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Catastrophe Risk Derivatives: A New Approach

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1 Introduction

We are in a century where the urbanism is in a constant expansion. The modernization and the use of technology in urban and rural areas require more and more investments and expenses. This proliferation is driving the expected value of the catastrophe to attain important sums, which makes it nowadays very hard for an insurance company to cover these costs. For instance, a big earthquake in Tokyo could engender a higher cost than all the capitalization of the Japanese insurance companies. One of the solutions to this problem is the securitization of the catastrophe risk, since the financial potential of the stock exchange is huge comparing with these loss amounts.

From their apparition in the CBOT (Chicago Board of Treading) in 1995 the catastrophe derivatives know an incomparable success. Hence, in the other hand the financial literacy is giving an increasing interest to the modeling of these derivatives.

A shared point between the catastrophe products is their common underlying that is an index representing the amount engendered by some catastrophes during a certain period of time. This index was created to solve a moral hazard problem since the veracity of the catastrophe's amount is not observable from the financial statement of an insurance company. A good catastrophe derivative pricing passes inevitably by a good modeling of the loss index. Catastrophe loss indexes displays the amount of the insurance claims that surpasses a certain amount. That is because the insurance institutions are interested only by the large amounts (the extraordinary events) as the small ones are covered by the reserve made for this purpose. For an example, the PCS (Property Claim Services) index used $5 million as a floor and nowadays this floor moved to $12 million.[1]

The occurrence of these claims seems to follow a jump process thus some authors modeled the index as a Poisson process like in Cox, Fairchild, and Pederson (2006)[5]. The inconvenience with this approach is that it assigns the same size to all claims, while in reality different catastrophes in an interval of time create gives different claim amounts thus different jump sizes.

[1] `http://www.iso.com/Products/Property-Claim-Services/Property-Claim-Services-PCS-info-on-losses-from-catastrophes.html`

To correct this, among Jaimungal and Wang (2006)[10] and others, Alexander (2003)[1] used a Compound Poisson process without specifying the distribution of the sizes. Others take the jump size to be an exponentially distributed random variable, which we will see is not appropriate to the insurance catastrophe modeling.

Unlike other papers, we do not assume here any probability distribution for the index claims and but will try to deduce it by using some useful statistical tools. The general methodology of this paper will be as follow: we will start by introducing a simple model and try to derive its induced option pricing formula. In a second time, we will gradually introduce some more sophisticated models concluding by their induced option pricing.

2 The Loss Index: A general Modeling

Let us denote by $\{L_t; t \geq 0\}$ the Loss index process, and by $Y_i, i = 1, 2, \ldots$ the random variable representing the i–th clam amount.

As usual $(\Omega, \mathcal{F}, \mathbb{P},)$ is the probability space with the statistical (real world) probability measure $\mathbb{P}$. Also we assume a constant interest rate that we denote as δ.

Following the existing PCS (Property Claim Services) index, the loss process is characterized by two distinguished periods:

- A loss period $[0, t_1)$: where t_1 represents the first maturity of the option, that is, for a call as an example, at the end of this period the index's level should be more important than the strike so that the option would have an eventual payoff.
- A development period $[t_1, t_2]$: since we can not assess the real amount of the sinister a period is left for the option investor so that the index shows the exact amount of claims engendered by the catastrophes happening between $[0, t_1)$. Note that only the re-estimations of the catastrophe sinister that happen during the Loss period is taken into account. Catastrophes that occur between $[t_1, t_2]$ are not considered.

For simplicity, we will start by modeling the loss period.

Let the Index process have the following form:

$$L_t = L_0 \exp\{X_t\}, \quad t \geq 0, \tag{1}$$

where X_t is a stochastic process (a Compound Poisson Process)satisfying the usual conditionsand such as :

$$X_t = \sum_{i=1}^{N_t} Y_i, \quad t \geq 0, \tag{2}$$

where N_t represents the number of catastrophes[2] occurring in the interval $[0, t)$. Hence the sum: $\sum_{i=1}^{N_t} Y_i$ represents the total amount of the claims to be hedged by the institution.

$\mathbb{F} = \{\mathcal{F}_t; t \geq 0\}$ symbolizes the filtration generated by the compound Poisson process X_t and the $\mathbb{P}$–null set of $\mathcal{F}$.

Here we are *not* making any assumption regarding the distribution of the claim amounts, but the classical *i.i.d.* assumption.

[2] Catastrophes are only the ones generating large sinister (extraordinary amounts).

Some scientific evidences show that the occurrence of an earthquake opens a hole in the earth. The opening of the hole may be done by a series of other earthquakes. This part is directly followed by the closing of the hole process via an other set of earthquakes. If the original earthquake has a big magnitude then the following provoked ones (the replicas) have an important probability to have also a big magnitude.

This is to say that one big earthquake can generate many others. In the probability of having many magnitude picks (and thus amounts that are measurable by the index) during an interval $[t_i, t_j], i \neq j$ where t_i is the time of an earthquake manifestation.

This observation is important in the sense that statistically it allows us to use the strong law of large numbers. As stated before the correct approach of the index should includes only the high claim amounts.

For this purpose, the *Extreme Value Theory* (EVT) seems to provide a very useful set of tools that could be exploited in our case to set our model.

A Distribution for the Index's Amounts

In this subsection, we will try to deduct the distribution of the amounts that are measured by the index starting from the sample of all the observable claims by using some statistical approaches.

This distribution differs depending on our definition of the "high" amount.

According to the EVT, two kinds of definitions could be used.

First, we define the extreme values during a period say $[0, T]$ as the set of the maximum amounts of each subset $[t_i, t_j], 0 \leq t_i < t_j \leq T$.

Let $M_n := \max\{Y_1, ..., Y_n\}$. According to the *Fisher, Tippet and Gnedenko Theorem of the Block Maxima,* if there exists a sequence of couples (a_n, b_n), $n = 1, 2, ...$ where $a_n > 0$ and

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{M_n - b_n}{a_n} \leq y\right) =: H(y),$$

where H is a non-degenerate cumulative distribution function, then H has a *Generalized Extreme Value* (GEV) distribution which is a distribution that includes one of the following distributions: *Weibull, Fréchet, or Gumbel* that is to say that the normalized maximum has only one of the distributions stated above.

The GEV distribution has the Following form:

$$H_\zeta(x) = \begin{cases} \exp\left\{-(1+\zeta x)^{-\frac{1}{\zeta}}\right\} & \text{if } \zeta \neq 0; \\ \exp\{-\exp(-x)\} & \text{if } \zeta = 0. \end{cases}$$

Choice of the GEV Distribution

The skewness of the Fréchet distribution is given by:

$$\text{Skewness}_{\text{Fréchet}} = \begin{cases} \dfrac{\Gamma\left(1-\frac{3}{\alpha}\right) - 3\Gamma\left(1-\frac{2}{\alpha}\right)\Gamma\left(1-\frac{1}{\alpha}\right) + 2\Gamma^2\left(1-\frac{1}{\alpha}\right)}{\sqrt{\left\{\Gamma\left(1-\frac{2}{\alpha}\right) - \Gamma^2\left(1-\frac{1}{\alpha}\right)\right\}^3}} & \text{if } \alpha > 3; \\ \infty & \text{otherwise,} \end{cases}$$

where $\alpha \in \mathbb{R}^{++}$ is Fréchet distribution's shape parameter. So the basically the shape of the distribution is determined depending on the value of our α. Thus the behavior of this distribution is not clear. Same is for the Weibull distribution.
To remedy this, we follow this reasoning: In one hand, following Proposition 3 of Fraga Alves and Neves (2010) [6], the skewness of the GEV distribution is positive for $\zeta > -2.8$, negative for $zeta < -2.8$, and approximately equal to 1.14 for $\zeta = 0$.

In the other hand the GEV corresponds to the *Fréchet* distribution for $\zeta > 0$, to the *Weibull* one in the case where $\zeta < 0$, and finally it represents the *Gumbel* law when $\zeta = 0$.

We deduct that the Fréchet distribution is right skewed. Affecting a Fréchet distribution to the extreme values of earthquake claims is equivalent to suppose that the maximum amounts are too close from the ones that appear in a regular situation. In other words this distribution is concentrated in its small values. While in reality the catastrophes amount happen to be unexpected, so out of the historical data and these amount are larger than we could have in a normal situation.

By the same reasoning, the Weibull distribution assigns a large probability to the values too far from the end of the original distribution of the claims. This seems to be unrealistic since the earthquake wave vanishes after a certain distance.

Contrariwise the Gumbel law offers a certain uniformity in its distribution thus to begin we will say that the claim's extreme values follow a Gumbel distribution.

As a reminder, the Gumbel Cumulative Distribution Function (CDF) and Probability Density Function (PDF) are defined consecutively as bellow:

$$F_Y(x) = \exp\left[-\exp\left\{-\left(\frac{x-\mu}{\beta}\right)\right\}\right], \quad -\infty < x < \infty;$$
$$f_Y(x) = \frac{1}{\beta}\exp\left\{-\left(\frac{x-\mu}{\beta}\right)\right\}\exp\left[-\exp\left\{-\left(\frac{x-\mu}{\beta}\right)\right\}\right], \quad -\infty < x < \infty.$$

Its mean ad variance are:

$$\mathbb{E}[Y] = \mu + \beta\gamma;$$
$$\mathrm{Var}[Y] = \sigma^2[Y] = \frac{\pi^2\gamma^2}{6}.$$

where $\gamma = 0.577...$ is the Euler–Mascheroni constant.

Starting from the claims observations we can estimate the parameters of the Gumbel distribution. The Maximum Likelihood Estimator (MLE) is discussed in Chapter 19 of Forbes, Evans, Hastings, and Peacock (2011) [7]. The system of equations for the shape and scale parameters that is issued by this method needs to be solved numerically. Many authors suggest instead the use of the Method of Moments (MM) approach. The MM yields to:

$$\widehat{\beta} = \frac{\sqrt{6}}{\pi}\overline{\sigma};$$
$$\widehat{\mu} = \overline{Y} - \widehat{\beta}\gamma,$$

where $\overline{Y}$ and $\overline{\sigma}^2$ are sample mean and variance, respectively.

Now that we have defined the probability distribution for the claims we can start investigating our option pricing.

3 Catastrophe Option pricing

We will start this suction by calculating the Moment generating function of the process X_t since it would be of an extreme necessity afterward.

Moment Generating Function of X_t

The Moment Generating Function (MGF) of X_t is given as follows:

$$
\begin{aligned}
M(z,t) &= \mathbb{E}\left[e^{zX_t}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[e^{z\sum_{i=1}^{N_t} Y_i} \,\middle|\, N_t\right]\right] \\
&= \mathbb{E}\left[\{\varphi_Y(z)\}^{N_t}\right] \\
&= \sum_{k=0}^{\infty} \{\varphi_Y(z)\}^k \frac{(\lambda t)^k}{k!} e^{-\lambda t} \\
&= e^{\lambda t \varphi_Y(z)} e^{-\lambda t} \\
&= \exp\left[\lambda t\{\Gamma(1-\beta z)e^{\mu z} - 1\}\right], \quad z < \frac{1}{\beta},
\end{aligned} \tag{3}
$$

where

$$
\varphi_Y(z) = \mathbb{E}\left[e^{zY}\right] = \Gamma(1-\beta z)e^{\mu z}, \quad z < \frac{1}{\beta}
$$

where is the MGF of Gumbel random variable Y.

Option Pricing by Esscher Transform

Our index is composed by a series of jumps which means that the market is incomplete. Consequently, there exists not only one measure for which the discounted option price is a martingale. Accordingly the pricing of our asset is not unique. Here, for the pricing of our catastrophe option, we propose to use the Esscher transform following Gerber and Shiu (1994) [9] in their seminal paper. For this purpose, we suppose that there are no transaction costs and no taxes as usual. We also give the pricing of a call option. The put option price afterward can be easily deducted by using the Call/Put parity.

The Esscher MGF

Recall that the Esscher Moment Generating Function (MGF) is given by:

$$
M(z,t,h) := \frac{M(z+h,t)}{M(h,t)}, \tag{4}
$$

where we denote $M(z,t)$ as X_t's MGF[3].

Therefore:

[3]Notice here that we are using the same setting as Gerber and Shiu (1994) [9] by modeling the loss as $L_t = L_0 \exp\{X_t\}$.

$$M(z,t,h) = \exp\left[\lambda t\left\{\Gamma(1-\beta(z+h))e^{\mu(z+h)} - e^{\mu h}\Gamma(1-\beta h)\right\}\right]$$
$$= \exp\left[\lambda t e^{\mu h}\left\{\Gamma(1-\beta(z+h))e^{\mu z} - \Gamma(1-\beta h)\right\}\right]. \tag{5}$$

The Risk Neutral Esscher Transform

According to the Fundamental Theorem of Asset Pricing (FTAP), a no–arbitrage pricing measure $\mathbb{P}^*$ satisfies:

$$\begin{aligned}
L_0 = \mathbb{E}^*\left[e^{-\delta t}L_t\right] \quad &\Longrightarrow \quad L_0 = e^{-\delta t}\mathbb{E}^*\left[L_0\exp\{X_t\}\right] \\
&\Longrightarrow \quad e^{\delta t} = \mathbb{E}^*\left[\exp\{X_t\}\right] \\
&\Longrightarrow \quad \delta t = \ln M(1,t,h^*) \qquad (6)\\
&\Longrightarrow \quad \delta = \ln M(1,1,h^*) \\
&\Longrightarrow \quad \delta = \lambda t e^{\mu h^*}\left[\Gamma(1-\beta(1+h^*))\left\{e^{\mu} - \Gamma(1-\beta h^*)\right\}\right] \qquad (7)
\end{aligned}$$

This equation has apparently no closed form solution for h^*. Gerber and Shiu (1994) [9] proved in their paper the existence and the uniqueness of the solution of our equation. We thus believe that depending on the cases some numerical approaches can provide us a solution to this equation.

This h^* represent the parameter for which the Esscher transform is a martingale. Notice that the uniqueness of h^* does not imply the uniqueness of the martingale measure. This is because the market is highly incomplete because of the jump process, thus, there exist other martingale measures that can induce us to a fair price. We have to think of the uniqueness of h^* as the existence of a unique Esscher risk neutral transform and that other pricing methods could lead to a different pricing formulas compared to the one we are trying to derive.

Having our martingale measure we can compute our option price. For a call option on the index we have we have :

$$C_0 := \mathbb{E}^*\left[e^{-\delta t}(L_t - K)^+\right] = e^{-\delta t}L_0\int_{\psi}^{\infty} e^x f(x,t,h^*)\mathrm{d}x - \int_{\psi}^{\infty} e^{-\delta t}Kf(x,t,h^*)\mathrm{d}x,$$

where K is the strike price, and ψ is issued from the argument:

$$L_t - K > 0 \quad \Longleftrightarrow \quad L_0\exp\{X_t\} - K > 0 \quad \Longleftrightarrow X_t > \ln\left(\frac{L_0}{K}\right) =: \psi.$$

Accordingly, we have

$$C_0 = L_0\left\{1 - F(\psi, t, h^*+1)\right\} - e^{-\delta t}K\left\{1 - F(\psi, t, h^*)\right\}. \tag{8}$$

This expression is familiar in the sense that it resembles the famous Black and Scholes option pricing formula (Black and Scholes (1973) [2]).

For the special case where $\beta \to 0+$, $\lim_{\beta\to 0+}\Gamma\left(1-\beta(z+h)\right) \approx 1$, as same as $\lim_{\beta\to 0+}\Gamma\left(1-\beta z\right) \approx 1$. By then,

$$M(z,t,h) = \exp\left[\lambda t e^{\mu h}\left\{e^{\mu z} - 1\right\}\right], \tag{9}$$

where the $\lambda t e^{\mu h}$ part would be independent of z. In this case, the Esscher transform has a Poisson distribution with an intensity $\Lambda = \lambda t e^{\mu h}$, or a Poisson process that has an intensity equal to $\Lambda = \lambda t$ multiplied by a constant $e^{\mu h}$. Then the Esscher CDF is written as:

$$F(\psi, t, h) = \frac{\Gamma\left(\lfloor \psi + 1 \rfloor, \lambda t e^{\mu h}\right)}{\lfloor \Psi \rfloor !}, \tag{10}$$

where $\Gamma(a, b)$ is the upper incomplete Gamma function.

For the general case where $\beta > 0$, the Esscher transform PDF is defined as, denoting $f(x, t)$ the X_t's PDF, by:

$$f(x, t, h) = \frac{e^{hx} f(x, t)}{\int_{-\infty}^{\infty} e^{hx} f(y, t) \mathrm{d}y} = \frac{e^{hx} f(x, t)}{M(h, t)} \tag{11}$$

Then the CDF is :

$$F(x, t, h) = \frac{\int_{-\infty}^{x} e^{uz} f(u, t) \mathrm{d}u}{M(h, t)}. \tag{12}$$

Correction of the Claim Amount

In general, in the catastrophe insurance and for the earthquake insurance especially, it is very hard –if not impossible– to assess the exact amount of the claim caused by a catastrophe on the date of the occurrence. This amount is continually revised with time. The first amount is most of the time only an approximation of the real ultimate amount.

In contradiction with Schradin (1996) [12] who supposes a re-estimation taking place only after the development period, we follow Jaimungal and Wang (2006) [10], and Biagini, Bregman, and Meyer–Brandis (2008a, 2008b) [3, 4]. Supposing that the re-estimation befall right after the catastrophe.

Unlike these authors who build their model as a convolution of a semi-martingale process and the claim size process, we model the re-estimation as a process that do not happen only once per catastrophe but we allow instead the claim to be re-corrected many times between two jumps.

Starting from this point the loss index is not a pure jump process but rather a jump diffusion process. We then propose this model:

$$Z_t = X_t + \left(\mu_c - \frac{1}{2}\alpha_c\right) t + \alpha_c W_t, \quad t \geq 0; \tag{13}$$

$$L_t = L_0 \exp\{Z_t\}, \quad t \geq 0, \tag{14}$$

where $\{W_t; t \geq 0\}$ is a Brownian motion satisfying: $W_t \sim N(0, t)$.

The correction is done through time with a mean μ_c and a standard deviation of α_c (≥ 0). First, let us compute the Moment Generating Function of the process X_t for $z = 1$.

$$\mathbb{E}\left[\exp\{X_t\}\right] = \exp\left\{\lambda t \left(\varphi_Y(1) - 1\right)\right\} \quad \Longrightarrow \quad \exp\left\{\lambda t \Gamma(1-\beta) e^{\mu}\right\} = M_{X_t}(1). \tag{15}$$

Proposition 1. *The process* $\left\{\frac{\exp\{X_t\}}{M_{X_t}(1)}; t \geq 0\right\}$ *is a* $\mathbb{P}$*–martingale.*

Proof. We obviously have to calculate the expectation $\frac{\exp\{X_t\}}{M_{X_t}(1)}$ of our process X_t conditionally on $\mathscr{F}_s$ the filtration generated by the compound Poisson process up to time s.

For $0 \leq s < t$, we have:

$$\mathbb{E}\left[\frac{\exp\{X_t\}}{M_{X_t}(1)}\bigg|\mathscr{F}_s\right] = \mathbb{E}\left[\exp\{X_t - \ln M_{X_t}(1)\}|\mathscr{F}_s\right]$$
$$= \mathbb{E}\left[\exp\left\{\sum_{i=1}^{N_t} Y_i - \ln M_{X_t}(1)\right\}\bigg|\mathscr{F}_s\right] \tag{16}$$

$$\mathbb{E}\left[\exp\left\{\sum_{i=1}^{N_t} Y_i - \ln M_{X_t}(1)\right\}\bigg|\mathscr{F}_s\right]$$
$$=\mathbb{E}\left[\exp\left\{\sum_{i=1}^{N_t} Y_i - \ln M_{X_t}(1) - \sum_{i=1}^{N_s} Y_i + \ln M_{X_s}(1)\right\}\bigg|\mathscr{F}_s\right]\cdot\mathbb{E}\left[\exp\left\{\sum_{i=1}^{N_s} Y_i - \ln M_{X_s}(1)\right\}\bigg|\mathscr{F}_s\right] \tag{17}$$
$$=\mathbb{E}\left[\exp\left\{\sum_{i=N_s+1}^{N_t} Y_i - \ln M_{X_t}(1) + \ln M_{X_s}(1)\right\}\bigg|\mathscr{F}_s\right]\cdot\exp\left\{\sum_{i=1}^{N_s} Y_i - \ln M_{X_s}(1)\right\}, \tag{18}$$

where the deviation from Eq. (17) to Eq. (18) is because the last part of the expression Eq. (18) is $\mathscr{F}_s$–measurable.

The other part is:

$$\mathbb{E}\left[\exp\left\{\sum_{i=N_s+1}^{N_t} Y_i - \ln M_{X_t}(1) + \ln M_{X_s}(1)\right\}\bigg|\mathscr{F}_s\right] = \mathbb{E}\left[\exp\left\{\sum_{i=N_s+1}^{N_t} Y_i - \lambda(s-t)\left(\varphi_Y(1)-1\right)\right\}\bigg|\mathscr{F}_s\right] \tag{19}$$
$$= \mathbb{E}\left[\exp\left\{\sum_{i=N_s+1}^{N_t} Y_i\right\}\bigg|\mathscr{F}_s\right]\exp\left\{-\lambda(s-t)\left(\varphi_Y(1)-1\right)\right\}$$

$\sum_{i=N_s+1}^{N_t} Y_i\}$ is the sum of the jumps with size Y_i that occurs in the time interval $(s,t]$, and because the jumps are independent between the intervals and the only thing that matters is the length of the interval we have that: $N_t - N_s \rightsquigarrow N_{t-s}$. This leads to he fact that: $\sum_{i=N_s+1}^{N_t} Y_i =_{\mathrm{d}} \sum_{i=1}^{N_{t-s}} Y_i$, So:

$$\mathbb{E}\left[\exp\left\{\sum_{i=N_s+1}^{N_t} Y_i\right\}\bigg|\mathscr{F}_s\right]\cdot\exp\left\{-\lambda(s-t)\left(\varphi_Y(1)-1\right)\right\}$$
$$=\mathbb{E}\left[\exp\left\{\sum_{i=1}^{N_{t-s}} Y_i\right\}\right]\cdot\exp\left\{-\lambda(s-t)\left(\varphi_Y(1)-1\right)\right\}$$
$$=M_{X_{t-s}}(1)\exp\left\{-\lambda(s-t)\left(\varphi_Y(1)-1\right)\right\}$$
$$=\exp\left\{-\lambda(t-s)\left(\varphi_Y(1)-1\right)\right\}\cdot\exp\left\{-\lambda(s-t)\left(\varphi_Y(1)-1\right)\right\} = 1.$$

Finally,

$$\mathbb{E}\left[\frac{e^{X_t}}{M_{X_t}(1)}\bigg|\mathscr{F}_s\right] = \exp\left\{\sum_{i=1}^{N_s} Y_i - \ln M_{X_s}(1)\right\}\cdot 1 = \frac{e^{X_s}}{M_{X_s}(1)}, \quad 0 \leq s < t. \tag{20}$$

□

Under the risk neutral probability measure $\mathbb{Q}$ the Brownian motion noted $\left\{W_t^{\mathbb{Q}}; t \geq 0\right\}$ has the following form:

$$W_t^{\mathbb{Q}} := W_t + \frac{\left(\mu_c - \delta\right)t - \ln M_{X_t}}{\alpha_c}, \quad t \geq 0.$$

It follows that the Radon-Nikodym derivative process is:

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{F}_t} = \exp\left\{-\frac{1}{2}\int_0^t \left(\frac{\mu_c-\delta+\omega}{\alpha_c}\right)^2 \mathrm{d}s - \int_0^t \left(\frac{\mu_c-\delta+\omega}{\alpha_c}\right)\mathrm{d}W_s\right\}, \quad t \geq 0, \tag{21}$$

where

$$\omega := \lambda\{\Gamma(1-\beta)\mathrm{e}^{\mu}-1\}.$$

Theorem 3.1. *The price of a catastrophe call on the loss index at time $t=0$ is given by:*

$$\sum_{k=0}^{\infty} \mathbb{E}^{\mathbb{Q}}\left[\kappa\left(t, L_0\exp\{\ln M_{X_t}(1)\}\exp\left\{\sum_{i=0}^{k} Y_i\right\}\right)\right]\mathrm{e}^{-\lambda}\frac{(\lambda t)^k}{k!}, \tag{22}$$

where

$$k(t,x) = xN(d_+(t,x)) - K\mathrm{e}^{-\delta t}N(d_-(t,x))$$

$$d_{\pm}(x,t) = \frac{1}{\alpha_c\sqrt{t}}\left[\ln\left(\frac{x}{K}\right) + \left(\delta \pm \frac{\alpha^2}{2}\right)t\right]$$

Proof. Recall that, from Eqs. (13), (14), we have:

$$L_t = L_0\exp\left\{\left(\mu_c - \frac{1}{2}\alpha_c^2\right)t + \alpha_c W_t\right\}\exp\{X_t\},$$

and

$$C_t = \mathbb{E}\left[\mathbb{E}^{\mathbb{Q}}\left[\left(L_0\exp\left\{\left(\mu_c-\frac{1}{2}\alpha_c^2\right)t\right\}\exp\left\{\sum_{j=1}^{N_t} Y_j\right\} - K\right)^+ \middle| \sigma\left(\exp\left\{\sum_{j=1}^{N_t} Y_j\right\}\right)\right]\right]$$

In one hand, since the jump size and the diffusion process are supposed to be independent of each other, we could separate the compound Poisson process exponent from the geometric Brownian motion to get two expectations. In the other hand, inside the second expectation (the risk neutral one)[4] is $\sigma\left(\exp\left\{\sum_{j=1}^{N_t} Y_j\right\}\right)$-measurable. Hence the quantity in the first expectation is the Black and Scholes formula for a call price (Black and Scholes (1973) [2]). Calculating the first expectation as usual for a Poisson process, we got our final pricing formula. □

This pricing formula needs to be calculated via simulation.

4 An Esscher Transform Pricing for the Claims with Correction

We proceed as same as with the firs case without continuous corrections.

$$\begin{aligned} M_Z(z,t) &= \mathbb{E}\left[\mathrm{e}^{zZ_t}\right] \\ &= \mathbb{E}\left[\mathrm{e}^{zX_t}\mathrm{e}^{z\alpha_c W_t}\exp\left\{z\left(\mu_c - \frac{1}{2}\alpha_c^2\right)t\right\}\right] \\ &= M_X(z,t)M_W(z\alpha_c,t)\exp\left\{z\left(\mu_c-\frac{1}{2}\alpha_c^2\right)t\right\} \\ &= \exp\left\{\lambda t\{\Gamma(1-\beta z)\mathrm{e}^{\mu z}-1\} + \mu_c tz + \frac{1}{2}\sigma^2 z^2\right\} \end{aligned}$$

[4] We used the $\mathbb{P}$–measure expectation here as the intensity does not change in the risk neutral world according to Merton (1976) [11]

Thus :

$$M(z,t,h) = \exp\left\{\lambda t e^{\mu h}\left\{\Gamma\left(1-\beta(z+h)\right)e^{\mu z} - \Gamma\left(1-\beta h\right)\right\} + \left(\alpha_c h - \mu_c\right) tz + \frac{1}{2}\alpha_c^2 t^2 z^2\right\},$$

we then find that:

$$M(1,1,h^*) = \delta = \exp\left\{\lambda e^{\mu h^*}\left\{\Gamma\left(1-\beta(1+h^*)\right)e^{\mu} - \Gamma\left(1-\beta h^*\right)\right\} + \left(\alpha_c h^* - \mu_c\right) + \frac{1}{2}\alpha_c^2\right\}$$

extracting h^* and replacing it in the formula will give us our option price.

5 Pricing in the Development Period

During the development period we assist only to the correction of the claims. The index on which our option is based, by convention, do not take into consideration the new jumps even if a catastrophe may happen during this period. Thus, it turns out that the loss index become a pure geometric Brownian motion during the development interval. We can then, rewrite our loss index ass follow:

- For the simple case (without corrections in the loss period):

$$L_t = L_0 e^{X_t^*}, \quad t \geq 0; \tag{23}$$

$$X_t^* = \mathbb{1}_{\{t\leq T_1\}} \sum_{i=1}^{N_t} Y_i + \mathbb{1}_{\{t>T_1\}}\left(\mu_c - \frac{1}{2}\alpha_c^2\right) t + \alpha_c W_t, \quad t \geq 0. \tag{24}$$

- For the case with corrections:

$$Z_t^* = \mathbb{1}_{\{t\leq T_1\}} \sum_{i=1}^{N_t} Y_i + \left(\mu_c - \frac{1}{2}\alpha_c^2\right) t + \alpha_c W_t, \quad t \geq 0. \tag{25}$$

Observe that, in this case, the Brownian motion is present with a probability equal to 1 a.s.

The pricing of the option during this period leads to the same results for the two models, since, during this interval, these two models are equal and are represented by a geometric Brownian motion only.

Pricing Formula for the Development Period

Using the *Esscher Transform Method*, we have the following result:

$$M(z,t,h) = \exp\left\{\left(\mu_c + h\alpha^2 z + \frac{1}{2}\alpha_c^2 z^2\right) t\right\}, \quad t \geq 0. \tag{26}$$

The transformed process is then normally distributed and our new h^* and call option price are:

$$h^* = \frac{\delta - \frac{1}{2}\alpha_c^2 - \mu_c}{\alpha_c^2}; \tag{27}$$

$$C_0 = L_{t_1}\varphi\left(\frac{-\psi + \left(\delta + \frac{1}{2}\alpha_c^2\right)\tau}{\alpha_c\sqrt{\tau}}\right) - e^{-\delta\tau} K \varphi\left(\frac{-\psi + \left(\delta - \frac{1}{2}\alpha^2\right)\tau}{\alpha\sqrt{\tau}}\right), \tag{28}$$

where $\tau := t - T_1$ for $t > T_1$.

This pricing formula is nothing else then the Black and Scholes formula for a call option (Black and Scholes (1973) [2]). Obviously, using the martingale approach will lead to the sane result[5].

References

[1] Alexander, M., *Actuarially Consistent Valuation of Catastrophe Derivatives*, Working Paper, 03–18, The Wharton Financial Institutions Center, University of Pennsylvania, July 2003.

[2] Black, F. and Scholes, M., The Pricing of Option and Corporate Liabilities, *Journal of Political Economy*, Volume 81, pp. 637–659, 1973.

[3] Biagini, F., Bregman, Y., and Meyer–Brandis, T., Pricing of Catastrophe Insurance Options Written on a Loss Index with Reestimation, *Insurance: Mathematics and Economics*, Volume 43, Issue 2, pp. 214–222, 2008a.

[4] Biagini, F., Bregman, Y., and Meyer–Brandis, T., Pricing of Catastrophe Insurance Options under Immediate Loss Reestimation, *Journal of Applied Probability*, Volume 45, Number 3, pp. 831–845, 2008b.

[5] Cox, S.H., Fairchild, J.R., and H.W. Pedersen, H.W., Valuation of Structured Risk Management Products, *Insurance: Mathematics and Economics*, Volume 34, Issue 2, pp. 259–272, 2004.

[6] Fraga Alves, M.I. and Neves, C., Extreme Value Distributions, in *International Encyclopedia of Statistical Science* (Lovric, M. Ed.), Volume 1, Springer, pp. 493–496, 2010.

[7] Forbes, C., Evans, M., Hastings, N., and Peacock, B., *Statistical Distributions*, 4th Ed., John Wiley & Sons, 2011.

[8] Fujita, T., Ishimura, N., and Tanaka, D., An Arbitrage Approach to the Catastrophe Options Involving the Cox Process, *Hitotsubashi Journal of Economics*, Volume 49, Issue 2, pp. 67–74, 2008.

[9] Gerber, H.U. and Shiu, E.S.W., Martingale Approach to Pricing Perpetual American Options, *ASTIN Bulletin*, Volume 24, Issue 2, pp. 195–220, 1994.

[10] Jaimungal, S. and Wang, T., Catastrophe Options with Stochastic Interest Rates and Compound Poisson Losse, *Insurance: Mathematics and Economics*, Volume 38, Issue 3, pp. 469–483, 2006.

[11] Merton, R.C., Option Pricing when Underlying Stock Returns are Discontinuous, *Journal of Financial Economics*, Volume 3, pp. 125–144, 1976.

[12] Schradin, H.R., *PCS Catastrophe Insurance Options – A New Instrument for Managing Catastrophe Risk –*, AFIR/ERM Colloquium, Nurenberg, Germany, October 1–3, 1996.

[13] Schradin, H.R. and Timpel, M., *Einsatz von Optionen auf den PCS Schadenindex in der Risikosteuerung von Versicherungsunternehen*, Mannheimer Manuskripte zu Versicherungsbetriebslehre, Finanzmanagement und Risikotheorie, 88, Institüt für Versicherungswissenschaft der Universität Mannheim, Mannheim, 1996.

[5] This is because in the development period we have only one source of randomness issued from the Brownian motion and one asset. Hence according to the meta theorem the market is complete and thus the martingale measure is unique