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Properties of q-Gaussian measures related to the isoperimetric and concentration profiles

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1 Introduction

This note is devoted to properties related to the isoperimetric profile and the concentration profile of a non-Gaussian probability measure, in particular q-Gaussian measures, on $\mathbb{R}^n$. We always assume that any measure and any set are Borel.

On one hand, the isoperimetric profile of a probability measure μ on $\mathbb{R}^n$ describes the relation between the volume $\mu(A)$ and the *boundary measure* $\mu^+(A) := \underline{\lim}_{\varepsilon\downarrow 0}\mu[A^\varepsilon\backslash A]/\varepsilon$ of $A\subset\mathbb{R}^n$, where $A^\varepsilon := \{x\in\mathbb{R}^n \mid \inf_{a\in A}|x-a|<\varepsilon\}$ denotes the *ε-open neighborhood* of A with respect to the standard Euclidean metric $|\cdot|$. To be precise, the *isoperimetric profile* $I[\mu]$ is the function on $[0,1]$ defined by

$$I[\mu](a) := \inf\left\{\mu^+(A) \mid A\subset\mathbb{R}^n \text{ with } \mu(A)=a\right\}.$$

We sometimes consider $I[\mu]$ only on $[0,1/2]$ since a given set and its complement may have the same boundary measure under suitable conditions.

On the other hand, the concentration profile of a probability measure μ on $\mathbb{R}^n$ estimates the volume of the r-open neighborhood of sets having measure $1/2$. To be precise, the *concentration profile* $C[\mu]$ is the function on $[0,\infty)$ defined as

$$C[\mu](r) := \sup\{1-\mu(A^r) \mid A\subset\mathbb{R}^n \text{ with } \mu(A)\geq 1/2\}.$$

Note that the both profiles can be defined for a probability measure on a metric space since the definition of the both profiles depend on only a probability measure and a distance function, where we do not take advantage of the Euclidean structure.

It is usually difficult to obtain the isoperimetric profile and the concentration profile of a given probability measure, however the both profiles of the Gaussian measure are known. Here the *Gaussian measure* γ_n is an absolutely continuous measure on $\mathbb{R}^n$ with density

$$\frac{d\gamma_n}{dx}(x) = (2\pi)^{-n/2}\exp\left(-\frac{|x|^2}{2}\right)$$

with respect to the Lebesgue measure.

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Theorem 1.1 ([3, Theorem 3.1], [11, Corollary 1]) *It holds for any* $a \in [0,1]$ *that*

$$I[\gamma_n](a) = I[\gamma_1](a) = G'(\Phi(a)),$$

where Φ *is the inverse function of* G *which is defined for* $r \in \mathbb{R}$ *by*

$$G(r) := \int_{-\infty}^{r} (2\pi)^{-1/2} \exp\left(-\frac{s^2}{2}\right) ds = \gamma_1(-\infty, r].$$

Theorem 1.1 easily induces

$$C[\gamma_n](r) = C[\gamma_1](r) = 1 - G(r) = \int_{r}^{\infty} (2\pi)^{-1/2} \exp\left(-\frac{s^2}{2}\right) ds \leq \exp\left(-\frac{r^2}{2}\right).$$

Since the isoperimetric profile and the concentration profile of γ_n are dimension free, we denote $I := I[\gamma_n]$ and $C := C[\gamma_n]$.

We say that a probability measure μ verifies *a Gaussian isoperimetric inequality* if there exists a positive constant c such that

$$I[\mu](a) \geq cI(a)$$

holds for any $a \in [0,1]$. Similarly, we say that a probability measure verifies a *Gaussian concentration inequality* if there exist positive constants c and λ such that

$$C[\mu](r) \leq c\exp\left(-\lambda r^2/2\right)$$

holds for any $r \geq 0$. If a probability measure verifies a Gaussian isoperimetric inequality, then the probability measure also verifies a Gaussian concentration inequality, which follows from Proposition 1.2 below and the fact that there exists a positive constant c such that

$$I(a) \geq ca\sqrt{\log 1/a}$$

holds for $a \in [0,1/2]$.

Proposition 1.2 ([6, Proposition 1.7]) *For a continuous function* $\sigma : [\log 2, \infty) \to [0,\infty)$, *let* α *be the inverse function of*

$$r \mapsto \int_{\log 2}^{r} \frac{1}{\tilde{\sigma}(s)} ds, \quad \tilde{\sigma}(s) = \begin{cases} \sigma(s) & \text{if } s \geq \log 2, \\ \sigma\left(-\log(1-e^{-s})\right) & \text{if } s < \log 2. \end{cases}$$

If a probability measure μ *on* $\mathbb{R}^n$ *verifies*

$$I[\mu](a) \geq a\sigma(\log 1/a)$$

on $[0,1/2]$, *then it holds for* $r \geq 0$ *that*

$$C[\mu](r) \leq \exp(-\alpha(r)).$$

More generally, we have the following implication from an isoperimetric inequality to a concentration inequality since the difference of the volumes between a set and its r-open neighborhood is roughly considered as an integral of the boundary measures of the t-open neighborhoods of the given set on $t \in (0, r)$.

Proposition 1.3 ([4, Corollary 2.2]) *Let μ be an absolutely continuous probability measure on $\mathbb{R}^n$ with respect to the Lebesgue measure. If there exists a strictly increasing, differentiable function v from an interval of $\mathbb{R}$ to $[0, 1]$ such that $I[\mu] \geq v' \circ u$ holds on $[0, 1]$, where u is the inverse function of v, then it holds for every $r > 0$ that*

$$C[\mu](r) \leq 1 - v(u(1/2) + r).$$

We thus find that a probability measure verifies a Gaussian concentration inequality if the probability measure verifies a Gaussian isoperimetric inequality.

There are several criteria for a probability measure to verify a Gaussian isoperimetric inequality. For example, given an absolutely continuous logarithmic concave probability measure μ on $\mathbb{R}^n$ with respect to the Lebesgue measure, namely there exists a convex function $V : \mathbb{R}^n \to (-\infty, \infty]$ such that $d\mu(x)/dx = \exp(-V(x))$ holds on $x \in \mathbb{R}^n$, the following equivalent condition is known.

Theorem 1.4 ([1, Theorem 1.3]) *For an absolutely continuous logarithmic concave probability measure μ on $\mathbb{R}^n$ with respect to the Lebesgue measure, the followings are equivalent to each other:*

- *μ verifies a Gaussian isoperimetric inequality.*
- *μ verifies a logarithmic Sobolev inequality, that is there exists a positive constant c such that*

  $$\int_{\mathbb{R}^n} f^2 \log(f^2) d\mu - \int_{\mathbb{R}^n} f^2 d\mu \log\left(\int_{\mathbb{R}^n} f^2 d\mu\right) \leq c \int_{\mathbb{R}^n} |\nabla f|^2 d\mu$$

  holds for every locally Lipschitz function f on $\mathbb{R}^n$ with its distributional gradient ∇f.
- *μ verifies a Herbst necessary condition, that is there exists a positive constant ε satisfying*

  $$\int_{\mathbb{R}^n} \exp(\varepsilon |x|^2) d\mu(x) < \infty.$$

Moreover, for an absolutely continuous probability measure μ on $\mathbb{R}^n$ with respect to the Lebesgue measure, if the Hessian of $-\log(d\mu/dx)$ is uniformly bounded below by some $K \in \mathbb{R}$, then verifying a Gaussian isoperimetric inequality is also equivalent to verifying a Gaussian concentration inequality. This was proved for a more general probability measure on a Riemannian manifold (see [7, Theorems 1.1, 1.2]), where the lower boundedness of the ∞-Ricci curvature is used instead of the uniform logarithmic concavity of a probability measure.

Definition 1.5 Let (M, g) be an n-dimensional complete connected Riemannian manifold without boundary and fix an arbitrary measure

$$\omega = e^{-f} \operatorname{vol}_g, \qquad f \in C^\infty(M),$$

where vol_g denotes the Riemannian volume measure of (M, g). Given $N \in (-\infty, 0) \cup [n, \infty]$ and $K \in \mathbb{R}$, we define the *N-Ricci curvature* of ω by

$$\operatorname{Ric}_N^\omega := \begin{cases} \operatorname{Ric} + \operatorname{Hess} f & \text{if } N = \infty, \\ \operatorname{Ric} + \operatorname{Hess} f - \dfrac{Df \otimes Df}{N - n} & \text{if } N \in (-\infty, 0) \cup (n, \infty), \\ \operatorname{Ric} + \operatorname{Hess} f - \infty \cdot (Df \otimes Df) & \text{if } N = n, \end{cases}$$

where by convention $\infty \cdot 0 = 0$.

We remark that the N-Ricci curvature is originally defined only for $N \in [n, \infty]$ and if $\operatorname{Ric}_N^\omega(v, v) \geq Kg(v, v)$ holds for every tangent vector v to M and for some $K \in \mathbb{R}$, $N \in [n, \infty)$ then (M, ω) behaves like a Riemannian manifold with dimension bounded above by N and Ricci curvature bounded below by K. We refer to [5],[10] and references therein for the details, and to [9] for the case of $N \in (-\infty, 0)$.

2 Probability measure on an admissible quadruple

It is known that if the ∞-Ricci curvature of ω is bounded below by some $K > 0$, then ω verifies a Gaussian isoperimetric inequality and hence a Gaussian concentration inequality (for instance, see [8, Theorem 5]). It is then natural to ask what kind of an isoperimetric inequality and a concentration inequality hold for a non-Gaussian probability measure whose ∞-Ricci curvature is not bounded from below. Moreover, under a suitable condition, are the two inequalities equivalent to each other? To discuss this, we deal with the following condition (see [9, Definition 4.3], where the condition is slightly different).

Definition 2.1 We say that a quadruple $(M, \omega, \varphi, \Psi)$ is *admissible* if all the following conditions hold:

- M is an n-dimensional complete connected Riemannian manifold with $n \geq 2$.
- φ is a non-decreasing, positive, continuous function on $(0, \infty)$ such that

$$\theta_\varphi := \sup_{s>0} \left\{ \frac{s}{\varphi(s)} \cdot \varlimsup_{\varepsilon \downarrow 0} \frac{\varphi(s+\varepsilon) - \varphi(s)}{\varepsilon} \right\} \in \left(0, \frac{n+1}{n}\right]$$

  and $\theta_\varphi \neq 1, 3/2$ with $\varphi(1) = 1$.

- Ψ is a function on M such that

$$M_\Psi^\varphi := \left\{ x \in M \,\middle|\, \Psi(x) \in \left(-\int_1^\infty \frac{1}{\varphi(s)} ds, -\int_1^0 \frac{1}{\varphi(s)} ds \right) \right\} \neq \emptyset$$

and $\Psi > -L_{\theta_\varphi}$ hold, where we set

$$L_{\theta_\varphi} := \begin{cases} (\theta_\varphi - 1)^{-1} & \text{if } \theta_\varphi > 1, \\ \infty & \text{if } \theta_\varphi \leq 1. \end{cases}$$

- ω is a positive measure on M satisfying $\mathrm{Ric}_N^\omega(v, v) \geq 0$ for $N = (\theta_\varphi - 1)^{-1}$ and for every tangent vector v to M_Ψ^φ.

Note that if φ is differentiable, then θ_φ is the upper bound of the differentiable coefficient of φ. We denote by δ_φ the quantity corresponding to the lower bound of the differentiable coefficient of φ, that is,

$$\delta_\varphi := \inf_{s>0} \left\{ \frac{s}{\varphi(s)} \cdot \overline{\lim_{\varepsilon \downarrow 0}} \frac{\varphi(s+\varepsilon) - \varphi(s)}{\varepsilon} \right\}.$$

We also define the *φ-exponential function* by

$$\exp_\varphi(\tau) := \sup \left\{ t > 0 \,\middle|\, \int_1^t \frac{1}{\varphi(s)} ds \leq \tau \right\},$$

where we set $\exp_\varphi(\tau) := 0$ for $\tau \leq \int_1^0 1/\varphi(s) ds$ by convention. Take for example, if $\varphi_q(s) = s^q$ with $q \neq 1$, then we have

$$\exp_q(\tau) := \exp_{\varphi_q}(\tau) = (1 + (1-q)\tau)_+^{1/(1-q)},$$

where we set $[\tau]_+ := \max\{\tau, 0\}$ and by convention $0^a := \infty$ for $a < 0$. Since $\exp_q$ recovers the usual exponential function when $q \to 1$, we set $\exp_1(\tau) := \exp(\tau)$.

We remark that if Ψ is K-convex for some $K > 0$ on M_Ψ^φ, then we may assume that the measure $\exp_\varphi(-\Psi)\omega$ on an admissible quadruple $(M, \omega, \varphi, \Psi)$ is a probability measure without loss of generality (see [9, Lemma 4.5]). In this case, the probability measure $\exp_\varphi(-\Psi)\omega$ verifies a non-Gaussian concentration inequality. Here the K-convexity of a function is roughly equivalent to that the Hessian of a function along any geodesic is bounded below by K (see [9, Definition 4.1] for the precise definition).

Proposition 2.2 ([9, Theorem 7.9]) *For an admissible quadruple $(M, \omega, \varphi, \Psi)$, we set $\mu := \exp_\varphi(-\Psi)\omega$ and $,_0 := \max\{1, \|\exp_\varphi(-\Psi)\|_\infty\}$. Suppose the K-convexity of Ψ for some $K > 0$ and $\mu[M] = 1$.*

(i) *If $\theta_\varphi < 1$ and $\delta_\varphi > 0$, then there exists a positive constant c_1 depending only on θ_φ and δ_φ such that we have for any $r > 0$*

$$C[\mu](r) \leq c_1 / \exp_{\delta_\varphi}\left(\frac{K}{4} ,_0^{\delta_\varphi - 1} r^2 \right).$$

(ii) *If $\theta_\varphi \in (1, 3/2)$, $\delta_\varphi > 3(\theta_\varphi - 1)$ and if $\omega[M] < \infty$, then there exist positive constants c_2, c_3 depending only on θ_φ and δ_φ such that we have for any $r > 0$*

$$C[\mu](r) \leq c_2 \exp_{2\theta_\varphi - \delta_\varphi}\left(-c_3 \frac{K}{2} ,_0^{\delta_\varphi - \theta_\varphi} \omega[M]^{1-\theta_\varphi} r^2 \right).$$

Moreover, when $\varphi(s) = s^q$ *and* $\theta_\varphi = \delta_\varphi = q \to 1$*, the two inequalities above recover a Gaussian concentration inequality.*

A fundamental and important example of an admissible quadruple is $\mathbb{R}^n$ $(n \geq 2)$ equipped with the Lebesgue measure and $\varphi_q(s) = s^q$ with $q \in (0, (n+1)/n]$ and $q \neq 1, 3/2$, $\Psi(x) = |x|^2/2$. In this case, there exists a constant $c(n,q)$ such that $1 + (1-q)c(n,q) > 0$ and

$$\int_{\mathbb{R}^n} \exp_q\left(-\frac{|x|^2}{2} + c(n,q)\right) dx = 1$$

(see [12] and Section 3 below for the explicit value of $c(n,q)$). In addition,

$$B_q^n := \left\{ x \in \mathbb{R}^n \;\middle|\; \exp_q\left(-\frac{|x|^2}{2} + c(n,q)\right) > 0 \right\}$$

contains the origin and is bounded (resp. unbounded) if $q < 1$ (resp. $q > 1$). An absolutely continuous probability measure γ_n^q on $\mathbb{R}^n$ with the density

$$\frac{d\gamma_n^q}{dx} = \exp_q\left(-\frac{|x|^2}{2} + c(n,q)\right)$$

with respect to the Lebesgue measure is called the *q-Gaussian measure.* According to [9, Theorem 5.7], the q-Gaussian measure can be regarded as an extremal element among all the probability measures $\exp_\varphi(-\Psi)\omega$ on an admissible quadruple $(M, \omega, \varphi, \Psi)$ as well as the Gaussian measure among all the probability measures on a Riemannian manifold whose ∞-Ricci curvature is bounded from below.

In this way, it turns out that a probability measure $\exp_\varphi(-\Psi)\omega$ on an admissible quadruple $(M, \omega, \varphi, \Psi)$ with certain conditions verifies a non-Gaussian isoperimetric inequality characterized by $\exp_{q(\varphi)}$, where $q(\varphi)$ depends on θ_φ and δ_φ. In particular, if $\varphi(s) = s^q$, then $q(\varphi) = q$ holds. However, as far as the author knows, the isoperimetric inequality for such a probability measure is not available in the literature, even for the case of the q-Gaussian measure.

3 Properties of φ-Gaussian measure

In this section, we provide some properties of the q-Gaussian measure, which are related to the concentration profile and may be useful to investigate the isoperimetric profile. We first discuss the logarithmic concavity of the q-Gaussian measure.

Proposition 3.1 *For any* $n \in \mathbb{N}$ *and any* $q \in (0, (n+1)/n]$ *with* $q \neq 3/2$*, define the function* V_q *on the open set*

$$B_q^n = \left\{ x \in \mathbb{R}^n \;\middle|\; \exp_q\left(-\frac{|x|^2}{2} + c(n,q)\right) > 0 \right\}$$

by

$$V_q(x) := -\log\left(\frac{d\gamma_n^q(x)}{dx}\right) = -\log\left(\exp_q\left(-\frac{|x|^2}{2} + c(n,q)\right)\right).$$

We moreover set $\lambda_q(n) := 1 + (1-q)c(n,q) > 0$. *Then for the smallest eigenvalue* $\lambda(x)$ *of the Hessian matrix of* V_q *at* $x \in B_q^n$ *satisfies*

$$\lambda(x) \geq \begin{cases} \dfrac{1}{\lambda_q(n)} & \text{if } q \leq 1, \\ -\dfrac{1}{8\lambda_q(n)} & \text{if } q > 1. \end{cases} \tag{3.1}$$

Proof. Consider the function on B_q^n of the form

$$f_q(x) := 1 + (1-q)\left(-\frac{|x|^2}{2} + c(n,q)\right) > 0.$$

We compute $f_q(0) = \lambda_q(n)$ and $\nabla f_q(x) = -(1-q)x$. It follows from the relation $V_q = -\log(f_q)/(1-q)$ that

$$\nabla V_q(x) = x/f_q(x),$$

moreover that the (i,j)-component of the Hessian matrix of V_q at x is given by

$$(1-q)\frac{x_i x_j}{f_q(x)^2} + \frac{\delta_{ij}}{f_q(x)},$$

where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$. It is easy to check that all the eigenvalue of $(H_{ij}(0))_{1\leq i,j\leq n}$ are $1/f_q(0) = 1/\lambda_q(n)$. In the case of $x \neq 0$, let $\{v_k\}_{k=1}^n$ be an orthogonal basis of $\mathbb{R}^n$ with $v_1 = x/|x|$. Then, for $k = 1, \ldots, n$, v_k is the eigenvector of $(H_{ij}(x))_{1\leq i,j\leq n}$ whose eigenvalue is

$$(1-q)\frac{|x|^2\delta_{1k}}{f_q(x)^2} + \frac{1}{f_q(x)}. \tag{3.2}$$

In the case of $q \leq 1$, it follows from $f_q \in (0, \lambda_q(n)]$ that

$$(1-q)\frac{|x|^2}{f_q(x)^2} + \frac{1}{f_q(x)} \geq \frac{1}{f_q(x)} \geq \frac{1}{\lambda_q(n)}.$$

For $q > 1$, we have $f_q \in [\lambda_q(n), \infty)$ and

$$\frac{1}{f_q(x)} \geq (1-q)\frac{|x|^2}{f_q(x)^2} + \frac{1}{f_q(x)} = \frac{\lambda_q(n) + (1-q)|x|^2/2}{(\lambda_q(n) - (1-q)|x|^2/2)^2} \geq -\frac{1}{8\lambda_q(n)}.$$

This completes the proof of the proposition. □

Remark 3.2 (1) Note that $\lambda_q(n) \to \lambda_1(n) = 1$ as $q \to 1$, and $\lambda(x) = \lambda_1(n) = 1$ on $\mathbb{R}^n$. On one hand, (3.1) recovers $\lambda(x) \geq 1$ as $q \nearrow 1$. On the other hand, when $q \searrow 1$, (3.1) does not recovers $\lambda(x) \geq 1$, however (3.2) recovers $\lambda(x) = 1$.
(2) Given any $q \in (0, (n+1)/n]$ with $q \neq 1, 3/2$, let $N_q \in (-\infty, 0) \cup (n, \infty)$ satisfy $1 - q \geq 1/(N_q - n)$. It then holds for any $v \in \mathbb{R}^n$ and $x \in B^n_q$ that

$$\begin{aligned}\operatorname{Hess} V_q(x)(v,v) - \frac{DV_q(x) \otimes DV_q(x)(v,v)}{N_q - n} &= (1-q)\frac{\langle v, x\rangle^2}{f_q(x)^2} + \frac{|v|^2}{f_q(x)} - \frac{\langle v, x\rangle^2}{(N_q - n) f_q(x)^2} \\ &\geq \frac{|v|^2}{f_q(x)}.\end{aligned}$$

This implies that, for $q > 1$ (hence N_q is negative), the N_q-Ricci curvature of γ^q_n on $\mathbb{R}^n$ equipped with the standard Euclidean metric is non-negative on the whole of $\mathbb{R}^n$, however little is known concerning a measure having the non-negative N-Ricci curvature for some negative N. For example, although a Poincaré type inequalities for γ^q_n are proved in [2], the condition $\omega(M) < \infty$ in Proposition 2.2(ii) does not hold for $\mathbb{R}^n$ equipped with the Lebesgue measure and then γ^n_q may not verify a concentration inequality in terms of the q-exponential function.

On the other hand, for $q < 1$, the N-Ricci curvature of γ^q_n on $\mathbb{R}^n$ equipped with the standard Euclidean metric is bounded below by K on B^n_q if $N \geq n + (1-q)^{-1}$ and $K \leq 1/f_q(0)$. There are many study about a measure whose N-Ricci curvature is bounded from below for some positive N, however we usually assume the positivity of a measure and the completeness of a metric space.

We finally estimate the smallest Lipchitz constant $L_q(n)$ of $T_{n,q}$ which pushes forward γ_n to γ^q_n. The existence of such a map $T_{n,q}$ is guaranteed for any $q \in (0,1)$ and $n \in \mathbb{N}$ by [13, Section 4]. To do this, set

$$R_q(n) := \sup\left\{ r \in \mathbb{R} \;\middle|\; \exp_q\left(-\frac{r^2}{2} + c(n,q)\right) > 0 \right\} = \left(\frac{2\lambda_q(n)}{1-q}\right)^{1/2} < \infty.$$

Proposition 3.3 *For any $q \in (0,1)$ and $n \in \mathbb{N}$, we have*

$$R_q(n)^{n+2/(1-q)} = \pi^{-n/2}\left(\frac{2}{1-q}\right)^{1/(1-q)} \Gamma\left(\frac{n}{2} + \frac{2-q}{1-q}\right) \Big/ \Gamma\left(\frac{2-q}{1-q}\right),$$

$$R_q(n)^2 \cdot \frac{(1-q)}{(n+2)(1-q)+2} \leq L_q(n)^2,$$

where Γ stands for the Gamma function.

Proof. The direct calculation gives

$$\begin{aligned}1 = \int_{\mathbb{R}^n} d\gamma^q_n(x) &= \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^{R_q(n)} \exp_q\left(-\frac{r^2}{2} + c(n,q)\right) r^{n-1} dr \\ &= \frac{2\pi^{n/2}}{\Gamma(n/2)} \lambda_q(n)^{1/(1-q)} R_q(n)^n \int_0^1 (1-s^2)^{1/(1-q)} s^{n-1} ds \\ &= \pi^{n/2} \lambda_q(n)^{1/(1-q)} R_q(n)^n \Gamma\left(\frac{2-q}{1-q}\right) \Big/ \Gamma\left(\frac{n}{2} + \frac{2-q}{1-q}\right),\end{aligned}$$

which implies the first equality. Similarly, we compute

$$\int_{R^n} |x|^2 d\gamma_n^q(x) = \frac{n\pi^{n/2}}{2}\lambda_q(n)^{1/(1-q)} R_q(n)^{n+2}\Gamma\left(\frac{2-q}{1-q}\right) \Big/ \Gamma\left(\frac{n}{2}+\frac{2-q}{1-q}+1\right)$$
$$= R_q(n)^2 \cdot \frac{n(1-q)}{(n+2)(1-q)+2}.$$

On the other hand, by the definition of the push-forward measure, we have

$$\int_{R^n} |x|^2 d\gamma_n^q(x) = \int_{R^n} |T_{n,q}(x)|^2 d\gamma_n(x) \le \int_{R^n} L_q(n)^2|x|^2 d\gamma_n(x) = nL_q(n)^2.$$

Combining the these implies

$$R_q(n)^2 \cdot \frac{(1-q)}{(n+2)(1-q)+2} \le L_q(n)^2.$$

□

From [13, Theorem 1.2] we deduce the another estimate of $L_q(n)$

$$(2\pi)^{1/2}L_q(n) \ge \lambda_q(n)^{-1/n(1-q)} = \left(\frac{1-q}{2}R_q(n)^2\right)^{-1/n(1-q)}$$
$$= \pi^{1/2}R_q(n)\left[\Gamma\left(\frac{2-q}{1-q}\right) \Big/ \Gamma\left(\frac{n}{2}+\frac{2-q}{1-q}\right)\right]^{1/n},$$

where the equalities follow from the equality in Proposition 3.3. This estimate is better than the estimate in Proposition 3.3. For simplicity, let us consider the case of $n = 2k$. We then have

$$\left(k+1+\frac{1}{1-q}\right)^k \ge \prod_{j=1}^k \left(k+1-j+\frac{1}{1-q}\right) = \Gamma\left(k+\frac{2-q}{1-q}\right) \Big/ \Gamma\left(\frac{2-q}{1-q}\right),$$

which implies

$$\frac{R_q(2k)^2}{2}\frac{1-q}{(k+1)(1-q)+1} \le \frac{R_q(2k)^2}{2}\left[\Gamma\left(\frac{2-q}{1-q}\right) \Big/ \Gamma\left(k+\frac{2-q}{1-q}\right)\right]^{1/k}.$$

The asymptotic behavior of $L_q(2k)$ as $k \to \infty$ is unknown, however we have

$$(2\pi)^{1/2}L_q(2k) \ge \left(\frac{1-q}{2}R_q(2k)^2\right)^{-1/2k(1-q)} = \pi^{1/a_k}\left(\frac{2}{1-q}\right)^{1/a_k} P_k^{-1/a_k} \to 1$$

as $k \to \infty$, where we set

$$P_k := \left[\prod_{j=1}^k \left(k-j+\frac{2-q}{1-q}\right)\right]^{1/k} \in \left[1+\frac{1}{1-q}, \frac{a_k}{2(1-q)}\right], \quad a_k := 2(k(1-q)+1).$$

It thus is enough to show $P_k^{-1/a_k} \to 1$, or equivalently $\log P_k^{-1/a_k} \to 0$, as $k \to \infty$. This follows from the observation that

$$0 = \lim_{k\to\infty} \frac{-1}{a_k} \log \frac{a_k}{2(1-q)} \leq \lim_{k\to\infty} \log P_k^{-1/a_k} \leq \lim_{k\to\infty} \frac{-1}{a_k} \log\left(1 + \frac{1}{1-q}\right) = 0.$$

This suggests that, for $q \in (0,1)$, the family $\{\gamma_n^q\}_{n\in\mathbb{N}}$ of the q-Gaussian measures may not have the Lévy property (for instance, see [4, Section 3.3] about the definition of the Lévy property) and then suggests how difficult and interesting to investigate the asymptotic behavior of the concentration profiles of $\{\gamma_n^q\}_{n\in\mathbb{N}}$.

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