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THURSTON THEORY ON TEICHMÜLLER SPACE WITH EXTREMAL LENGTH

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1. Introduction and Background

The purpose of this note is to survey a recent progress on the extremal length geometry on Teichmüller space. Indeed, we will discuss the "Thurston theory" with extremal length by comparing our results with facts in the "original" Thurston theory.

We first give basics in the Thurston theory. Let S be a closed surface of genus $g\geq 2$. After an appropriate modification, the argument here is available for hyperbolic surface of finite area. We fix a reference hyperbolic structure on S. Let $\mathcal{S}$ be the set of homotopy classes of non-trivial and non-peripheral simple closed curves on S. The *intersection number function* on $\mathcal{S}$ is defined by

$$i(\alpha,\beta)=\min\{^{\#}(\alpha'\cap\beta')\mid\alpha'\in\alpha,\beta'\in\beta\}.$$

for $\alpha,\beta\in\mathcal{S}$. Consider the space $\mathcal{R}=\mathbb{R}_{\geq 0}^{\mathcal{S}}$ of non-negative functions on $\mathcal{S}$. We topologize $\mathcal{R}$ with the topology of pointwise convergence. A *weighted simple closed curve* is a formal product of a non-negative number and an element in $\mathcal{S}$. We denote by $\mathcal{WS}$ the set of weighted simple closed curves on S. The *space of measured laminations* $\mathcal{ML}=\mathcal{ML}(S)$ is defined as the closure of the embedding

$$\mathcal{WS}\ni t\alpha\mapsto[\mathcal{S}\ni\beta\mapsto t\cdot i(\alpha,\beta)]\in\mathcal{R}.$$

Thurston showed that $\mathcal{ML}$ is homeomorphic to the Euclidean space of the same (real) dimension as that of the Teichmüller space of S. The space $\mathcal{ML}$ is a *cone* in the sense that the space admits a canonical $\mathbb{R}_{+}$-action:

$$\alpha\mapsto t\alpha$$

for $\alpha\in\mathcal{ML}$ and $t>0$. The projective space $\mathcal{PML}=(\mathcal{ML}-\{0\})/\mathbb{R}_{+}$ is called the *space of projective measured foliations* on S. The space $\mathcal{PML}$ is homeomorphic to the sphere. $\mathcal{R}$ also has a canonical $\mathbb{R}_{+}$-action. We define $\mathcal{PR}=(\mathcal{R}-\{0\})/\mathbb{R}_{+}$ and the projection $\mathbf{pr}\colon\mathcal{ML}-\{0\}\to\mathcal{PML}$ is the restriction of the projection

$$\mathbf{pr}\colon\mathcal{R}-\{0\}\to\mathcal{PR}$$

to $\mathcal{ML} - \{0\}$.

By definition, $\mathcal{WS}$ is dense in $\mathcal{ML}$. We define the intersection number for two curves in $\mathcal{WS}$ by

$$i(t\alpha, s\beta) = ts \cdot i(\alpha, \beta).$$

It is known that the intersection number function on $\mathcal{WS}$ extends continuously on $\mathcal{ML}$. The space of measured laminations and the intersection number function are an important mathematical object in the Thurston theory for Teichmüller space.

2. Teichmüller space

2.1. **Teichmüller space.** The *Teichmüller space* $T(S)$ of S is the set of equivalence classes of marked Riemann surfaces, where a *marked Riemann surface* (Y, f) is a pair of a Riemann surface Y and an orientation preserving homeomorphism $f\colon \mathrm{Int}(S) \to Y$. Two marked Riemann surfaces (Y_1, f_1) and (Y_2, f_2) are *Teichmüller equivalent* if there exists a conformal mapping $h\colon Y_1 \to Y_2$ such that $h \circ f_1$ is homotopic to f_2.

2.2. **Length spectum distance.** Let $y = (Y, f) \in T(S)$. From the assumption, any Riemann surface Y admits a unique hyperbolic structure comparable with the conformal structure on Y. For $\alpha \in \mathcal{S}$, we define the *hyperbolic length* $\ell_y(\alpha)$ of α on y as the hyperbolic length of simple closed geodesic homotopic to $f(\alpha)$. For $t\alpha \in \mathcal{WS}$, we set $\ell_y(t\alpha) = t\ell_y(\alpha)$. It is known that the hyperbolic length function ℓ_y extends continuously on $\mathcal{ML}$ (cf. [2]).

For $y_1, y_2 \in T(S)$, we define the *Thurston's asymmetric metric* d_{Th} on $T(S)$ by

$$d_{Th}(y_1, y_2) = \log \sup_{\alpha \in \mathcal{S}} \frac{\ell_{y_2}(\alpha)}{\ell_{y_1}(\alpha)}. \tag{2.1}$$

The function d_{Th} is not a distance function. Indeed, d_{Th} satisfies the axiom of the distance function except for the symmetricity. The Thurston's asymmetric metric is represented as the infimum of the logarithms of the Lipschitz constants of Lipschitz mappings from y_1 to y_2 respecting the marking. For detail, see [14].

We also consider the symmetrization of the Thurston's asymmetric metric, called the *length spectrum metric*

$$\begin{aligned} d_{ls}(x, y) &= \max\{d_{Th}(x, y), d_{Th}(y, x)\} \\ &= \log \sup_{\alpha \in \mathcal{S}} \left\{ \frac{\ell_{y_2}(\alpha)}{\ell_{y_1}(\alpha)}, \frac{\ell_{y_1}(\alpha)}{\ell_{y_2}(\alpha)} \right\}. \end{aligned}$$

2.3. **Teichmüller distance.** Let $y = (Y, f) \in T(S)$. The *extremal length* $\mathrm{Ext}_y(\alpha)$ of $\alpha \in \mathcal{S}$ on y is, by definition, the infimum of the reciprocals of the embedded annuli whose cores are homotopic to α. We set $\mathrm{Ext}_y(t\alpha) = t^2 \mathrm{Ext}_y(\alpha)$. Then, it is known that the extremal length function Ext_y extends continuously on $\mathcal{ML}$ (cf. [6]). Recently, we know that Ext_y is right-differentiable with respect to the piecewise linear structure on $\mathcal{ML}$ (cf. [11] and [12]).

For $y_1, y_2 \in T(S)$, we define the *Teichmüller distance* d_T on $T(S)$ by

$$d_T(y_1, y_2) = \frac{1}{2} \log \sup_{\alpha \in \mathcal{S}} \frac{\mathrm{Ext}_{y_2}(\alpha)}{\mathrm{Ext}_{y_1}(\alpha)}. \tag{2.2}$$

The Teichmüller distance is originally defined as the half of the infimum of the logarithms of the maximal dilatations of quasiconformal mappings from y_1 to y_2 respecting the marking. The presentation (2.2) is called the *Kerckhoff's formula* of the Teichmüller distance (cf. [6]).

3. Realizations of Teichmüller space

3.1. **Thurston compactification.** Let $y = (Y, f) \in T(S)$. We define a mapping

$$\tilde{\Phi}_{Th} \colon T(S) \ni y \mapsto [\mathcal{S} \ni \alpha \mapsto \ell_y(\alpha)] \in \mathcal{R}, \tag{3.1}$$

and set

$$\Phi_{Th} \colon T(S) \ni y \mapsto \mathbf{pr} \circ \tilde{\Phi}_{Th}(y) \in \mathcal{PR}. \tag{3.2}$$

It is known that Φ_{Th} is injective and the image $\Phi_{Th}(T(S))$ is relatively compact in $\mathcal{PR}$. The closure $\overline{T(S)}^{Th}$ of the image is called the *Thurston compactification* of $T(S)$. The *Thurston boundary* $\partial_{Th}T(S)$ is, by definition, the complement of the image. It is known that the Thurston boundary coincides with the space $\mathcal{PML}$ of projective measured laminations, and the Thuston compactification is homeomorphic to the closed ball (cf. [2]).

3.2. **Gardiner-Masur compactification.** Let $y = (Y, f) \in T(S)$. As above, we also define a mapping

$$\tilde{\Phi}_{GM} \colon T(S) \ni y \mapsto [\mathcal{S} \ni \alpha \mapsto \mathrm{Ext}_y(\alpha)^{1/2}] \in \mathcal{R}, \tag{3.3}$$

and set

$$\Phi_{GM} \colon T(S) \ni y \mapsto \mathbf{pr} \circ \tilde{\Phi}_{GM}(y) \in \mathcal{PR}. \tag{3.4}$$

As the case of the Thuston compactification, Φ_{GM} is injective and the image $\Phi_{GM}(T(S))$ is relatively compact in $\mathcal{PR}$ (cf. [3]). The closure $\overline{T(S)}^{GM}$ of the image is called the *Gardiner-Masur compactification* of $T(S)$. It is known that the Gardinar-Masur boundary contains the space of projective measured foliations. Almost no topological property of the Gardiner-Masur compactification is known.

3.3. **Metric d_∞ on $\mathcal{R}_0$.** Let

$$\mathcal{R}_0 = \{(x_\alpha)_{\alpha\in\mathcal{S}} \in \mathcal{R} \mid x_\alpha > 0 \text{ for } \alpha \in \mathcal{S}\}$$

For $\mathbf{x} = (x_\alpha)_{\alpha\in\mathcal{S}}, \mathbf{y} = (y_\alpha)_{\alpha\in\mathcal{S}} \in \mathcal{R}_0$, we define

$$\begin{aligned} d_\infty(\mathbf{x},\mathbf{y}) &= \left|\log \sup_{\alpha\in\mathcal{S}} \frac{y_\alpha}{x_\alpha}\right|. \\ d_\infty^{sym}(\mathbf{x},\mathbf{y}) &= \max\{d_\infty(\mathbf{x},\mathbf{y}), d_\infty(\mathbf{y},\mathbf{x})\} \\ &= \max\left\{\left|\log \sup_{\alpha\in\mathcal{S}} \frac{y_\alpha}{x_\alpha}\right|, \left|\log \sup_{\alpha\in\mathcal{S}} \frac{x_\alpha}{y_\alpha}\right|\right\}. \end{aligned}$$

Then, $(\mathcal{R}_0, d_\infty)$ is an asymmetric metric space and $(\mathcal{R}_0, d_\infty^{sym})$ is a metric space. However, each space has infinitely many components. Notice that the images of the embeddings $\tilde{\Phi}_{Th}$ and $\tilde{\Phi}_{GM}$ is contained in $\mathcal{R}_0$. The metrics d_∞ and d_∞^{sym} have a *universal property* in our geometries in the sense that

$$\begin{aligned} d_{Th}(x,y) &= d_\infty(\tilde{\Phi}_{Th}(x), \tilde{\Phi}_{Th}(y)) = (\tilde{\Phi}_{Th})^* d_\infty(x,y) \\ d_{ls}(x,y) &= d_\infty^{sym}(\tilde{\Phi}_{Th}(x), \tilde{\Phi}_{Th}(y)) = (\tilde{\Phi}_{Th})^* d_\infty^{sym}(x,y) \\ d_T(x,y) &= d_\infty(\tilde{\Phi}_{GM}(x), \tilde{\Phi}_{GM}(y)) = (\tilde{\Phi}_{GM})^* d_\infty(x,y) = (\tilde{\Phi}_{GM})^* d_\infty^{sym}(x,y). \end{aligned}$$

Namely, our distances are represented as pull-back distances on $T(S)$.

4. Cones

4.1. Geodesic currents and Bonahon's theory.

4.1.1. *Geodesic currents.* Let Γ be the Fuchsian group of S acting on the upper-half plane $\mathbb{H}$. Let

$$\mathcal{G} = (\partial\mathbb{H} \times \partial\mathbb{H}/(\text{diagonal}))/\mathbb{Z}_2,$$

where $\mathbb{Z}_2$ acts on the product space by interchanging the coordinates. The space $\mathcal{G}$ is recognized as the space of non-oriented geodesic on $\mathbb{H}$. A *geodesic current* on S is a Γ-invariant Radon measure on $\mathcal{G}$. The space of geodesic currents on S is denoted by $\mathcal{C}(S)$. By definition, the space $\mathcal{C}(S)$ of geodesic currents is a *convex cone* in the sense that

- $t\nu \in \mathcal{C}(S)$ for all $\nu \in \mathcal{C}(S)$ and $t \geq 0$;
- $t\nu_1 + (1-t)\nu_2 \in \mathcal{C}(S)$ for all $\nu_1, \nu_2 \in \mathcal{C}(S)$ and $0 \leq t \leq 1$.

The space of meaured laminations is canonically embedded into $\mathcal{C}(S)$. Indeed, for $t\alpha \in \mathcal{WS}$, let a_1 and a_2 be the endpoint of a lift of α to $\mathbb{H}$. Let $\delta_{(a_1,a_2)}$ be the Dirac measure with atom at $(a_1, a_2) \in \mathcal{G}$. Then, we define

$$t\sum_{\gamma\in\Gamma}\gamma^*\delta_{(a_1,a_2)} \tag{4.1}$$

is a Γ-invariant Radon measure on $\mathcal{G}$. Then, we have a mapping

$$\mathcal{WS} \ni t\alpha \mapsto t\sum_{\gamma\in\Gamma}\gamma^*\delta_{(a_1,a_2)} \in \mathcal{C}(S)$$

is injective and extends continuously to $\mathcal{ML}$ (the extension is injective). The series (4.1) is also defined for non-trivial (non-simple) closed curves. Hence, any non-trivial closed curves is canonically recognized as a geodesic current on S. Indeed, the set of non-trivial closed curves is dense in $\mathcal{C}(S)$ (cf. [1]).

The interesection numbers on the set of closed curves extends continuously on $\mathcal{C}(S)$. Namely, there is a continuous function

$$i(\cdot,\cdot)\colon \mathcal{C}(S) \times \mathcal{C}(S) \to \mathbb{R}$$

which coincides with the geometric interesection number for any pair of non-trivial closed curves on S. We call the function $i(\cdot,\cdot)$ the *intersection number function* on $\mathcal{C}(S)$.

The *Liouville measure* on $\mathcal{G}$ is defined by

$$L([a,b]\times[c,d]) = \left|\log\left|\frac{(a-c)(b-d)}{(a-d)(b-c)}\right|\right|$$

for any pair of disjoint intervals in $\partial\mathbb{H}$. We can see that the measure L is Γ-invariant: $L(\gamma(E)) = L(E)$ for any measurable set $E \subset \mathcal{G}$.

4.1.2. *Bonahon's theory.* The Teichmüller space $T(S)$ of S is canonically identified with the Teichmüller space $T(\Gamma)$ of the Fuchsian group Γ as follows: Let $\mathrm{QC}(\Gamma)$ is the set of normalized quasiconformal automorphisms w on $\mathbb{H}$ comparable with Γ in the sense that for any γ, $w\circ\gamma\circ w^{-1} \in \mathrm{PSL}_2(\mathbb{R})$, where a quasiconformal automorphism w is said to be *normalized* if it fixes 0, 1 and ∞. Two quasiconformal automorphisms w_1 and w_2 are *equivalent* if $w_1 = w_2$ on $\partial\mathbb{H}$. The *Teichmüller space* $T(\Gamma)$ of Γ is defined to be the quotient space of $\mathrm{QC}(\Gamma)$ under the equivalence relation.

Let $y = (Y, f) \in T(S)$. We may take f as a quasiconformal mapping from S to Y. Let $\tilde{f}\colon \mathbb{H} \to \mathbb{H}$ be a lift of f. After post-composing an appropriate isometry on $\mathbb{H}$ to the lift, we may assume that $\tilde{f}$ is a normalized quasiconformal automorphism on $\mathbb{H}$. Then, we can see that

$$T(S) \ni y = (Y, f) \mapsto [\tilde{f}] \in T(\Gamma) \tag{4.2}$$

is bijective (cf. [5]). Especially, to any $y \in T(S)$, we can assign the boundary value of a quasiconformal mapping w_y associated by the isomorphism (4.2).

The boundary value w_y induces a homeomorphism $\tilde{w}_y$ on $\mathcal{G}$ which is equivariant under the action of Γ. The *Bonahon's embedding* of $T(S)$ to $\mathcal{C}(S)$ is defined by

$$\tilde{\Phi}_{Bo} \colon T(S) \ni y \mapsto L_y = (\tilde{w}_y)^* L \in \mathcal{C}(S).$$

We call L_y the *Liouville current* of $y \in T(S)$. Bonahon observed that $\tilde{\Phi}_{Bo}$ is proper (cf. [1]). The Liouville current satisfies the following remarkable properties:

$$i(L_y, L_y) = \pi^2 |\chi(S)| \tag{4.3}$$

$$i(L_y, \alpha) = \ell_y(\alpha) \tag{4.4}$$

for all $y \in T(S)$ and $\alpha \in \mathcal{ML} \subset \mathcal{C}(S)$.

The space $\mathcal{C}(S)$ admits a canonical $\mathbb{R}_+$-action. Let

$$\mathbf{pr} \colon \mathcal{C}(S) - \{0\} \to \mathcal{PC}(S) = (\mathcal{C}(S) - \{0\})/\mathbb{R}_+$$

be the projection. Then, the mapping

$$\Phi_{Bo} \colon T(S) \ni y \mapsto \mathbf{pr}(L_y) \in \mathcal{PC}(S)$$

is an embedding and the image is relatively compact. We call the closure of the image under Φ_{Bo} the *Bonahon's compactification* (see [1]).

The embedding Φ_{Bo} induces a homeomorphism from the Thurston compactification to the Bonahon's compactification. Indeed, Let

$$\begin{aligned} \mathcal{C}_{Th} &= \mathbf{pr}^{-1}(\overline{T(S)}^{Th}) \cup \{0\} \\ &= \{\mathfrak{a} \in \mathcal{R} \mid \mathbf{pr}(\mathfrak{a}) \in \overline{T(S)}^{Th}\} \cup \{0\}. \end{aligned}$$

The image of the Thurston embedding $\tilde{\Phi}_{Th}$ is contained in $\mathcal{C}_{Th}$. We define a mapping Ξ_H on $\mathcal{C}(S)$ to $\mathcal{R}$ by

$$\Xi_H \colon \mu \mapsto [\mathcal{S} \ni \alpha \mapsto i(\alpha, \mu)] \in \mathcal{R}$$

("H" stands for the initial letter of "Hyperbolic"). From (4.4), we have the following relation

$$\Xi_H \circ \tilde{\Phi}_{Bo}(y) = \tilde{\Phi}_{Th}(y) \tag{4.5}$$

for all $y \in T(S)$.

4.2. **Cone $\mathcal{C}_{GM}$.** Aiming for a counterpart for Bonahon's theory, we attempt to unify the extremal length geometry via intersection number in cones.

Notice that the Gardiner-Masur compactification $\overline{T(S)}^{GM}$ is contained in the projective space $\mathcal{PR}$. Let

$$\begin{aligned}\mathcal{C}_{GM} &= \mathbf{pr}^{-1}(\overline{T(S)}^{GM}) \cup \{0\} \\ &= \left\{\mathfrak{a} \in \mathcal{R} \mid \mathbf{pr}(\mathfrak{a}) \in \overline{T(S)}^{GM}\right\} \cup \{0\}.\end{aligned}$$

By definition, the set $\mathcal{C}_{GM}$ is a *cone* in the sense that $t\mathfrak{a} \in \mathcal{C}_{GM}$ for all $\mathfrak{a} \in \mathcal{C}_{GM}$. However, to the author's knowledge, it is not known whether $\mathcal{C}_{GM}$ is convex or not. Since $\mathcal{PML} \subset \partial_{GM}T(S) \subset \overline{T(S)}^{GM}$, we see

$$\mathcal{ML} \subset \mathcal{C}_{GM}.$$

Notice from the definition that the image of the embedding $\tilde{\Phi}_{GM}\colon T(S) \to \mathcal{R}$ is contained in $\mathcal{C}_{GM}$.

From the following theorem, the cone $\mathcal{C}_{GM}$ is recognized as (a subset of) the stage for developing the Thurston theory with respect to the extremal length geometry.

Theorem 4.1 ([10]). *There is a unique continuous function*

$$i(\cdot\,,\cdot)\colon \mathcal{C}_{GM} \times \mathcal{C}_{GM} \to \mathbb{R}$$

with the following properties.

(i) *For any $y \in T(S)$, the projective class of the function $\mathcal{S} \ni \alpha \mapsto i(\tilde{\Phi}_{GM}(y), \alpha)$ is exactly the image of y under the Gardiner-Masur embedding. Actually, it holds*

$$i(\tilde{\Phi}_{GM}(y), \alpha) = \mathrm{Ext}_y(\alpha)^{1/2}$$

for all $\alpha \in \mathcal{S}$.

(ii) *For $\mathfrak{a}, \mathfrak{b} \in \mathcal{C}_{GM}$, $i(\mathfrak{a}, \mathfrak{b}) = i(\mathfrak{b}, \mathfrak{a})$.*

(iii) *For $\mathfrak{a}, \mathfrak{b} \in \mathcal{C}_{GM}$ and $t, s \geq 0$, $i(t\mathfrak{a}, s\mathfrak{b}) = ts\, i(\mathfrak{a}, \mathfrak{b})$.*

(iv) *For any $y, z \in T(S)$,*

$$i(\tilde{\Phi}_{GM}(y), \tilde{\Phi}_{GM}(z)) = \exp(d_T(y, z)).$$

In particular, we have $i(\tilde{\Phi}_{GM}(y), \tilde{\Phi}_{GM}(y)) = 1$ for $y \in T(S)$.

(v) *For $F, G \in \mathcal{MF} \subset \mathcal{C}_{GM}$, the value $i(F, G)$ is equal to the geometric intersection number.*

We fix a base surface $x_0 = (S, id) \in T(S)$. Let

$$\tilde{\Psi}_{GM}(y) = e^{-d_T(x_0,y)}\tilde{\Phi}_{GM}(y)$$

for $y \in T(S)$. From (iii) and (iv) in Theorem 4.1, we can see

$$\begin{aligned}\exp(-2\langle y \mid z\rangle_{x_0}) &= \exp(d_T(x_0,y)+d_T(x_0,z)-d_T(y,z)) \\ &= \exp(d_T(x_0,y)+d_T(x_0,z))i(\tilde{\Phi}_{GM}(y),\tilde{\Phi}_{GM}(z)) \\ &= i(\tilde{\Psi}_{GM}(y),\tilde{\Psi}_{GM}(z)),\end{aligned}$$

where $\langle y \mid z\rangle_{x_0}$ is the *Gromov product* on $T(S)$ which is defined by

$$\langle y \mid z\rangle_{x_0} = \frac{1}{2}(d_T(x_0,y)+d_T(x_0,z)-d_T(y,z)).$$

We can see that $\tilde{\Psi}_{GM}\colon T(S) \to \mathcal{C}_{GM} \subset \mathcal{R}$ extends continuously on the Gardiner-Masur compactification and satisfys that $\mathbf{pr}\circ\tilde{\Psi}_{GM}$ is the "identity mapping" on $\overline{T(S)}^{GM}$ (cf. [9] and [10]). Therefore, we have the following corollary.

Corollary 4.1. *The Gromov product* $\langle\cdot\mid\cdot\rangle_{x_0}$ *extends continuously on the Gardiner-Masur compactification.*

In fact, one can check that the extension of the Gromov product satisfies

$$\langle [F] \mid [G]\rangle_{x_0} = \frac{i(F,G)}{\mathrm{Ext}_{x_0}(F)^{1/2}\mathrm{Ext}_{x_0}(F)^{1/2}}$$

for all $[F],[G] \in \mathcal{PML} \subset \partial_{GM}T(S)$ Corollary 4.1 links the analytic aspect to the topological aspect of Teichmüller space. Indeed, we can obtain an alternative approach to the characterization of the isometry group of $(T(S),d_T)$ via the Gromov product (cf. [10]).

5. Conclusion: Ressemblances

The hyperbolic length and the exremal length are important and useful geometric quantities in the Teichmüller theory. The "original" Thurston theory is accomplished with the hyperbolic geometry and the geometry of simple closed curves via the intersection number function. From Theorem 4.1, we may expect that our extremal length geometry is carried out with the intersection number function.

We give a table on the (expected) ressemblances between two geometries. Papadopoulos and Su also discussed ressemblances (cf. [13]). In the following table, we assume $y \in T(S)$ and $\alpha \in \mathcal{S}$.

Hyperbolic geometry	Extremal length geometry
Thurston's asymmetric metric d_{Th}	Teichmüller distance d_T
Length spectrum distance d_{ls}	
$\mathcal{C}(S)$	(*)
$\tilde{\Phi}_{Bo}\colon T(S) \to \mathcal{C}(S)$	(*)
$\Xi_H\colon \mathcal{C}(S) \to \mathcal{R}$	(*)
$\Xi_H \circ \tilde{\Phi}_{Bo} = \tilde{\Phi}_{Th}$	(*)
$\mathcal{C}_{Th}$	$\mathcal{C}_{GM}$ (*)
$\tilde{\Phi}_{Th}\colon T(S) \to \mathcal{C}_{Th} \subset \mathcal{R}$	$\tilde{\Phi}_{GM}\colon T(S) \to \mathcal{C}_{GM} \subset \mathcal{R}$
$d_{Th} = (\tilde{\Phi}_{Th})^* d_\infty$	$d_T = (\tilde{\Phi}_{GM})^* d_\infty = (\tilde{\Phi}_{GM})^* d_\infty^{sym}$
$d_{ls} = (\tilde{\Phi}_{Th})^* d_\infty^{sym}$	
Intersection number function $i(\cdot,\cdot)$ on cones	
L_y	(*)
$\tilde{\Phi}_{Th}(y)(\alpha) = \ell_y(\alpha)$	$\tilde{\Phi}_{GM}(y)(\alpha) = \mathrm{Ext}_y(\alpha)^{1/2}$
$i(L_y, L_y) = \pi^2 \lvert\chi(S)\rvert$	$i(\tilde{\Phi}_{GM}(y), \tilde{\Phi}_{GM}(y)) = 1$ (*)

(*). Notice that our cone $\mathcal{C}_{GM}$ seems to be *artificial.* Namely, it is possible to exist a "geometrically natural" cone containing $\mathcal{C}_{GM}$ on which the intersection number is defined. Here, by "geometrically natural", we mean that the element $\tilde{\Phi}_{GM}(y)$ could be represented as some geometric object. For instance, in the column on the hyperbolic geometry, the cone $\mathcal{C}_{Th}$ is (essentially) contained in $\mathcal{C}(S)$. Furthermore, from (4.5), the Thurston embedding $\tilde{\Phi}_{Th}$ is related to the Bonahon's embedding by

$$i(\tilde{\Phi}_{Bo}(y), \alpha) = \tilde{\Phi}_{Th}(y)(\alpha)(= \ell_y(\alpha))$$

for all $y \in T(S)$ and $\alpha \in \mathcal{S}$. We expect to find the geometric objects corresponding to the Liouville measures (geodesic currents) and Bonahon's embedding. The cone and the embedding, if exist, will be canonical stage and realization of Teichmüller space for developing the extremal length geometry on Teichmüller space.

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