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The Dual Theory of the Smooth Ambiguity Model *

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Abstract

This paper studies the "dual" theory of the smooth ambiguity model introduced by Klibanoff, Marinacci and Mukerji (2005). Unlike the original model, we characterize the attitude toward ambiguity captured by second-order beliefs in the dual model. First, we give a set of axioms to derive a dual representation of the smooth ambiguity model. Second, we present a characterization of ambiguity aversion. Last, as an application to our dual theory to a standard portfolio problem, we conduct comparative static predictions which give sufficient conditions to guarantee that an increase in smooth ambiguity aversion decreases the optimal portfolio.

Keywords: Ambiguity, Ambiguity aversion, Comparative statics, Smooth ambiguity model
JEL Classification D800 · D810

1 Introduction

Many experiments provide clear evidence that the expected utility model cannot capture actual behavior under risk and uncertainty. One of the most famous classical experiments was conducted by Ellsberg (1961). His experiment denied that probabilistic beliefs are not used for decision making under risk and uncertainty. Many alternatives, including Gilboa and Schmeidler (1989), Schmeidler (1989) and so on, have been proposed to explain the choices in Ellsberg's experiment. One explanation of the choices that is consistent with the Ellsberg experiment is based on second-order beliefs, which are probabilistic beliefs over probabilistic beliefs. Klibanoff, Marinacci and Mukerji (2005) gave a preference functional based on this idea, named smooth ambiguity aversion. The smooth ambiguity model has an advantage in tractability, since it is easier to apply to results in the expected utility model under circumstances with ambiguity. Gollier (2011) presented the systematic comparative statics technique to measure the effects of smooth ambiguity aversion. This paper develops a "dual" theory of the smooth ambiguity model, representation, comparison and comparative statics. We note that the term "dual" apparently comes from Yaari (1986, 1987).

The dual extension of the original smooth ambiguity model has not only theoretical but also behavioral contributions. The smooth ambiguity model represents the double expected utility form with respect to first-order belief (risk) and second-order belief (uncertainty). Similarly, the dual model of the smooth

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ambiguity model also shares the analog form of the expected utility model. This basically means that experimental methods under risk can be applied to ambiguity. The cumulative prospect theory of Tversky and Kahneman (1992) combines loss aversion with rank dependent probability. Since then, measuring the rank dependent probability experimentally has been well developed. We note that Yaari's dual theory can also be seen as rank dependent probability with linear utility, so our development of the dual theory of the smooth ambiguity model is also useful for measuring the attitude toward ambiguity experimentally. Smooth ambiguity aversion is motivated by the descriptive viewpoint, so that the experimental application of the dual model has an essential meaning. A systematic method of comparative statics is another contribution of this paper. By the similarity in form between the expected utility model and the smooth ambiguity model, it is natural to apply comparative statics results from the expected utility model to the smooth ambiguity model. Gollier (2011) developed a comparative statics method by transforming the decision problem with ambiguity to that without ambiguity. The transformation is accomplished using an artificial probability weighted by smooth ambiguity aversion. Unlike Gollier (2011), our method uses second-order belief directly. We note that this tool gives another proof of Gollier's result. We conduct a comparative statics analysis under ambiguity applying a useful method for the expected utility model from Jewitt (1987) and Athey (2002). As in the expected utility model, this method also becomes to be a powerful tool for comparative statics under ambiguity. The dual representation of smooth ambiguity aversion also has merit from a theoretical viewpoint. Our results insist that smooth ambiguity aversion is easier to analyze under the original smooth ambiguity model in some situations, but that it is much easier to analyze under the dual model in other situations.

The organization of this paper is as follows. The next section introduces the notation and the setting. In Section 3, we derive the dual representation of the smooth ambiguity model. In Section 4, we characterize the notion of smooth ambiguity aversion and more smooth ambiguity aversion in our dual theory. In Section 5, we display the effect of smooth ambiguity aversion on the portfolio choice through comparative static analysis.

2 Preliminaries

We begin with the setting and notation for our model. The setting is essentially constructed by combining Yaari (1987) and Klibanoff, Marinacci and Mukerji (2005), henceforth KMM. Thus, we follow the same notation as KMM. The state space $S = \Omega \times (0,1]$ consists of the separable metric space, Ω, and $(0,1]$. Let $\mathscr{A}$ be the Borel σ-algebra of Ω, $\mathscr{B}_1$ be the Borel σ-algebra of $(0,1]$ and $\Sigma = \mathscr{A} \otimes \mathscr{B}_1$. Let $f : S \to \mathscr{C}$ be a Savage act, where $\mathscr{C} \subset \mathbb{R}$ is the set of consequences. We denote by $\succeq$ a preference over the set of Savage acts and by $\mathscr{F}$ the set of all bounded Σ-measurable Savage acts.

Since $f \in \mathscr{F}$, we can assume that $\mathscr{C} \subseteq [0,1]$ without loss of generality. Thus we assume that $\mathscr{C} = [0,1]$.

An act $l \in \mathscr{F}$ is called a lottery if

$$l(\omega_1, r) = l(\omega_2, r) \quad \forall \omega_1, \omega_2 \in \Omega,\ r \in (0,1], \tag{1}$$

and is also Riemann integrable. Let $\mathscr{L}$ be the set of all such lotteries. For $f \in \mathscr{L}$ and $r \in (0,1]$, we define

$$f(r) = f(\omega, r) \quad \forall \omega \in \Omega.$$

We assume that the following assumption that is derived by Grandmont (1972) holds.

Assumption 1 (Expected Utility on Lotteries) *There exists a unique $u : \mathscr{C} \to \mathscr{C}$, continuous, strictly increasing, and normalized so that $u(0) = 0$ and $u(1) = 1$ such that*

$$f \succeq g \iff \int_{(0,1]} u(f(r))\mathrm{d}r \geq \int_{(0,1]} u(g(r))\mathrm{d}r \quad \forall f, g \in \mathscr{L}.$$

Let $\pi:\Sigma\to[0,1]$ be a probability measure satisfying

$$\pi(A\times B)=\pi(A\times(0,1])\lambda(B)\quad \forall A\in\mathscr{A},\quad B\in\mathscr{B}_1, \tag{2}$$

where $\lambda:\mathscr{B}_1\to[0,1]$ is the Lebesgue measure. Let Δ denote the set of all such probability measures. Let $\mathscr{C}(S)$ denote the set of all continuous and bounded real-valued functions on S.

Definition 1 *A second-order act is any bounded $\sigma(\Delta)$-measurable function $\mathfrak{f}:\Delta\to\mathscr{C}$.*

Let $\mathfrak{F}$ be the set of all second-order acts and $\succeq^2$ be the preference ordering on $\mathfrak{F}$.

Assumption 2 (Subjective Expected Utility on Second Order Acts) *There exists a probability measure $\mu:\sigma(\Delta)\to[0,1];\ 0<\mu(J)<1\,\forall J\in\sigma(\Delta)$, and a continuous, strictly increasing function $v:\mathscr{C}\to\mathbb{R}$ such that*

$$\mathfrak{f}\succeq^2\mathfrak{g}\iff\int_\Delta v(\mathfrak{f}(\pi))\mathrm{d}\mu\geq\int_\Delta v(\mathfrak{g}(\pi))\mathrm{d}\mu\quad\forall\mathfrak{f},\mathfrak{g}\in\mathfrak{F}.$$

As to the necessary and sufficient conditions such that the above assumption holds, we can refer to, e.g., Theorem 10.3 of Fishburn (1982).

For each $\mathfrak{f}\in\mathfrak{F}$,

$$G_{\mathfrak{f}}(t):=\mu(\mathfrak{f}(\pi)>t),\quad t\in[0,1].$$

Axiom 1 (Neutrality) *For each $\mathfrak{f},\mathfrak{g}\in\mathfrak{F}$, if $G_{\mathfrak{f}}=G_{\mathfrak{g}}$, then $\mathfrak{f}\sim^2\mathfrak{g}$.*

$$\Gamma=\{G:[0,1]\to[0,1]\,|\,G\text{ is nonincreasing, right-continuous and }G(1)=0\}.$$

From Axiom 1, the preference relation $\succeq^2$ on Γ is constructed by

$$G=G_{\mathfrak{f}}\succeq^2 H=G_{\mathfrak{g}}\iff\mathfrak{f}\succeq^2\mathfrak{g},$$

for $G,H\in\Gamma$ and $\mathfrak{f},\mathfrak{g}\in\mathfrak{F}$.

For each $G\in\Gamma$ and each $t\in[0,1]$, let $\hat{G}(t)$ be defined by

$$\hat{G}(t)=\{x|G(t)\leq x\leq G(t-)\}.$$

We define the inverse of G by

$$G^{-1}(p)=\min\{t|p\in\hat{G}(t)\}.$$

We note that $G^{-1}\in\Gamma$ and that $(G^{-1})^{-1}=G$.

We define a mixture of functions contained in Γ. For each $G,H\in\Gamma$ and $\alpha\in[0,1]$

$$\alpha G\boxplus(1-\alpha)H:=\left(\alpha G^{-1}+(1-\alpha)H^{-1}\right)^{-1}. \tag{3}$$

Axiom 2 (Dual Independence) *If $G,G',H\in\Gamma$ and $\alpha\in[0,1]$, then*

$$G\succeq^2 G'\Longrightarrow\alpha G\boxplus(1-\alpha)H\succeq^2\alpha G'\boxplus(1-\alpha)H.$$

We have the following lemma.

Lemma 1 *$\succeq^2$ satisfies Assumption2 and Axioms 1-2 if and only if there exists a continuous strictly increasing function $\varphi:[0,1]\to[0,1]$ such that*

$$\mathfrak{f}\succeq^2\mathfrak{g}\iff\int_{(0,1]}\varphi(G_{\mathfrak{f}}(t))\,\mathrm{d}t\geq\int_{(0,1]}\varphi(G_{\mathfrak{g}}(t))\,\mathrm{d}t\quad\forall\mathfrak{f},\mathfrak{g}\in\mathfrak{F}.$$

Let $\pi_f : \mathscr{B}_1 \to [0,1]$ be defined by

$$\pi_f(B) = \pi\left(f^{-1}(B)\right) \quad \forall B \in \mathscr{B}_1.$$

Lemma 2 *For all $f \in \mathscr{F}$ and $\pi \in \Delta$, there exists a non-decreasing lottery act $l_f(\pi) \in \mathscr{L}$ such that*

$$\lambda\left(l_f(\pi) \in B\right) = \pi_f(B) \quad \forall B \in \mathscr{B}_1.$$

Let δ_x be the constant act for $x \in \mathscr{C}$. Let $c_f(\pi)$ denote the certainty equivalent of the lottery act $l_f(\pi)$ such that $\delta_{c_f(\pi)} \sim l_f(\pi)$.

Definition 2 *Given $f \in \mathscr{F}$, $f^2 \in \mathfrak{F}$ is a second-order act associated with f if*

$$f^2(\pi) = u\left(c_f(\pi)\right) \quad \forall \pi \in \Delta.$$

Assumption 3 (Consistency with preferences over associated second-order acts) *Given $f, g \in \mathscr{F}$ and $f^2, g^2 \in \mathfrak{F}$,*

$$f \succeq g \iff f^2 \succeq^2 g^2.$$

3 Representation Theory

Theorem 1 presents the dual theory of the smooth ambiguity model.

Theorem 1 *Given Assumptions 1 and 2 and Axioms 1-5, $\succeq$ is represented by $V : \mathscr{F} \to \mathbb{R}$,*

$$V(f) = \int_0^1 \varphi\left[G_{u^\pi(f)}(t)\right] \mathrm{d}t, \tag{4}$$

where

$$u^\pi(f) : \Pi \to \mathscr{C}; \quad u^\pi(f) = \int_S u(f(s))\, \mathrm{d}\pi(s).$$

If we define $F^\varphi_{u^\pi(f)}(t) = 1 - \varphi\left[G_{u^\pi(f)}(t)\right]$, then integration by parts allows us to rewrite (4) as

$$V(f) = \int_0^1 t \mathrm{d}F^\varphi_{u^\pi(f)}(t). \tag{5}$$

Since $F^\varphi_{u^\pi(f)}$ is a cumulative distribution function of $u^\pi(f)$, we can consider a probability measure μ^φ over Δ induced by $F^\varphi_{u^\pi(f)}$. Hence we obtain the following Corollary.

Corollary 1 *Let $\mathbb{E}^\varphi_\mu$ denote the expectation under the probability measure μ^φ. Under the same assumptions and axioms as Theorem 1, the functional $V : \mathscr{F} \to \mathbb{R}$ is represented by*

$$V(f) = \mathbb{E}^\varphi_\mu\left[\int u(f(s)) \mathrm{d}\pi(s)\right] = \mathbb{E}^\varphi\left[u \circ f\right], \tag{6}$$

where $\mathbb{E}^\varphi$ denotes the expectation under the probability measure defined by

$$\mathbb{E}^\varphi_\mu[\mathbb{E}_\pi[1_{A\times B}]] \quad \forall A \in \mathscr{A},\ B \in \mathscr{B}_1.$$

4 Ambiguity Attitude

Let μ_f be the induced distribution defined by

$$\mu_f(u(B)) = \mu((f^2)^{-1}(B)) \quad \forall B \in \mathscr{B}_1.$$

We denote by Π the support of μ. In addition to the earlier assumptions, we impose the following assumption in order to discuss ambiguity attitude of decision makers.

Assumption 4 *Fix a family of preference relationships $\{\succ_\Pi, \succ_\Pi^2\}_{\Pi \subseteq \Delta}$ for a given decision maker. Both the restriction of $\succ_\Pi$ to lottery acts and the risk preferences derived from $\succ_\Pi^2$ remain the same for every $\Pi \subseteq \Delta$.*

This assumption guarantees that the same φ may be used to represent each $\succ_\Pi$ for a decision maker as the support of her subjective belief varies.

According to KMM, smooth ambiguity aversion is defined as follows.

Definition 3 *A DM (Decision Maker) displays smooth ambiguity aversion at (f, Π) if*

$$\delta_{u^{-1}e(\mu_f)} \succ_\Pi f,$$

where

$$e(\mu_f) = \int_{(0,1]} x \mathrm{d}\mu_f,$$

and μ has a support Π. A DM displays smooth ambiguity aversion if she displays smooth ambiguity aversion at (f, Π) for all $f \in \mathscr{F}$ and all supports $\Pi \subseteq \Delta$.

Proposition 1 *Under Assumptions 1-3, the following conditions are equivalent:*

(a) The function $\varphi : [0,1] \to [0,1]$ is convex.

(b) The DM displays smooth ambiguity aversion.

Let A and B be two DMs whose families of preferences share the same probability measure μ_Π for each support Π. According to KMM, we define the statement "A is more ambiguity averse than B" as follows.

Definition 4 *A is more ambiguity averse than B if*

$$f \succeq_\Pi^A l \Longrightarrow f \succeq_\Pi^B l \quad \forall f \in \mathscr{F},\ l \in \mathscr{L},\ \Pi \in \Delta. \tag{7}$$

Theorem 2 *Let A and B be two DMs whose families of preferences share the same probability measure μ_Π for each support Π. Then A is more ambiguity averse than B if and only if they share the same (von Neumman-Morgenstern) utility function u and*

$$\varphi_A = h \circ \varphi_B$$

for some strictly increasing, continuous and convex $h : [0,1] \to [0,1]$.

5 Application to the Portfolio Problem and Comparative Statics

5.1 Preliminaries

In this section, we apply the dual theory of the smooth ambiguity model to the simple portfolio problem consisting of one risk-free asset and one risky asset. The risk-free rate of return is normalized to zero without loss of generality. The risky asset return is ambiguous in the sense that its return is indexed by $\theta \in \Theta = [0,1]$. We assume that the investor's belief over Θ is represented by a decumulative function $G(\theta)$. The excess return of the risky asset indexed by θ is denoted by $\tilde{x}(\theta)$. To avoid technical difficulties, we assume that $\tilde{x}(\theta)$ takes a value in a bounded interval. Given an endowment wealth w, the investor chooses the share α of the risky asset. Then she receives the (conditional) expected utility $U(\alpha,\theta) = \mathbb{E}_\theta\left[u\left(w+\alpha\tilde{x}(\theta)\right)\right]$. Here u is the (von Neumman-Morgenstern) utility function which is assumed to be increasing and concave, and $\mathbb{E}_\theta$ denotes the conditional expectation given by θ.

We also assume that the expected utility $U(\alpha,\theta)$ is ranked by ascending order in θ. In other words, the excess asset return is ranked by the first- and/or the second-order stochastic dominance for $\alpha > 0$. By this assumption, there exists $\theta^*(\alpha,t) \in \Theta$ such that $U(\alpha,\theta^*(\alpha,t)) = t$ for each $t \in [0,1]$. Then, from Theorem 1, the investor computes the welfare $V(\alpha)$ through the transformation function φ:

$$V(\alpha) = \int_0^1 \varphi\left[G(\theta^*(\alpha,t))\right]\mathrm{d}t, \tag{8}$$

where φ is increasing and satisfies $\varphi(0) = 0$ and $\varphi(1) = 1$. From Proposition 1 the smooth ambiguity aversion is captured by the convexity of φ. In this representation, comparison of smooth ambiguity aversion is defined as follows: if φ_2 is more smoothly ambiguity averse than φ_1, then there exists an increasing, convex transformation function h such that $\varphi_2 = h \circ \varphi_1$.

At first, we confirm that the sign of the risky asset holding coincides with that of the expectation of the risky asset if the signs of $\mathbb{E}_\theta[\tilde{x}(\theta)]$ are the same for all $\theta \in \Theta$. The first-order condition is given by

$$V'(\alpha) = \int_0^1 \varphi'(G(\theta^*(\alpha,t))G'(\theta^*(\alpha,t))\frac{\partial\theta^*(\alpha,t)}{\partial\alpha}\mathrm{d}t = 0. \tag{9}$$

For each $t \in [0,1]$, totally differentiating $U(\alpha,\theta)$ gives

$$\begin{aligned} \left.\frac{\partial\theta^*(\alpha,t)}{\partial\alpha}\right|_{\alpha=0} &= -\left.\frac{\mathbb{E}_\theta[u'(w+\alpha\tilde{x}(\theta))\tilde{x}(\theta)]}{\partial U(\alpha,\theta)/\partial\theta}\right|_{\alpha=0} \\ &= -\frac{u'(w)\mathbb{E}_\theta[\tilde{x}(\theta)]}{\partial U(\alpha,\theta)/\partial\theta|_{\alpha=0}}. \end{aligned}$$

Since $\varphi'(\cdot)G'(\cdot) \le 0$, by the concavity of the objective function, the following relation holds:

$$\mathbb{E}_\theta\left[\tilde{x}(\theta)\right] \ge (\le) 0 \Longrightarrow \alpha^* \ge (\le) 0. \tag{10}$$

In the remainder of this section, we assume that $\mathbb{E}_\theta[\tilde{x}(\theta)]$ is positive for each $\theta \in \Theta$, that is, the risky asset holding is positive. We obtain similar results by changing the sign for the case when the risky asset holding is negative. Thus we omit them.

5.2 Two indexes case

In this subsection, we consider a simple case in which the set of indexes Θ consists of two elements $\{0,1\}$. The investor believes that $\theta = 0$ and that $\theta = 1$ are given by $1-p$ and p respectively. The excess risky asset returns indexed by them are given by $\tilde{x}_0 = \tilde{x}(0)$ and $\tilde{x}_1 = \tilde{x}(1)$ respectively. We assume that $\tilde{x}_1$ dominates $\tilde{x}_0$ in the sense of first-order stochastic dominance (FSD), so that $\mathbb{E}\left[u\left(w+\alpha\tilde{x}_0\right)\right] \leq \mathbb{E}\left[u\left(w+\alpha\tilde{x}_1\right)\right]$ for any $\alpha > 0$. In this setting, the investor maximizes the welfare:

$$\max_{\alpha}(1-\varphi(p))\mathbb{E}\left[u\left(w+\alpha\tilde{x}_0\right)\right]+\varphi(p)\mathbb{E}\left[u\left(w+\alpha\tilde{x}_1\right)\right]. \tag{11}$$

The first-order condition is given by $\alpha = \alpha^*$ satisfying

$$(1-\varphi(p))\mathbb{E}\left[u'\left(w+\alpha^*\tilde{x}_0\right)\tilde{x}_0\right]+\varphi(p)\mathbb{E}\left[u'\left(w+\alpha^*\tilde{x}_1\right)\tilde{x}_1\right]=0. \tag{12}$$

The second-order condition is also satisfied by the concavity of the utility function. We examine the effect of increasing smooth ambiguity aversion on the optimal portfolio in our dual theory.

Let an investor with her transformation function φ_1 be more ambiguity averse than an investor with her transformation function φ_2. From Theorem 2, we can readily observe that $\varphi_2(p) > \varphi_1(p)$. It is natural to conjecture that an increase in smooth ambiguity aversion decreases the holding of the risky asset. However, as illustrated by Gollier (2011) for the original smooth ambiguity model, this does not hold.

Proposition 2 *In the two indexes case, if the investor has the preference that her relative risk aversion is less than unity, an increase in smooth ambiguity aversion decreases the share of the risky asset.*

We note that if the investor's relative risk aversion is not less than unity, it may be possible that an increase in smooth ambiguity aversion increases the share of the risky asset since it may occur that $\alpha_0 > \alpha_1$ as shown in an example of a quadratic utility function in Fishburn and Porter (1976).

5.3 The general case

In this subsection, we consider a general case in which the set of indexes is defined by the interval $\Theta = [0,1]$. Let an investor with transformation function φ_1 be more ambiguity averse than an investor with transformation function φ_2. The optimal portfolio of the risky asset for the investor with φ_1 is given by the following first-order condition:

$$V_1'(\alpha) = \int_0^1 \varphi_1'(G(\theta^*(\alpha,t))G'(\theta^*(\alpha,t))\frac{\partial\theta^*(\alpha,t)}{\partial\alpha}dt = 0. \tag{13}$$

To determine a condition such that an increase in smooth ambiguity aversion decreases the optimal portfolio of the risky asset, we have to find a condition that satisfies the following inequality:

$$V_2'(\alpha) = \int_0^1 \varphi_2'(G(\theta^*(\alpha,t))G'(\theta^*(\alpha,t))\frac{\partial\theta^*(\alpha,t)}{\partial\alpha}dt \geq 0. \tag{14}$$

To prove this inequality, we use the variation diminishing property introduced by Karlin and Novikoff (1963) and Karlin (1968). Jewitt (1987) and Athey (2002) have discussed its economic and financial applications.

Property 1 *Suppose that $g : \mathbb{R} \to \mathbb{R}$ satisfies the single crossing condition that there exists a $x_0 \in \mathbb{R}$ satisfying $(x-x_0)g(x) \geq 0$ for all $x \in \mathbb{R}$. Then the following two conditions are equivalent:*

- $\phi : \mathbb{R}^2 \to \mathbb{R}$ *is log-supermodular (LSPM).*
- $\mathbb{E}[g(x)\phi(x,\theta_L)] = 0 \Longrightarrow \mathbb{E}[g(x)\phi(x,\theta_H)] \geq 0 \quad \forall\theta_H \geq \theta_L.$

To use this property, we begin with the following lemma.

Lemma 3 *Suppose* $\phi : \mathbb{R}^2 \to \mathbb{R}^+$ *is differentiable with respect to its first argument. Then* ϕ *is LSPM if and only if* $\frac{\partial\phi(x,y)/\partial x}{\phi(x,y)}$ *is non-decreasing in* y.

Proposition 3 *Suppose that an investor follows the dual theory of smooth ambiguity aversion. An increase in smooth ambiguity aversion reduces the optimal share of the risky asset if*

$$g(t) = G'(\theta^*(\alpha,t))\frac{\partial\theta^*(\alpha,t)}{\partial\alpha}$$

has the single crossing property at t_0*;* $(t-t_0)g(t) \geq 0$.

Theorem 3 *Suppose that an investor follows the dual theory of smooth ambiguity aversion. If the investor has the preference that her relative risk aversion is less than unity and the risky asset returns are ranked by first-order stochastic dominance, then an increase in smooth ambiguity aversion decreases the share of the risky asset.*

Theorem 4 *Suppose that an investor follows the dual theory of smooth ambiguity aversion. If the investor has the preference that her relative prudence is greater than zero and less than two and the risky asset returns are ranked by the second-order stochastic dominance, an increase in smooth ambiguity aversion decreases the share of the risky asset.*

A similar theorem holds for central dominance, which is necessary and sufficient for increasing the share of the risky asset for all risk averse investors. In addition, a similar theorem also holds for any stochastic dominance that is stronger than central dominance. See Chapter 6 in Gollier (2001) for an explanation of central dominance and the related stochastic dominance.

Theorem 5 *Suppose that an investor follows the dual theory of smooth ambiguity aversion. An increase in smooth ambiguity aversion decreases the share of the risky asset when the risky asset returns are ranked by central dominance.*

6 Concluding Remarks

This paper studies the dual theory of the smooth ambiguity model introduced by KMM. This can be viewed as an extension of Yaari's dual theory of the expected utility model for ambiguity. In our dual theory, the preferences for ambiguity capture second-order beliefs. Adding some axioms to the original model, we present the preference functional for the dual representation of the smooth ambiguity model. We characterized ambiguity aversion and its comparison in our dual theory. Lastly, we determined a set of sufficient conditions that guarantee that smooth ambiguity aversion decreases the optimal portfolio in an application of our model to the standard portfolio problem.

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