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# Recent progress on Takhtajan-Zograf and Weil-Petersson metrics

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**Abstract**

We will survey recent progress on Weil-Petersson and Takhtajan-Zograf metric. After reviewing the backgrounds and the known results for those metrics, a new estimate of the asymptotic behavior of the Takhtajan-Zograf metric near the boundary of the moduli space of punctured Riemann surfaces is stated without proof.

CONTENTS



## 1 Backgrounds on Weil-Petersson and Takhtajan-Zograf metrics

$T_{g,n}$ denotes the **Teichmüller space** of Riemann surfaces of genus $g$ with $n$ marked points $(2g-2+n>0)$. Let $C_{g,n}$ be the **Teichmüller curve** over $T_{g,n}$ with the projection $\pi : C_{g,n} \to T_{g,n}$ which has $n$ sections $\mathbf{P}_1, \ldots, \mathbf{P}_n$ corresponding to $n$ marked points. Consider $\Omega^1_{C_{g,n}}$(resp. $\Omega^1_{T_{g,n}}$) the sheaf of holomorphic 1-forms on $C_{g,n}$ (resp. $T_{g,n}$). The sheaf of **relative differential forms** on $C_{g,n}$ is defined as

$$\omega_{C_{g,n}/T_{g,n}} := \Omega^1_{C_{g,n}}/\pi^*\Omega^1_{T_{g,n}}. \tag{1.1}$$

Then the **determinant line bundle** $\lambda_l$ on $T_{g,n}$ ($l \in \mathbf{N}$) is defined as

$$\lambda_l := \bigwedge^{\max} R^0\pi_*\omega_{C_{g,n}/T_{g,n}}^{\otimes l}\big((l-1)(\mathbf{P}_1 + \cdots + \mathbf{P}_n)\big). \tag{1.2}$$

For a point $s \in T_{g,n}$, $S := \pi^{-1}(s)$ is a compact Riemann surface. Set $S^0 := S - \{\mathbf{P}_1(s), \ldots, \mathbf{P}_n(s)\}$ and $P_p := \mathbf{P}_p(s)$ $(p = 1, \ldots, n)$.
Here we can see

$$R^0\pi_*\omega_{C_{g,n}/T_{g,n}}^{\otimes l}\big((l-1)(\mathbf{P}_1+\cdots+\mathbf{P}_n)\big)|_s = \Gamma(S, K_S^{\otimes l} \otimes \mathcal{O}_S(P_1+\cdots+P_n)^{\otimes(l-1)})$$

$\simeq$\{meromorphic $l$ differentials on $S$ with possibly poles of order at most $l-1$ only at the marked points\}.

Pick a basis of local holomorphic sections $\phi_1, \ldots, \phi_{d(l)}$ for $R^0\pi_*\omega_{C_{g,n}/T_{g,n}}^{\otimes l}\big((l-1)(\mathbf{P}_1 + \cdots + \mathbf{P}_n)\big)$, where

$$d(l) = \begin{cases} g & (l = 1) \\ (2l-1)(g-1) + (l-1)n & (l > 1). \end{cases}$$

$$\langle \phi_i, \phi_j \rangle := \iint_{S^0} \phi_i\, \overline{\phi_j}\, \rho_{S^0}^{-(l-1)} \quad (i, j = 1, \ldots, d(l)) \tag{1.3}$$

is called the **Petersson product**, where $\rho_{S^0}$ is the hyperbolic area element on $S^0$.

We set

$$\|\phi_1 \wedge \cdots \wedge \phi_{d(l)}\|_{L^2} := |\det(\langle \phi_i, \phi_j \rangle)|^{1/2}, \tag{1.4}$$

$$\|\phi_1 \wedge \cdots \wedge \phi_{d(l)}\|_Q := \|\phi_1 \wedge \cdots \wedge \phi_{d(l)}\|_{L^2}\, Z_{S^0}(l)^{-\frac{1}{2}} \tag{1.5}$$

($l \geq 2$. For $l = 1$, employ $Z'_{S^0}(1)$ in place of $Z_{S^0}(1) = 0$). Here, $Z_{S^0}(l)$ denotes the special value of $Z_{S^0}(\cdot)$ on $S^0$ at $l$ integer, which will be defined below. Then $\lambda_l \to T_{g,n}$ is a Hermitian holomorphic line bundle equipped with the **Quillen metric** $\|\cdot\|_Q$ (see [7]). Here

$$Z_{S^0}(s) := \prod_{\{\gamma\}} \prod_{m=1}^{\infty} \left(1 - e^{-(s+m)L(\gamma)}\right) \tag{1.6}$$

is the **Selberg Zeta function** for $S^0$, Re $(s) > 1$, where $\gamma$ runs over all oriented primitive closed geodesics on $S^0$, and $L(\gamma)$ denotes the hyperbolic length of $\gamma$. It extends meromorphically to the whole plane in $s$.

In the late 80's, we have discovered the following important formulas for the curvature forms of the determinant line bundles with respect to the Quillen metrics.

**Theorem 1.1** (Belavin-Knizhnik+Wolpert(1986), [1], [8]).

$$c_1(\lambda_l, \|\cdot\|_Q) = \frac{6l^2 - 6l + 1}{12\pi^2}\omega_{WP} \quad (n = 0).$$

**Theorem 1.2** (Takhtajan-Zograf (1988, 1991), [7] ).

$$c_1(\lambda_l, \|\cdot\|_Q) = \frac{6l^2 - 6l + 1}{12\pi^2}\omega_{WP} - \frac{1}{9}\omega_{TZ} \; (n > 0).$$

*Here, $\omega_{WP}, \omega_{TZ}$ are the Kähler forms of the Weil-Petersson, the Takhtajan-Zograf metrics respectively.*

Here remind us of the definitions of the Weil-Petersson and the Takhtajan-Zograf metrics. By the deformation theory of Kodaira-Spencer and the Hodge theory, for $[S^0] \in T_{g,n}$, we have

$$T_{[S^0]}T_{g,n} \simeq HB(S^0), \tag{1.7}$$

where $HB(S^0)$ is the space of harmonic Beltrami differentials on $S^0$.

By the Serre duality, one has

$$T^*_{[S^0]}T_{g,n} \simeq Q(S^0), \tag{1.8}$$

where $Q(S^0)$ is the space of holomorphic quadratic differentials on $S^0$ with finite the Petersson-norm, which is dual to $HB(S^0)$.

The inner product of the **Weil-Petersson metric** at $T_{[S^0]}T_{g,n}$ is defined to be

$$\langle \alpha, \beta \rangle_{WP}([S^0]) := \iint_{S^0} \alpha\overline{\beta}\; \rho_{S^0}, \tag{1.9}$$

where $\alpha, \beta$ are in $HB(S^0) \simeq T_{[S^0]}T_{g,n}$.

The inner products of the **Takhtajan-Zograf metrics** are defined to be

$$\langle \alpha, \beta \rangle_p([S^0]) := \iint_{S^0} \alpha\overline{\beta}E_p(\cdot, 2)\; \rho_{S^0}, \quad (p = 1, \ldots, n). \tag{1.10}$$

Here, $E_p(\cdot, 2)$ is the Eisenstein series associated with the $p$-th marked point with index 2. Moreover, we set

$$\langle \alpha, \beta \rangle_{TZ}([S^0]) := \sum_{p=1}^{n} \langle \alpha, \beta \rangle_p([S^0]). \tag{1.11}$$

The **Eisenstein series** associated with the $p$-th marked point with index 2 is defined to be

$$E_p(z, 2) := \sum_{A \in \Gamma_p \backslash \Gamma} \left\{ \mathrm{Im}(\sigma_p^{-1} A(z)) \right\}^2, \quad \text{for } z \in \mathbf{H}, \tag{1.12}$$

where $\mathbf{H}$ is the upper-half plane, $\Gamma$ is a uniformizing Fuchsian group for $S^0$ and $\Gamma_p$ is the parabolic subgroup associated with the $p$-th marked point, and $\sigma_p \in \mathrm{PSL}(2, \mathbf{R})$ is a normalizer. $E_p(z, 2)$ assumes the infinity at the $p$-th marked point and vanishes at the other marked points. In addition, the Eisenstein series satisfy

$$\Delta_{hyp}\, E_p(z, 2) = 2E_p(z, 2), \tag{1.13}$$

where $\Delta_{hyp}$ is the negative hyperbolic Laplacian on $S^0$. Especially $E_p(z, 2)$ is a positive subharmonic function on $S^0$.

$\mathrm{Mod}_{g,n}$ denotes the **mapping class group** of surfaces of genus $g$ with $n$ marked points. Then the **moduli space** $\mathcal{M}_{g,n}$ of Riemann surfaces of genus $g$ with $n$ marked points is described as $\mathcal{M}_{g,n} = T_{g,n}/\mathrm{Mod}_{g,n}$. $\lambda_l$ and all metrics we defined are compatible with the action of $\mathrm{Mod}_{g,n}$, thus they all naturally descend down to $\mathcal{M}_{g,n}$ as orbifold line sheaves and orbifold metrics respectively.

Let $\overline{\mathcal{M}}_{g,n}$ denote the **Deligne-Mumford compactification** of $\mathcal{M}_{g,n}$. We have known the relations of the $L^2$−cohomology of $\mathcal{M}_{g,n}$ with respect to the Weil-Petersson metric and the second cohomology of $\overline{\mathcal{M}}_{g,n}$.

**Theorem 1.3** (Saper (1993) [6] ). *For $g > 1, n = 0$,*

$$H^*_{(2)}(\mathcal{M}_g, \omega_{WP}) \simeq H^*(\overline{\mathcal{M}}_g, \mathbf{R}).$$

*Here, the left hand side is the $L^2$−cohomology with respect to the Weil-Petersson metric.*

## 2 Known results for the asymptotic behaviors of the Weil-Petersson and Takhtajan-Zograf metrics

The proof of Theorem 1.3 is based on the asymptotic behavior of the Weil-Petersson metric near the boundary of the moduli space which we will review now.

Here we set $D := \overline{\mathcal{M}}_{g,n} \backslash \mathcal{M}_{g,n}$ the compactification divisor. Now take $X_0 \in D$ a degenerate Riemann surface of genus $g$ with $n$ marked points and $k$ nodes (we regard the marked points as deleted from the surface).

Each node $q_i$ $(i = 1, 2, \ldots, k)$ has a neighborhood

$$N_i = \{(z_i, w_i) \in \mathbf{C}^2 \mid |z_i|, |w_i| < 1, z_i w_i = 0\}.$$

$X_t$ denotes the smooth surface gotten from $X_0$ after cutting and pasting $N_i$ under the relation $z_i w_i = t_i, |t_i|$ small. Then, $D$ is locally described as $\{t_1 \cdots t_k = 0\}$ (see 3. in more details).

$D$ has locally the pinching coordinate $(t, s) = (t_1, \ldots, t_k, s_{k+1}, \ldots, s_{3g-3+n})$ around $[X_0]$. Set $\alpha_i = \partial/\partial t_i, \beta_\mu = \partial/\partial s_\mu \in T_{(t,s)}(T_{g,n})$. We define the Riemannian tensors for the Weil-Petersson metric

$$g_{i\bar{j}}(t, s) := \langle \alpha_i, \alpha_j \rangle_{WP}(t, s),$$

$$g_{i\bar{\mu}}(t, s) := \langle \alpha_i, \beta_\mu \rangle_{WP}(t, s),$$

$$g_{\mu\bar{\nu}}(t, s) := \langle \beta_\mu, \beta_\nu \rangle_{WP}(t, s),$$

$(i, j = 1, 2, \ldots, k,\ \mu, \nu = k + 1, \ldots, 3g - 3 + n)$.

Furthermore, we define the Riemannian tensors for the Takhtajan-Zograf metric

$$h_{i\bar{j}}(t, s) := \langle \alpha_i, \alpha_j \rangle_{TZ}(t, s),$$

$$h_{i\bar{\mu}}(t, s) := \langle \alpha_i, \beta_\mu \rangle_{TZ}(t, s),$$

$$h_{\mu\bar{\nu}}(t, s) := \langle \beta_\mu, \beta_\nu \rangle_{TZ}(t, s),$$

$(i, j = 1, 2, \ldots, k,\ \mu, \nu = k + 1, \ldots, 3g - 3 + n)$.

The following theorem is a pioneering result for the asymptotic behavior of the Weil-Petersson metric near the boundary of the moduli space.

**Theorem 2.1** (Masur (1976), [2]). *As $t_i, s_\mu \to 0$,*

$$
\begin{aligned}
&i)\quad g_{i\bar{i}}(t,s) \approx \frac{1}{|t_i|^2(-\log|t_i|)^3} \quad \textit{for } i \le k,\\
&ii)\quad g_{i\bar{j}}(t,s) = O\left(\frac{1}{|t_i||t_j|(\log|t_i|)^3(\log|t_j|)^3}\right)\\
&\qquad \textit{for } i,j \le k, i \ne j,\\
&iii)\quad g_{i\bar{\mu}}(t,s) = O\left(\frac{1}{|t_i|(-\log|t_i|)^3}\right)\\
&\qquad \textit{for } i \le k, \mu \ge k+1,\\
&iv)\quad g_{\mu\bar{\nu}}(t,s) \longrightarrow g_{\mu\bar{\nu}}(0,0) \quad \textit{for } \mu,\nu \ge k+1.
\end{aligned}
$$

Recently, we updated Masur's result by improving Wolpert's formula for the asymptotic of the hyperbolic metric for degenerating Riemann surfaces.

**Theorem 2.2** (Obitsu and Wolpert (2008), [5]). *We can improve iv) in Theorem 2.1 as follows;*

$$
iv)'\ g_{\mu\bar{\nu}}(t,s) = g_{\mu\bar{\nu}}(0,s) + \frac{4\pi^4}{3}\sum_{i=1}^{k}(\log|t_i|)^{-2}\Big\langle \beta_\mu, (E_{i,1}+E_{i,2})\beta_\nu\Big\rangle_{WP}(0,s)
$$
$$
+\, O\Big(\sum_{i=1}^{k}(\log|t_i|)^{-3}\Big)
$$

*as $t \to 0$, for $\mu,\nu \ge k+1$.*

*Here, $E_{i,1}, E_{i,2}$ denote a pair of the Eisenstein series with index 2 associated with the i-th node of the limit surface $X_0$.*

That is, the Takhtajan-Zograf metrics have appeared from degeneration of the Weil-Petersson metric. On the other hand, we have a result for asymptotics of the Takhtajan-Zograf metric near the boundary of the moduli space. Before stating the result, we need the following definition.

**Definition 2.3.** *Let $X_0$ be a degenerate Riemann surface with n punctures $p_1, \cdots, p_n$ and m nodes $q_1, \cdots, q_m$.*

*A node $q_i$ is said to be* **adjacent to punctures (resp. a puncture** $p_j$) *if the component of $X_0 \backslash \{q_1, \cdots, q_{i-1}, q_{i+1}, \cdots, q_m\}$ containing $q_i$ also contains at least one of the $p_j$'s (resp. the puncture $p_j$). Otherwise, it is said to be non-adjacent to punctures (resp. the puncture $p_j$).*

**Theorem 2.4** (Obitsu-To-Weng (2008), [3]). *As $(t,s) \to 0$, we observe the followings:*
*i) For any , $> 0$, there exists a constant $C_{1,\varepsilon}$ such that*

$$h_{i\bar{i}}(t,s) \leq \frac{C_{1,\varepsilon}}{|t_i|^2(-\log|t_i|)^{4-\varepsilon}} \quad \textit{for } i \leq k;$$

*For any* , $> 0$, *there exists a constant* $C_{2,\varepsilon}$ *such that*

$$h_{i\bar{i}}(t,s) \geq \frac{C_{2,\varepsilon}}{|t_i|^2(-\log|t_i|)^{4+\varepsilon}} \quad \textit{for } i \leq k$$

*and the node* $q_i$ *adjacent to punctures*;

*ii)* $h_{i\bar{j}}(t,s) = O\left(\frac{1}{|t_i||t_j|(\log|t_i|)^3(\log|t_j|)^3}\right)$
*for* $i, j \leq k, i \neq j$;

*iii)* $h_{i\bar{\mu}}(t,s) = O\left(\frac{1}{|t_i|(-\log|t_i|)^3}\right)$
*for* $i \leq k, \mu \geq k+1$;

*iv)* $h_{\mu\bar{\nu}}(t,s) \longrightarrow h_{\mu\bar{\nu}}(0,0)$ *for* $\mu, \nu \geq k+1$.

# 3 Degenerate families of punctured Riemann surfaces and A test Eisenstein series

First of all, let us review the construction of degenerating punctured hyperbolic surfaces. We recall the construction of the plumbing family (see 2 [5] ). Considerations begin with the *plumbing variety* $\mathcal{V} = \{(z,w,t) \mid zw = t,\ |z|,|w|,|t| < 1\}$. The defining function $zw - t$ has differential $z\,dw + w\,dz - dt$. Consequences are that $\mathcal{V}$ is a smooth variety, $(z,w)$ are global coordinates, while $(z,t)$ and $(w,t)$ are not. Consider the projection $\Pi : \mathcal{V} \to D$ onto the $t$-unit disc. The projection $\Pi$ is a submersion, except at $(z,w) = (0,0)$; we consider $\Pi : \mathcal{V} \to D$ as a (degenerate) family of open Riemann surfaces. The $t$-fiber, $t \neq 0$, is the hyperbola germ $zw = t$ or equivalently the annulus $\{|t| < |z| < 1, w = t/z\} = \{|t| < |w| < 1, z = t/w\}$. The 0-fiber is the intersection of the unit ball with the union of the coordinate axes in $\mathbb{C}^2$; on removing the origin the union becomes $\{0 < |z| < 1\} \cup \{0 < |w| < 1\}$. Each fiber of $\mathcal{V}_0 = \mathcal{V} - \{0\} \to D$ has a complete hyperbolic metric.

Consider $X_0$ a finite union of hyperbolic surfaces with cusps. A plumbing family is the fiberwise gluing of the complement of cusp neighborhoods in $X_0$ and the plumbing variety $\mathcal{V} = \{(z,w,t) \mid zw = t, |z|, |w|, |t| < 1\}$. For a positive constant $c_* < 1$ and initial surface $X_0$, with puncture $p$ with cusp coordinate $z$ and puncture $q$ with cusp coordinate $w$, we construct a family $\{X_t\}$. For $|t| < c_*^4$ the resulting surface $X_t$ will be independent of $c_*$;

the constant $c_*$ will serve to specify the overlap of coordinate charts and to define a *collar* in each $X_t$.

We first describe the gluing of fibers. For $|t| < c_*^4$, remove from $X_0$ the punctured discs $\{0 < |z| \le |t|/c_*\}$ about $p$ and $\{0 < |w| \le |t|/c_*\}$ about $q$ to obtain a surface $X^*_{t/c_*}$. For $t \neq 0$, form an identification space $X_t$, by identifying the annulus $\{|t|/c_* < |z| < c_*\} \subset X^*_{t/c_*}$ with the annulus $\{|t|/c_* < |w| < c_*\} \subset X^*_{t/c_*}$ by the rule $zw = t$. The resulting surface $X_t$ is the *plumbing* for the prescribed value of $t$. We note for $|t| < |t'|$ that there is an inclusion of $X^*_{t'/c_*}$ in $X^*_{t/c_*}$; the inclusion maps provide a way to compare structures on the surfaces. The inclusion maps are a basic feature of the plumbing construction. We next describe the plumbing family. Consider the variety $\mathcal{V}_{c_*} = \{(z,w,t) \mid zw = t, |z|, |w| < c_*, |t| < c_*^4\}$ and the disc $D_{c_*} = \{|t| < c_*^4\}$. The complex manifolds $M = X^*_{t/c_*} \times D_{c_*}$ and $\mathcal{V}_{c_*}$ have holomorphic projections to the disc $D_{c_*}$. The variables $z, w$ denote prescribed coordinates on $X^*_{t/c_*}$ and on $\mathcal{V}_{c_*}$. There are holomorphic maps of subsets of $M$ to $\mathcal{V}_{c_*}$, commuting with the projections to $D_{c_*}$, as follows

$$(z,t) \xrightarrow{\hat{F}} (z, t/z, t) \text{ and } (w,t) \xrightarrow{\hat{G}} (w, t/w, t).$$

The identification space $\mathcal{F} = M \cup \mathcal{V}_{c_*}/\{\hat{F}, \hat{G} \text{ equivalence}\}$ is the *plumbing family* $\{X_t\}$ with projection to $D_{c_*}$ (an analytic fiber space of Riemann surfaces in the sense of Kodaira. For $0 < |t| < c_*^4$, the $t$-fiber of $\mathcal{F}$ is the surface $X_t$ constructed by overlapping annuli $N_t$.

We set two anului

$$\Omega_t^1 := \left\{ z \in \mathbf{C} \;\middle|\; \frac{|t|}{e^{a^0} c^*} < |z| < e^{a^0} c^* \right\} \quad \text{for } |t| < (c^*)^4, \tag{3.1}$$

$$\Omega_t^2 := \left\{ w \in \mathbf{C} \;\middle|\; \frac{|t|}{e^{a^0} c^*} < |w| < e^{a^0} c^* \right\} \quad \text{for } |t| < (c^*)^4. \tag{3.2}$$

Here $0 < c^* < 1, a_0 < 0$ are the constants in [5].

When $t \neq 0$, on can identify as an annulus via coordinate projections as

$$N_t \longleftrightarrow \Omega_t^1 \longleftrightarrow \Omega_t^2. \tag{3.3}$$

And we may write $N_t = N_t^1 \cup N_t^2$, where

$$N_t^1 = \{ z \in \mathbf{C} \mid |t|^{\frac{1}{2}} \le |z| < e^{a^0} c^* \}, N_t^2 = \{ w \in \mathbf{C} \mid |t|^{\frac{1}{2}} \le |w| < e^{a^0} c^* \}. \tag{3.4}$$

For $t = 0$, define the cusp neighborhood

$$N_0 := \Omega_0^1 \cup \Omega_0^2. \tag{3.5}$$

In another word, we may consider that $\Omega_t^1$ embed into $X_t$ holomorphically for $t, z$. (See 2 in [5])

Here, remember the test function which is defined in [3]. For $t \neq 0$ one defines for $z \in \Omega_t^1$,

$$E_t^*(z) := \frac{-\pi}{\log|t| \sin\left(\frac{\pi \log|z|}{\log|t|}\right)}, \quad \rho_t^*(z) := \frac{\pi^2}{|z|^2 \log^2|t| \sin^2\left(\frac{\pi \log|z|}{\log|t|}\right)},$$

for $t = 0, z \in \Omega_t^1$,

$$E_0^*(z) := \frac{-1}{\log|z|}, \quad \rho_0^*(z) := \frac{1}{|z|^2 \log^2|z|}.$$

It is easy to see that for $t \neq 0$, $E_t^*, \rho_t^*$ have similar expressions for $w$ in $\Omega_t^2$ via the rule $zw = t$. Thus, $E_t^*, \rho_t^*$ can be considered as functions on the manifolds $N_t$ for $t \neq 0$. And one defines for $w \in \Omega_0^2$, $E_0^*(w), \rho_0^*(w)$ as the same expression as $E_0^*(z), \rho_0^*(z)$. Furthermore, we can easily observe that

$$\rho_0^* \leq \rho_t^* \quad \text{on } N_t \quad \text{for } |t| < (c^*)^4. \tag{3.6}$$

Masur showed in (6.5) [2] that there exists a positive constant $K$ such that

$$\rho_t^* \leq K\rho_0^* \quad \text{on } N_t \quad \text{for } |t| < (c^*)^4. \tag{3.7}$$

From now, we always assume that the smooth surfaces $X_t$ have at least one punctures. We are ready to consider a function

$$\varphi_t := \frac{E_t}{E_t^*}, \quad \text{on } N_t, \quad \text{for } |t| < (c^*)^4,$$

where $E_t$ is the intrinsic Eisenstein series on a punctured hyperbolic surface $X_t$ associated with a puncture.

We have already seen in the proof of Proposition 4.2.2 in [3] that on $\Omega_t^1$,

$$\Delta E_t(z) = 2\rho_t(z) E_t(z), \tag{3.8}$$

$$\Delta E_t^*(z) = \left(1 + \cos^2\left(\frac{\pi \log|z|}{\log|t|}\right)\right) \rho_t^*(z) E_t^*(z), \tag{3.9}$$

where $\Delta := 4\frac{\partial^2}{\partial z \partial \bar{z}}$, $\rho_t(z)$ is the intrinsic hyperbolic area element on $X_t$, and $\rho_t^*(z)$ is the restriction to $\Omega_t^1$ of the complete hyperbolic metric $r(z)|dz|^2$ of an annulus $\{z \in \mathbf{C} \mid |t| < |z| < 1\}$. It should be noted that $\rho_t^*(z)$ on $\Omega_t^1$ is strictly smaller than the complete hyperbolic metric of $\Omega_t^1$. Now a straightforward calculation leads the following proposition (see [4] for the proof).

**Proposition 3.1.** *The function $\varphi_t(z)$ satisfies the following equation on $\Omega_t^1$*

$$-\Delta\varphi_t(z) + \frac{\pi}{\log|t|}\cot\Big(\frac{\pi\log|z|}{\log|t|}\Big)\Big(\frac{2}{z}\frac{\partial\varphi_t(z)}{\partial\bar z} + \frac{2}{\bar z}\frac{\partial\varphi_t(z)}{\partial z}\Big)$$

$$+\Big\{2\rho_t(z) - \Big(1+\cos^2\Big(\frac{\pi\log|z|}{\log|t|}\Big)\Big)\rho_t^*(z)\Big\}\varphi_t(z) = 0.$$

We need the following result which is a special case of [5] Theorem 1.

**Theorem 3.2.** *On $N_t$, $\rho_t$ has the expansion for $t\to 0$,*

$$\rho_t = \rho_t^*\Big(1 + \frac{4\pi^4}{3}\big(E_{t,1}^\dagger + E_{t,2}^\dagger\big)\frac{1}{(\log|t|)^2} + Q(t)\Big),$$

*where $Q(t)$ has the estimate*

$$Q(t) = O\Big(\frac{1}{(\log|t|)^3}\Big) \quad \text{for } t\to 0.$$

*The function $E_{t,1}^\dagger, E_{t,2}^\dagger$ is the modified Eisenstein series. The O-term refers to the intrinsic $C^1$-norm of a function on $X_t$. The bounds depend on the choice of $c^*, a_0$ and a lower bound for the injectivity radius for the complement of the cusp regions in $X_0$.*

The functions $E_{t,1}^\dagger, E_{t,2}^\dagger$ are constructed as follows (see Definition 1 in [5]). First, consider the case where the pinching curve is non-dividing. Now we may assume that for $t=0$, our coordinates $z, w$ are so-called the *standard* coordinate (see Remark-Definition 2.1.2 in [3]). Take the two Eisenstein series $E_{0,1}, E_{0,2}$ on $X_0$ associated with the node. Set $E_{0,1}^\sharp = E_{0,1} - (\log|z|)^2$ on $\Omega_0^1$, $E_{0,1}^\sharp = E_{0,1}$ otherwise. $E_{0,2}^\sharp = E_{0,2} - (\log|w|)^2$ on $\Omega_0^2$, $E_{0,2}^\sharp = E_{0,2}$ otherwise. Set $E_{t,1}^\dagger = E_{0,1}^\sharp(z) + E_{0,1}^\sharp(\frac{t}{z})$ on $N_t$. Similarly set $E_{t,2}^\dagger = E_{0,2}^\sharp(w) + E_{0,2}^\sharp(\frac{t}{w})$ on $N_t$. These functions are smooth, bounded and strictly positive on $N_t$ for $|t| < (c^*)^4$. In the dividing case, we consider $E_{0,1}$ be just 0 on the other component, follow the construction in the non-dividing case. It should be noted that $E_{0,1}^\sharp, E_{0,2}^\sharp$ on $N_t$ is independent of $t$. Furthermore, we should remark that in the construction of [5], $E_{0,1}^\sharp, E_{0,2}^\sharp$ are modified except for the factor $(\log|z|)^2$, $(\log|w|)^2$ just on $\{e^{a_0}c^* < |z| < c^*\} \simeq \{\frac{|t|}{c^*} < |w| < \frac{|t|}{e^{a_0}c^*}\}$ and $\{\frac{|t|}{c^*} < |z| < \frac{|t|}{e^{a_0}c^*}\} \simeq \{e^{a_0}c^* < |w| < c^*\}$ so that the modified function be smooth, thus in our case, $E_{0,1}^\sharp, E_{0,2}^\sharp$ is exactly $E_{0,1}, E_{0,2}$ on $X_0$ except for the factor $(\log|z|)^2, (\log|w|)^2$ respectively.

**Remark 3.3.** *As mentioned before, $\rho_t^*$ is strictly smaller than the complete hyperbolic metric of $\Omega_t^1$. Thus, the claim of Theorem 3.2 does not contradict the implication of the classical Schwarz lemma.*

## 4 A new estimate for the Takhtajan-Zograf metric

We are ready to state a new estimate of the intrinsic Eisenstein series which is an improvement of Proposition 4.2.2 in [3]. Detailed proofs will appear in [4]. Here we quote a lemma ( Lemma 1[5]).

**Lemma 4.1.** *There exist a positive constant $C^*$ such that*

$$E_0 \leq C^* E_0^* \quad on\, \Omega_0^1.$$

We are now in a position to generalize Lemma 4.1 for any $t$.

**Proposition 4.2.** *Assume that in the family $\{X_t\}$, $N_0$ has the intersection with the component attached to the cusp where the Eisenstein series $E_0$ has a singularity. Then there exists a positive constant $C, C'$ independent of $t$ such that*

$$E_t \leq C E_t^* \quad on\, N_t \quad for\, |t|\, sufficiently\, small, \tag{4.1}$$

$$E_t \leq C' E_0^* \quad on\, N_t \quad for\, |t|\, sufficiently\, small. \tag{4.2}$$

Applying Proposition 4.2, we can improve $(i)$ of Theorem 1 in [3].

**Theorem 4.3.** *For the simplicity of description, we assume that the degenerating family of a punctured hyperbolic surface $X_t$ has only one pinching curve. Then there exists a positive constant $C$ such that the Takhtajan-Zograf inner product has the estimate*

$$g^{TZ}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) \leq \frac{C}{|t|^2(\log|t|)^4} \quad for\, t \to 0.$$

*That is, we have removed , in $(i)$ of Theorem 1 in [3].*

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