}}^{m+2,1}} = \mathbb{H}_{\mathbb{R}}^{m+2,1} \cup S^{m+1,1}.$$

From this viewpoint, the pseudo-hyperbolic action of $PO(m+2,2)$ on $\mathbb{H}_{\mathbb{R}}^{m+2,1}$ extends to *conformal action* of $S^{m+1,1}$. We obtain *conformally flat Lorentz geometry* $(PO(m+2,2), S^{m+1,1})$. This is of course non-Riemannian homogeneous geometry.

Let $(1,0,\ldots,0,1) \in V_0$ be a null vector. Put $\hat{\infty} = P(1,0,\ldots,0,1) \in S^{m+1,1}$ which is called *the point at infinity.* The stabilizer $PO(m+2,2)_{\hat{\infty}}$ is $\mathbb{R}^{m+2} \rtimes (O(m+1,1) \times \mathbb{R}^+)$ up to conjugacy. When $h \in PO(m+2,2)_{\hat{\infty}}$, the differential map $h_* : T_{\hat{\infty}}S^{m+1,1} \to T_{\hat{\infty}}S^{m+1,1}$ is an isomorphism, $h_* \in \mathrm{Aut}(T_{\hat{\infty}}S^{m+1,1}) = O(m+1,1) \times \mathbb{R}^+$. Thus the structure group of $(PO(m+2,2), S^{m+1,1})$ is $O(m+1,1) \times \mathbb{R}^+$. Originally as a G-structure, conformal Lorentz structure is an $O(m+1,1) \times \mathbb{R}^+$-structure. In addition, an integrable $O(m+1,1) \times \mathbb{R}^+$-structure is *conformally flat Lorentz structure.* (Equivalently, the Weyl conformal curvature tensor vanishes.) When $\{\infty\}$ is the point at infinity of $S^m = \partial\mathbb{H}_{\mathbb{R}}^{m+1}$, we can consider the minimal parabolic group $O(m+1,1)_\infty$ which is an amenable Lie subgroup of $O(m+1,1)$. We remark that $O(m+1,1)_\infty$ is isomorphic to the similarity group $\mathrm{Sim}(\mathbb{R}^m)$.

Definition 4.1. If the structure group of a conformally flat Lorentz $(m+2)$-manifold M belongs to $O(m+1,1)_\infty \times \mathbb{R}^+$, then M is said to be a *conformally flat Lorentz parabolic* manifold.

We study a special class of conformally flat Lorentz parabolic manifolds called *Lorentz similarity manifold* of dimension $m+2$ and *Fefferman-Lorentz manifold* of dimension $2n+2$.

5. Lorentzian similarity geometry

Recall that $\mathbb{R}^{m+2}$ is the euclidean space with Lorentz inner product sitting in $S^{m+1,1}-\{\hat{\infty}\}$. Then $\mathrm{PO}(m+2,2)_{\hat{\infty}} = \mathbb{R}^{m+2} \rtimes (\mathrm{O}(m+1,1)\times \mathbb{R}^+)$. We define $\mathrm{Sim}_{\mathrm{L}}(\mathbb{R}^{m+2}) = \mathbb{R}^{m+2} \rtimes (\mathrm{O}(m+1,1)\times \mathbb{R}^+)$. The pair $(\mathrm{Sim}_{\mathrm{L}}(\mathbb{R}^{m+2}), \mathbb{R}^{m+2})$ is said to be *Lorentz similarity geometry.* In [3] we proved the following.

Theorem 5.1. *If M is a compact complete Lorentz similarity manifold of dimension $m+2$, then the fundamental group of M is virtually polycyclic. Furthermore, M is diffeomorphic to an infrasolvmanifold.*

This theorem is originally proved by T. Aristide. Once $\pi_1(M)$ turns out to be virtually polycyclic, the holonomy group $L(\pi_1(M))$ belongs to either $\mathrm{O}(m+1,1)_\infty \times \mathbb{R}^+$ or $\mathrm{O}(m+1)\times\mathrm{O}(1)\times\mathbb{R}^+$. Here $\mathrm{O}(m+1,1)_\infty = \mathrm{Sim}(\mathbb{R}^m) = \mathbb{R}^m \rtimes (\mathrm{O}(m)\times\mathbb{R}^*)$. Since Γ acts freely as a' ne motions on $\mathbb{R}^{m+2}$, the matrix of holonomy group has no eigenvalue 1. The latter case shows that $L(\pi_1(M)) \leq \mathrm{O}(m+1)\times\mathrm{O}(1)$ so that M reduces to a compact euclidean space form. Then $\pi_1(M)$ is a Bieberbach group.

Corollary 5.2. *A finite cover of a compact complete Lorentz similarity manifold M is a conformally flat Lorentz parabolic manifold.*

We shall give a sketch of proof of Theorem 5.1. Put $M = \mathbb{R}^{m+2}/\Gamma$ where $\Gamma \leq \mathrm{Sim}_{\mathrm{L}}(\mathbb{R}^{m+2})$. There is the exact sequence: $1 \to \mathbb{R}^{m+2} \to \mathrm{Sim}_{\mathrm{L}}(\mathbb{R}^{m+2}) \xrightarrow{L} \mathrm{O}(m+1,1)\times\mathbb{R}^+ \to 1$. If $\mathbb{R}^{m+2}\cap\Gamma$ is nontrivial, say $\mathbb{Z}^k$, then a properly discontinuous action of Γ induces a properly discontinuous action of $L(\Gamma)$ on $\mathbb{R}^{m-k}$ as in the same argument of [3, (1) Proposition 2.2]. Then Γ is virtually polycyclic by induction. So we assume

$$\mathbb{R}^{m+2}\cap\Gamma = \{1\}. \tag{5.1}$$

Note also that $(\mathbb{R}^{m+2}\rtimes\mathbb{R}^+)\cap\Gamma = \{1\}$ because each element has the form $(a, \lambda\cdot I)$. As Γ acts freely on $\mathbb{R}^{m+2}$, $\lambda = 1$. It follows $(\mathbb{R}^{m+2}\rtimes\mathbb{R}^+)\cap\Gamma = \mathbb{R}^{m+2}\cap\Gamma$.

Consider the following exact sequence:

$$1 \to \mathbb{R}^{m+2}\rtimes\mathbb{R}^+ \to \mathrm{Sim}_{\mathrm{L}}(\mathbb{R}^{m+2}) \xrightarrow{\mathsf{p}} \mathrm{O}(m+1,1) \to 1. \tag{5.2}$$

If $\mathsf{p}(\Gamma)$ is discrete in $\mathrm{O}(m+1,1)$, then the cohomological dimension $\mathrm{cd}\,\mathsf{p}(\Gamma) \leq m+1$. As $\mathbb{R}^{m+2}/\Gamma$ is compact, $\mathrm{cd}\,\Gamma = m+2$. On the other hand, $\Gamma \cong \mathsf{p}(\Gamma)$ by (5.1), $\mathrm{cd}\,\Gamma = \mathrm{cd}\,\mathsf{p}(\Gamma)$ which yields a contradiction.

Suppose that $\mathsf{p}(\Gamma)$ is indiscrete in $\mathrm{O}(m+1,1)$. Then the identity component of the closure $\overline{\mathsf{p}(\Gamma)}^0$ is solvable in $\mathrm{O}(m+1,1)$.

Case I. If it is noncompact, then it belongs to the maximal amenable subgroup $\mathrm{Sim}(\mathbb{R}^m)$ up to conjugate. The normalizer of $\overline{\mathsf{p}(\Gamma)}^0$ is contained in $\mathrm{Sim}(\mathbb{R}^m)$. In particular, $\mathsf{p}(\Gamma) \leq \mathrm{Sim}(\mathbb{R}^m)$. (5.2) induces an exact sequence:

$$1 \to \mathbb{R}^{m+2} \rtimes \mathbb{R}^+ \to \mathsf{p}^{-1}(\mathrm{Sim}(\mathbb{R}^m)) \xrightarrow{\mathsf{p}} \mathrm{Sim}(\mathbb{R}^m) \to 1$$

in which $\mathsf{p}^{-1}(\mathrm{Sim}(\mathbb{R}^m))$ is an amenable Lie subgroup. Any discrete subgroup of an amenable Lie group is virtually polycyclic so is Γ.

Case II. Suppose that $\overline{\mathsf{p}(\Gamma)}^0$ is compact, say T^ℓ. We consider actions of subgroups of $\mathrm{O}(m+1,1)$ on $\mathbb{H}_{\mathbb{R}}^{m+1} \cup S^m$. If T^ℓ has no fixed point in S^m, then T^ℓ has a unique fixed point $0 \in \mathbb{H}_{\mathbb{R}}^{m+1}$ so that $\mathsf{p}(\Gamma) \leq \mathrm{O}(m+1) \times \mathrm{O}(1)$. Thus $\Gamma \leq \mathrm{Sim}(\mathbb{R}^{m+2})$. $\mathbb{R}^{m+2}/\Gamma$ turns out to be a compact complete similarity manifold and so Γ is virtually abelian (a Bieberbach group).

Suppose that T^ℓ has the fixed point set S^k in S^m for some $k < m$. As $\overline{\mathsf{p}(\Gamma)}$ leaves invariant the complement $S^m - S^k = \mathbb{H}_{\mathbb{R}}^{k+1} \times S^{m-k-1}$. It follows $\overline{\mathsf{p}(\Gamma)} \leq \mathrm{O}(k+1,1) \times \mathrm{O}(m-k)$ for which $T^\ell = \overline{\mathsf{p}(\Gamma)}^0 \leq \mathrm{O}(m-k)$. If $\mathrm{Pr} : \mathrm{O}(k+1,1) \times \mathrm{O}(m-k) \to \mathrm{O}(k+1,1)$ is the canonical projection, then $\mathrm{Pr}(\mathsf{p}(\Gamma))$ is discrete. Note that $\mathrm{Ker}\,\mathrm{Pr} \circ \mathsf{p} = \mathbb{R}^{m+2} \rtimes (\mathrm{O}(m-k) \times \mathbb{R}^+)$. Put

$$\Delta = (\mathbb{R}^{m+2} \rtimes (\mathrm{O}(m-k) \times \mathbb{R}^+)) \cap \Gamma. \tag{5.3}$$

It is nontrivial, because if trivial, $\Gamma \cong \mathrm{Pr}(\mathsf{p}(\Gamma))$ so $m+2 = \mathrm{cd}\,\Gamma = \mathrm{cd}\,\mathrm{Pr}(\mathsf{p}(\Gamma))$ but $\mathrm{cd}\,\mathrm{Pr}(\mathsf{p}(\Gamma)) \leq k+1$ which is impossible by the inequality $k < m$.

Since T^ℓ is a maximal torus in $\mathrm{O}(m-k)$, $N_{\mathrm{O}(m-k)}(T^\ell)/T^\ell$ is finite for the normalizer $N_{\mathrm{O}(m-k)}(T^\ell)$. As $\overline{\mathsf{p}(\Gamma)}$ normalizes T^ℓ, there exists a finite index normal subgroup H of $\overline{\mathsf{p}(\Gamma)}$ which centralizes T^ℓ with $H^0 = T^\ell$. Note that $H \cap \mathsf{p}(\Gamma)$ is of finite index in $\mathsf{p}(\Gamma)$. Put $\Gamma_1 = \mathsf{p}^{-1}(H \cap \mathsf{p}(\Gamma))$ which is a finite index subgroup of Γ.

Let $\mathsf{p}_1 : \mathbb{R}^{m+2} \rtimes (\mathrm{O}(m-k) \times \mathbb{R}^+) \to \mathrm{O}(m-k)$ be the projection. Then $\mathsf{p}_1(\Delta) \leq \mathrm{O}(m-k)$ such that $\overline{\mathsf{p}_1(\Delta)}^0$ is a torus in $\mathrm{O}(m-k)$ from (5.3). Since $\overline{\mathsf{p}_1(\Delta)}$ is a finite extension of $\overline{\mathsf{p}_1(\Delta)}^0$, we choose Δ_1 such that $\mathsf{p}_1(\Delta_1) = \overline{\mathsf{p}_1(\Delta)}^0 \cap \mathsf{p}_1(\Delta)$. As $\mathsf{p}_1(\Delta_1) \leq \mathsf{p}(\Gamma)$, $\overline{\mathsf{p}_1(\Delta_1)}^0 \leq \overline{\mathsf{p}(\Gamma)}^0$. Noting that $\mathsf{p}(\Gamma_1) \leq H$ and $\mathsf{p}(\Delta_1) \leq \overline{\mathsf{p}(\Gamma)}^0$ for which H centralizes $T^\ell = \overline{\mathsf{p}(\Gamma)}^0$ as above, it follows that $\mathsf{p}(\Gamma_1)$ centralizes $\mathsf{p}(\Delta_1)$.

Note that $\mathsf{p}_1 : \Delta \to \mathsf{p}_1(\Delta)$ is injective. In fact, if not, then $\Delta \cap \mathbb{R}^{m+2} \rtimes \mathbb{R}^+ \neq \{1\}$, so $\Gamma \cap \mathbb{R}^{m+2} \rtimes \mathbb{R}^+ \neq \{1\}$ which is impossible by the remark below (5.1). Since Γ normalizes Δ, it is easy to see that Γ_1 centralizes Δ_1. Consider the exact sequences:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{R}^{m+2} & \longrightarrow & \mathbb{R}^{m+2} \rtimes \mathrm{O}(m-k) & \xrightarrow{\mathsf{p}} & \mathrm{O}(m-k) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 1 & \longrightarrow & \Delta_1 & \xrightarrow{\mathsf{p}} & \mathsf{p}(\Delta_1) & \longrightarrow & 1
\end{array}$$

where $\overline{\mathsf{p}(\Delta_1)}^0 = T^s$ for some $s \leq m-k$. It is well known that the abelian discrete subgroup belongs to the following group (cf. [8]):

$$(5.4) \qquad \Delta_1 \leq V \times T^s = \{\left(\begin{bmatrix} a \\ 0 \end{bmatrix}, \begin{pmatrix} I & 0 \\ 0 & C \end{pmatrix}\right) \mid C \in T^s, a \in V\}$$

such that $V \times T^s/\Delta_1$ is compact. Here $V \cong \mathbb{R}^{k+2}$.

Let $\Gamma_1 \leq \mathbb{R}^{m+2} \rtimes (\mathrm{O}(k+1,1) \times \mathrm{O}(m-k) \times \mathbb{R}^+)$ be as before and choose an arbitrary element $\gamma = \left(\begin{bmatrix} x \\ y \end{bmatrix}, \lambda \cdot \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right)$ and take an element $\alpha = \left(\begin{bmatrix} a \\ 0 \end{bmatrix}, \begin{pmatrix} I & 0 \\ 0 & C \end{pmatrix}\right)$ from Δ_1. As Γ_1 centralizes Δ_1, the equation $\gamma\alpha\gamma^{-1} = \alpha$ implies that

$$\lambda \cdot Aa = a, BCB^{-1} = C \text{ and } y - BCB^{-1}y = 0.$$

The projection $P : \mathbb{R}^{k+2} - \{0\} \to \mathbb{RP}^{k+1}$ maps the cone V_0 onto S^k. We observe that if $\langle a, a \rangle = 0$ with respect to the Lorentz inner product, then $P(a) = [a] \in S^k$. Put $[a] = \infty \in S^k$ up to conjugacy. The equality $\lambda \cdot Aa = a$ implies $A\infty = \infty$ so $A \in \mathrm{O}(k+1,1)_\infty$. This holds for arbitrary elements of Γ_1. It follows

$$(5.5) \qquad \Gamma \leq \mathbb{R}^{m+2} \rtimes (\mathrm{O}(k+1,1)_\infty \times \mathrm{O}(m-k) \times \mathbb{R}^+)$$

which is an amenable Lie subgroup. Thus Γ is virtually polycyclic. When $\langle a, a \rangle \neq 0$, as $\langle a, a \rangle = \langle \lambda Aa, \lambda Aa \rangle = \lambda^2 \langle a, a \rangle$, it follows $\lambda = 1$. Thus

$$(5.6) \qquad \Gamma \leq \mathbb{R}^{m+2} \rtimes (\mathrm{O}(k+1,1) \times \mathrm{O}(m-k)) \leq \mathrm{E}(m+1,1).$$

$\mathbb{R}^{m+2}/\Gamma$ becomes a compact complete Lorentz flat space form. It is well known that Γ is virtually polycyclic. This proves the theorem 5.1.

6. Conformally flat Fefferman-Lorentz geometry

Let $(\mathrm{O}(2n+2,2), S^1 \times S^{2n+1})$ be the conformally flat Lorentz geometry (which is a 2-fold cover.) There is the natural embedding $\mathrm{U}(n+1,1) \to \mathrm{O}(2n+2,2)$. $\mathrm{U}(n+1,1)$ acts transitively on $S^1 \times S^{2n+1}$ so we have a subgeometry $(\mathrm{U}(n+1,1), S^1 \times S^{2n+1})$.

Proposition 6.1. *A manifold locally modelled on* $(\mathrm{U}(n+1,1), S^1 \times S^{2n+1})$ *admits a Lorentz prabolic structure.*

Proof. We see that

$$\hat{\mathrm{U}}(n+1,1)\cap \mathrm{O}(2n+2,2)_{\infty} = \mathbb{R}^{2n+2} \rtimes (\mathrm{O}(2n+1,1)\times \mathbb{R}^*.$$

Then the intersection $\hat{\mathrm{U}}(n+1,1)_{\infty} = \mathcal{N} \rtimes (\mathrm{U}(n)\times \mathbb{R}^+)$ which is amenable. Here $\mathcal{N}$ is the Heisenberg Lie group. So $\hat{\mathrm{U}}(n+1,1)_{\infty}$ belongs to the maximal amenable group $\mathbb{R}^{2n+2} \rtimes (\mathrm{O}(2n+1,1)_{\infty}\times \mathbb{R}^*)$. Thus the structure group of $(\mathrm{U}(n+1,1), S^1\times S^{2n+1})$ belongs to the parabolic group $\mathrm{O}(2n+1,1)_{\infty}\times \mathbb{R}^*$. So does any manifold modelled on $(\mathrm{U}(n+1,1), S^1\times S^{2n+1})$. □

Definition 6.2. A manifold locally modelled on $(\mathrm{U}(n+1,1), S^1 \times S^{2n+1})$ is said to be a conformally flat Fefferman-Lorentz parabolic manifold.

To the rest of this section we shall give our recent results concerning compact conformally flat Fefferman-Lorentz parabolic manifolds. The details will be given elesewhere.

Recall that the center S^1 acts freely on the 2-fold covering $S^1\times S^{2n+1}$ of $S^{2n+1,1}$, there is the equivariant principal bundle:

(6.1) $(S^1,S^1)\rightarrow(\mathrm{U}(n+1,1),S^1\times S^{2n+1}) \xrightarrow{(P,\mathrm{p})} (\mathrm{PU}(n+1,1),S^{2n+1})$.

Let X be a domain of $S^1\times S^{2n+1}$. If h is an element of the group of conformal Lorentz transformations $\mathrm{Conf}(X)$, then $h: X\rightarrow X$ extends uniquely to a conformal diffeomorphism of $S^1\times S^{2n+1}$ by Liouville's theorem. We assume that

(6.2) $$\mathrm{Conf}(X)\leq \mathrm{U}(n+1,1).$$

Suppose that a discrete subgroup Γ of $\mathrm{U}(n+1,1)$ acts properly discontinuously on X such that the quotient X/Γ is compact. Note that there is a covering group extension:

(6.3) $$1 \longrightarrow \Gamma \longrightarrow N_{\mathrm{Conf}(X)}(\Gamma) \xrightarrow{\ \nu\ } \mathrm{Conf}(X/\Gamma) \longrightarrow 1.$$

We shall determine X/Γ when X/Γ admits a 1-parameter subgroup H whose lift H to $\mathrm{U}(n+1,1)$ is not the center $\mathcal{Z}\mathrm{U}(n+1,1)$.

Theorem 6.3. *Let X/Γ be a $2n+2$-dimensional compact conformally flat Fefferman-Lorentz parabolic manifold. If X/Γ admits a 1-parameter subgroup H whose lift H to $\mathrm{U}(n+1,1)$ is not the center $\mathcal{Z}\mathrm{U}(n+1,1)$, then X/Γ is a Seifert fiber space over a spherical CR-orbifold. Moreover X/Γ is either one of* (i), …, (v). *As a consequence, a finite covering of such X/Γ is a Fefferman-Lorentz manifold.*

(i) $X/\Gamma = S^1\times_{\mathbb{Z}_\ell} S^{2n+1}$ *where* $\mathbb{Z}_\ell\leq T^{n+1}$.

(ii) $S^1\rightarrow X/\Gamma \longrightarrow \mathcal{N}/Q$ *where* $Q\leq \mathcal{N}\rtimes \mathrm{U}(n)$.

(iii) $S^1 \to X/\Gamma \longrightarrow S^{2n}\times_F S^1$ *where* $F \leq \mathrm{U}(n)$.
(iv) $S^1 \to X/\Gamma \longrightarrow S^{2n+1}/F$ *where* $F \leq T^{n+1}$.
(v) $S^1 \to X/\Gamma \longrightarrow (S^{2n+1} - L(Q))/Q$ *where*
$Q \leq \mathrm{P}(\mathrm{U}(k,1) \times \mathrm{U}(n-k+1))$ $(k = 1, \ldots, n)$.

The idea of proof is as follows. Let $S^1 = \mathcal{Z}\mathrm{U}(n+1,1)$ be the center of $\mathrm{U}(n+1,1)$. Then $S^1 \cdot \tilde{\mathsf{H}} \leq \mathrm{U}(n+1,1)$. There is an equivariant fibration:

$$(6.4) \qquad (S^1, S^1) \longrightarrow (S^1 \cdot \tilde{\mathsf{H}}, \Gamma, X) \xrightarrow{(P,p)} (G, Q, W)$$

where we put $G = S^1 \cdot \tilde{\mathsf{H}}/S^1$, $Q = \Gamma/S^1 \cap \Gamma$ and $W = X/S^1$. As $Q, G \leq \mathrm{PU}(n+1,1)$, the quotient W/Q is a spherical CR-orbifold with CR-action G. To determine X/Γ reduces to the classification of CR-manifolds (Q, W) with the 1-parameter group G of CR-transformations. The classification is accomplished by the result in [4].

When $\dim X/\Gamma = 4$, then $Q \leq \mathrm{U}(1,1)$ so that $L(Q) \subset S^1$ $(k = 1)$. According to whether $L(Q)$ is a Cantor set in S^1 or $L(Q) = S^1$, it is well known that $S^3 - L(Q)/Q = S^1 \times S^2 \# \cdots \# S^1 \times S^2$ or some finite cover of $S^3 - L(Q)/Q = V^3_{-1}/Q$ is a principal S^1-bundle with nonzero euler class over a closed surface of genus $g \geq 2$.

6.1. **Non Fefferman-Lorentz manifold.** It is conceivable whether some finite cover of any compact conformally flat Fefferman-Lorentz parabolic manifold is a Fefferman-Lorentz manifold. It is not true in general. It will be shown

Proposition 6.4. *There exists a compact conformally flat Fefferman-Lorentz parabolic manifold P of dimension $2n+2$ $(n \geq 1)$ but no finite covering is a Fefferman-Lorentz manifold.*

This manifold P supports a principal fiber space: $T^2 \to P \longrightarrow \mathbb{H}^n_{\mathbb{C}}/Q_0$ where $\mathbb{H}^n_{\mathbb{C}}/Q_0$ is a compact complex hyperbolic manifold.

7. Representation space

Let X/Γ be a compact conformally flat Lorentz manifold with S^1-action so that $X \subset \mathbb{R}\times S^{2n+1}$, $\Gamma, \tilde{S}^1 \leq \widetilde{\mathrm{O}(m+2,2)}$. If $p : \widetilde{\mathrm{O}(m+2,2)} \to \mathrm{O}(m+2,2)$ is the covering homomorphism, put $G = \overline{p(\tilde{S}^1)} \leq \mathrm{O}(m+2,2)$. We will prove that

- If G is compact, then $m = 2n$ and $G = S^1$, $C_{\mathrm{O}(m+2,2)}(S^1) = \mathrm{U}(n+1,1)$. $(S^1, X/\Gamma)$ is locally modelled on $(\mathrm{U}(n+1,1), S^1 \times S^{2n+1})$ where $S^1 = \mathcal{Z}\mathrm{U}(n+1,1)$, i.e. X/Γ is a conformally flat Fefferman-Lorentz parabolic manifold.

- If G is noncompact, the either $\Gamma \leq \mathbb{R}^{m+2} \rtimes \mathrm{O}(m+1,1)$ and $G = \mathbb{R}$ or $\Gamma \leq \mathrm{O}(m+1,1) \times \mathbb{R}^+$ and $G = \mathbb{R}^+$.

Proposition 7.1. *Let $X/\Gamma = S^1 \times \mathcal{N}^3/\Delta$ which is a conformally flat Lorentz parabolic manifold and $\Gamma = \mathbb{Z} \times \Delta \leq \widetilde{\mathrm{O}(4,2)}$, $\# \subset \mathbb{R} \times S^3$. There are exactly two distinct faithful representations up to conjugate in $\mathrm{O}(4,2)$:*

$$\begin{aligned} &\rho_1 : \Gamma \to \mathbb{R} \times (\mathcal{N} \rtimes \mathrm{U}(1)), \ S^1 \textit{ is lightlike.} \\ &\rho_2 : \Gamma \to \mathbb{R}^3 \times (\mathbb{R}^2 \rtimes \mathrm{O}(2)) \leq \mathbb{R}^4 \rtimes \mathrm{O}(3,1), \ S^1 \textit{ is spacelike.} \end{aligned} \tag{7.1}$$

Then the space of discrete faithful representations $\mathsf{R}(\Gamma, \widetilde{\mathrm{O}(4,2)})$ consists of two components $\mathsf{R}(\Gamma, \mathbb{R} \times (\mathcal{N} \rtimes \mathrm{U}(1)))$, $\mathsf{R}(\Gamma, \mathbb{R}^3 \rtimes (\mathbb{R}^2 \rtimes \mathrm{O}(2)))$.

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