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On an operations research game related to a search problem on a linear graph

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1. Introduction

Suppose there are optimization problems which have the same structure and in which parameters are different partly. Decision-makers in these problems may agree to consider a problem cooperatively by merging those problems and pursueing some efficiency. If they have any saving of cost by doing so, the merger is meaningful. In this note we consider redistribution of the saving among the decision-makers in original problems. For example, we find such example in [Owen 1975, p.231-233] in which the original problem is linear production. In [Borm,Hamers and Hendrickx 2001], mergers are considered for several kinds of optimization problems. An optimization problem in this note is a search problem on a finite and connected graph. There is an (immobile) object in a node except for a specified node, with a priori probabilities. A seeker starts at the specified node and examines each node until he finds an object, traveling along edges. Associated with an examination of a node is the examination cost, and associated with a movement from a node to a node is a traveling cost. A strategy for the seeker is an ordering of nodes in which the seeker examines each node in that order. The purpose of the seeker is to find a strategy which minimizes the expected search cost. In [Kikuta 1990], this problem is studied when the graph is a rooted tree with two branches. In a monograph [Alpern and Gal 2003], this problem is commented. [Lössner and Wegener 1982] studies a more general problem, including this problem as a special case. [Gluss 1961] analyzes this problem when the underlying graph is linear. [Kikuta 1991] examines some optimality of strategies given in [Gluss 1961]. In this note, we try to construct a merger model for this search problem. Then we derive a cooperative game by defining the cost function for the merger and examine the subadditivity of the cost function and treats the existence problem of the core for the derived cooperative game. [Ruckle 1983] introduces many search games on graphs. In [Kikuta 2012] we tried to construct a merger model for search games on a cyclic graph.

2. A search problem on a finite graph

In this section we state in detail a search problem on a finite and connected graph with traveling cost and examination cost. Let (N, E) be a finite, connected and undirected graph where $N = \{0, 1, \ldots, n\}, n \geq 2$, is the set of nodes and $E \subseteq N \times N$ is the set of edges. The node 0 is specified. A path between i_0 and i_s is an ordered $(s+1)$-tuple $\pi = (i_0, i_1, \ldots, i_s)$ such that $(i_{r-1}, i_r) \in E$ for $r = 1, \ldots, s$. Each edge $(i, j) \in E$ is associated with a positive real number $d(i, j) > 0$, called a traveling cost of $(i, j) \in E$. The traveling cost of a path is the sum of the traveling costs of the edges in the path. We assume $d(i, j) = d(j, i)$ for all $(i, j) \in E$ and $d(i, i) = 0$ for all $i \in N$. If $i, j \in N$ and $(i, j) \notin E$, we define $d(i, j)$ by the minimum of the traveling costs of the paths between i and j. We assume $d(i, k) \leq d(i, j) + d(j, k)$ for all $i, j, k \in N$.

First we state a search problem on (N, E). There is an (immobile) object in a node except for the node 0, with a priori probabilities $p_i, i \in N \setminus \{0\}$. A seeker starts at the node 0 and examines each node until he finds an object, traveling along edges. He finds an object certainly (with probability 1) if he examines the right node. Associated with an examination of $i \in N \setminus \{0\}$ is the examination cost c_i, and associated with a movement from a node $i \in N$ to a node $j \in N$ is a traveling cost $d(i, j)$. A strategy for the seeker is a permutation σ on N with $\sigma(0) = 0$, which means that the seeker examines each node in the order of $\sigma(1), \ldots, \sigma(n)$, starting at the node 0. This is alternatively denoted by $[\sigma(1), \ldots, \sigma(n)]$. It must be noted that the seeker may pass through some nodes without examination when $(\sigma(i), \sigma(i+1)) \notin E$. Σ is the set of all permutations on N such that $\sigma(0) = 0$. We let $\Gamma = ((N, E), \Sigma, \{c_j\}_{j \in N \setminus \{0\}}, \{d(i, j)\}_{(i,j) \in E}, \{p_j\}_{j \in N \setminus \{0\}})$. For $i \in N \setminus \{0\}$ and $\sigma \in \Sigma$, $f(i, \sigma)$ is the cost of finding the object at the node i when the seeker takes a strategy σ:

$$f(i, \sigma; \Gamma) = \sum_{x=0}^{\sigma^{-1}(i)-1} \{d(\sigma(x+1), \sigma(x)) + c_{\sigma(x+1)}\}. \tag{1}$$

For $\sigma \in \Sigma$, $f(\sigma; \Gamma)$ is the expected cost of finding the object, starting at the node 0:

$$f(\sigma; \Gamma) = \sum_{i \in N \setminus \{0\}} p_i f(i, \sigma; \Gamma) = \sum_{i \in N \setminus \{0\}} p_{\sigma(i)} f(\sigma(i), \sigma; \Gamma). \tag{2}$$

A strategy $\sigma^* \in \Sigma$ is said to be optimal if

$$f(\sigma^*; \Gamma) = \min_{\sigma \in \Sigma} f(\sigma; \Gamma).$$

The purpose of the seeker is to find an optimal strategy. This problem is denoted by (f, Γ).

3. A merger of search problems on a finite graph

In this section we derive a cooperative game by merging search problems mentioned in the previous section. There are m seekers $\bar{1}, \ldots, \bar{m}$ and they are facing search problems $\Gamma^{\bar{1}}, \ldots, \Gamma^{\bar{m}}$ respectively. Here, $\Gamma^{\bar{t}} = ((N, E), \Sigma, \{c_j^{\bar{t}}\}_{j \in N \setminus \{0\}}, \{d^{\bar{t}}(i, j)\}_{(i,j) \in E}, \{p_j^{\bar{t}}\}_{j \in N \setminus \{0\}}), 1 \leq t \leq m$. Let $M = \{\bar{1}, \ldots, \bar{m}\}$ be the set of seekers. Each of m search problems is defined on the same graph (N, E). Each seeker starts at the node 0. The examination cost, the traveling cost, and a priori probabilities depend on each seeker. Let $C(\bar{t}) = \min_{\sigma \in \Sigma} f(\sigma; \Gamma^{\bar{t}}), 1 \leq t \leq m$ be expected search costs for seekers $\bar{t}, 1 \leq t \leq m$ under optimal strategies for these search problems, $\Gamma^{\bar{t}}, 1 \leq t \leq m$.

Now, an interpretation of a priori probabilities is that it is a result in the same situation of search which occurs many times. We imagine that seekers in the set $S \subseteq M$ hire an agent who really behaves instead of every seeker in S. The set S is called a coalition. Let $\{c_j^S\}_{j \in N \setminus \{0\}}, \{d^S(i, j)\}_{(i,j) \in E}$ be examination and traveling costs for this agent. These costs do not depend on each seeker in the coalition S. Let

$$\Gamma^{S,\bar{t}} = ((N, E), \Sigma, \{c_j^S\}_{j \in N \setminus \{0\}}, \{d^S(i, j)\}_{(i,j) \in E}, \{p_j^{\bar{t}}\}_{j \in N \setminus \{0\}}), 1 \leq t \leq m.$$

We assume that the agent faces to a problem $(f, \Gamma^{S,\bar{t}}), \bar{t} \in S$ at random one at a time. After he finds an object for one problem and returns to the node 0, he faces the next problem. Let $C(S, \bar{t})$ be the optimal value of the problem $(f, \Gamma^{S,\bar{t}})$. That is, $C(S, \bar{t}) = \min_{\sigma \in \Sigma} f(\sigma; \Gamma^{S,\bar{t}})$ is the expected search cost for the agent when he faces the problem of the seeker $\bar{t}$. We let $C(S) = \sum_{\bar{t} \in S} C(S, \bar{t})$ and we shall consider that this is the cost for the merger when all members in S cooperate. We let $C(\emptyset) = 0$.

We consider the pair (M, C) as a TU-game where M is the set of searchers and C is the cost function on the power set of M. When C is subadditive, that is, if C satisfies the condition

$$C(S) + C(T) \geq C(S \cup T), \forall S, T : S \cap T = \emptyset, \tag{3}$$

then we say the TU-game (M, C) is subadditive. The left hand side of (3) is the sum of expected costs when the coalitions S and T play search games separately, while the right hand side is the expected cost when the coalitions S and T play a search game jointly.

The subadditivity (3) states that it is advantages in cost for two coalitions to behave cooperatively.

Now, if the TU-game (M, C) is subadditive, then all seekers will agree to cooperate, to hire an agent and to pay $C(M)$ to this agent. How much should each seeker pay ? Suppose each seeker $\bar{t} \in M$ agrees to pay $x_t, \bar{t} \in M$. Then we must have

$$\sum_{\bar{t} \in M} x_t = C(M). \tag{4}$$

There are many payoff vectors $x = (x_1, \ldots, x_m)$ which satisfy (4). We consider payoff vectors $x = (x_1, \ldots, x_m)$ which furthermore satisfy the condition:

$$\sum_{\bar{t} \in S} x_t \leq C(S), \quad \forall S \subseteq M, \tag{5}$$

The left hand side of (5) is the sum of payoffs for the members of S when all seekers cooperate and pay $C(M)$ to an agent. The right hand side is the expected cost for the merger when all members in S cooperate. If a payoff vector $x = (x_1, \ldots, x_m)$ is proposed which satisfies (5), then all seekers will cooperate strongly and pay $C(M)$ to an agent. The set of payoff vectors which satisfy the conditions (4) and (5) is called the core for the TU-game (M, C) and it is denoted by $\mathcal{C}(M, C)$.

In this note we assume

$$\begin{aligned} &d^{\bar{t}}(j,k) = 1, \forall \bar{t} \in M, \forall (j,k) \in E, \\ &d^S(j,k) = \frac{1}{|S|} \sum_{\bar{t} \in S} d^{\bar{t}}(j,k) = 1, \forall S \subseteq M, \forall (j,k) \in E, \\ &c_j^S = \frac{1}{|S|} \sum_{\bar{t} \in S} c_j^{\bar{t}}, \forall S \subseteq M, \forall j \in N. \end{aligned} \tag{6}$$

where $|S|$ is the number of members in the set S. The second and third equations in the assumption (6) state that there is no saving in traveling cost for each edge and in examination cost for each node. The next proposition says a relation between $C(S \cup T)$ and $C(S), C(T)$ such as $S \cap T = \emptyset$. We would study properties of the cost function on this line later.

Proposition 3.1. For $S, T \subseteq M$ such that $S \cap T = \emptyset$ and for $\bar{t} \in S \cup T$,

$$C(S \cup T, \bar{t}) = \begin{cases} \min_{\sigma \in \Sigma}\{f(\sigma; \Gamma^{S,\bar{t}}) + \frac{|T|}{|S \cup T|} \sum_{i \in N \setminus \{0\}} p_i^{\bar{t}} \sum_{x=0}^{\sigma^{-1}(i)-1} (c_{\sigma(x+1)}^T - c_{\sigma(x+1)}^S)\}, & \bar{t} \in S, \\ \\ \min_{\sigma \in \Sigma}\{f(\sigma; \Gamma^{T,\bar{t}}) + \frac{|S|}{|S \cup T|} \sum_{i \in N \setminus \{0\}} p_i^{\bar{t}} \sum_{x=0}^{\sigma^{-1}(i)-1} (c_{\sigma(x+1)}^S - c_{\sigma(x+1)}^T)\}, & \bar{t} \in T. \end{cases} \tag{7}$$

Proof: This is because

$$
\begin{aligned}
d^{S\cup T}(i,j) &= d^S(i,j) = d^T(i,j), \quad \forall i,j \in N \\
c_j^{S\cup T} &= \frac{|S|}{|S\cup T|}c_j^S + \frac{|T|}{|S\cup T|}c_j^T, \quad \forall j \in N \\
&= c_j^S + \frac{|T|}{|S\cup T|}(c_j^T - c_j^S) = c_j^T + \frac{|S|}{|S\cup T|}(c_j^S - c_j^T), \quad \forall j \in N
\end{aligned}
\tag{8}
$$

□

4. A merger of search problems on a linear graph

4.1. On a linear graph with 3 nodes

In this subsection we analyze a TU-game on a linear graph with 3 nodes. The set of nodes is $N = \{0,1,2\}$ and the set of edges is $E = \{(0,1),(1,2)\}$. The seeker has two pure strategies $[12],[21]$. The cost matrix A for the agent of a coalition S is

$$
A = \begin{matrix} \\ 1 \\ 2 \end{matrix}
\begin{matrix} \quad [12] \qquad\qquad [21] \\ \begin{pmatrix} 1 + c_1^S & 3 + c_1^S + c_2^S \\ 2 + c_1^S + c_2^S & 2 + c_2^S \end{pmatrix} \end{matrix}
\tag{9}
$$

where each component of A is $f(i,\sigma;\Gamma^{S,\bar{t}}), i = 1,2$ and $\sigma = [12],[21]$, The expected costs for two pure strategies $[12],[21]$ are

$$
\begin{aligned}
f([12];\Gamma^{S,\bar{t}}) &= p_1^{\bar{t}}(1 + c_1^S) + p_2^{\bar{t}}(2 + c_1^S + c_2^S) \\
f([21];\Gamma^{S,\bar{t}}) &= p_1^{\bar{t}}(3 + c_1^S + c_2^S) + p_2^{\bar{t}}(1 + c_2^S).
\end{aligned}
\tag{10}
$$

By definition

$$
\begin{aligned}
C(S,\bar{t}) &= \min\{f([12];\Gamma^{S,\bar{t}}), f([21];\Gamma^{S,\bar{t}})\} \\
C(S) &= \sum_{\bar{t}\in S} C(S,\bar{t}).
\end{aligned}
\tag{11}
$$

By $f([12];\Gamma^{S,\bar{t}}) \le f([21];\Gamma^{S,\bar{t}})$ we have

$$
c_2^S \ge \frac{p_2^{\bar{t}}}{p_1^{\bar{t}}}c_1^S - 2. \tag{12}
$$

The equality holds on the line in the (c_1^S, c_2^S)-plane in the problem $(f,\Gamma^{S,\bar{t}})$, which is shown in the next figure, and we see roughly that the agent must choose $[12]$ when c_2^S is large and $p_1^{\bar{t}}$ is large, that is, $p_2^{\bar{t}}$ is small.

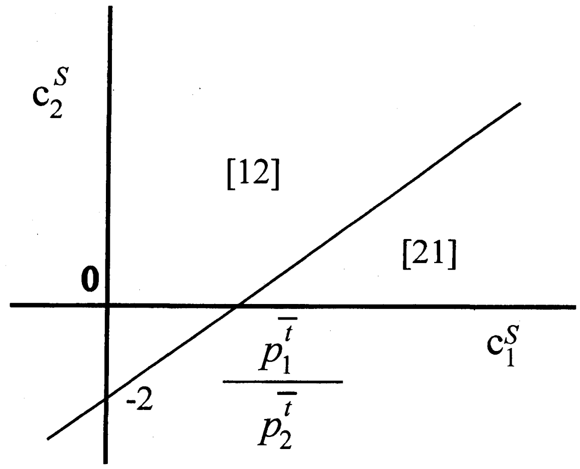

Figure 4.1 : Optimal strategies in $(f, \Gamma^{S,\bar{t}})$

Noting $p_2^{\bar{t}} = 1 - p_1^{\bar{t}}$ in (12), it holds $f([12]; \Gamma^{S,\bar{t}}) \leq f([21]; \Gamma^{S,\bar{t}})$ if and only if

$$p_1^{\bar{t}} \geq p_*^S \equiv \frac{c_1^S}{2 + c_1^S + c_2^S}. \tag{13}$$

Example 4.1. Let $S = \{\bar{1}, \bar{2}\}$.

$$\begin{aligned} c_1^{\bar{1}} = c_2^{\bar{1}} = 1, c_1^{\bar{2}} = \frac{3}{4}, c_2^{\bar{2}} = \frac{1}{2}, \\ p_1^{\bar{1}} = \frac{3}{4}, p_1^{\bar{2}} = \frac{1}{2}. \end{aligned} \tag{14}$$

Then $c_1^S = \frac{7}{8}$ and $c_2^S = \frac{3}{4}$. By (10) and (11),

$$\begin{aligned} C(\bar{1}) = C(\{\bar{1}\}, \bar{1}) = \min\{\frac{5}{2}, \frac{9}{2}\} = \frac{5}{2}, C(\bar{2}) = C(\{\bar{2}\}, \bar{2}) = \min\{\frac{5}{2}, \frac{27}{8}\} = \frac{5}{2}, \\ C(S, \bar{1}) = \min\{\frac{37}{16}, \frac{133}{32}\} = \frac{37}{16}, C(S, \bar{2}) = \min\{\frac{11}{4}, \frac{59}{16}\} = \frac{11}{4}, \\ C(S) = \frac{37}{16} + \frac{11}{4} = \frac{81}{16} > \frac{5}{2} + \frac{5}{2} = C(\bar{1}) + C(\bar{2}). \end{aligned} \tag{15}$$

The last inequality shows that the cost function is not subadditive.

Let $S_1, \ldots, S_r$ be a family of nonempty and different subsets of M. Let $\lambda_1, \ldots, \lambda_r$ be positive numbers. When this family and positive numbers satisfy the following conditions, the family $\{S_1, \ldots, S_r\}$ is called a balanced set, and positive numbers $\lambda_1, \ldots, \lambda_r$ is called balancing coefficient.

$$\sum_{j:\bar{t} \in S_j} \lambda_j = 1, \quad \forall \bar{t} \in M. \tag{16}$$

It is known that there exists a payoff vector $\{x_t, \bar{t} \in M\}$ satisfying equations (4) and (5) if and only if the cost function satisfies

$$C(M) \leq \sum_{j=1}^{r} \lambda_j C(S_j) \tag{17}$$

for every balanced set and balancing coefficient (See [5], Chapter X). From this, (11) and (13) we obtain the next proposition.

Proposition 4.1. Let (M, C) be the TU-game defined by (11). Assume either $p_1^{\bar{t}} > \max_{S:\bar{t}\in S}\{p_*^S\}$ for all $\bar{t} \in M$ or $p_1^{\bar{t}} < \min_{S:\bar{t}\in S}\{p_*^S\}$ for all $\bar{t} \in M$. Then there exists a payoff vector $x = (x_1, \ldots, x_m)$ satisfying equations (4) and (5) if and only if

$$\sum_{j=1}^{r} \lambda_j c_2^{S_j} \frac{\sum_{\bar{t}\in S_j} p_2^{\bar{t}}}{\sum_{\bar{t}\in M} p_2^{\bar{t}}} \geq c_2^M, \tag{18}$$

for every balanced set $\{S_1, \ldots, S_r\}$ and balancing coefficient $\lambda_1, \ldots, \lambda_r$, if $p_1^{\bar{t}} > \max_{S:\bar{t}\in S}\{p_*^S\}$ for all $\bar{t} \in M$, and

$$\sum_{j=1}^{r} \lambda_j c_1^{S_j} \frac{\sum_{\bar{t}\in S_j} p_1^{\bar{t}}}{\sum_{\bar{t}\in M} p_1^{\bar{t}}} \geq c_1^M, \tag{19}$$

for every balanced set $\{S_1, \ldots, S_r\}$ and balancing coefficient $\lambda_1, \ldots, \lambda_r$, if $p_1^{\bar{t}} < \min_{S:\bar{t}\in S}\{p_*^S\}$ for all $\bar{t} \in M$.

Proof: Assume $p_1^{\bar{t}} > \max_{S:\bar{t}\in S}\{p_*^S\}$ for all $\bar{t} \in M$. From (11) and (13), $C(S, \bar{t}) = f([12]; \Gamma^{S,\bar{t}})$ for all $\bar{t} \in S$. From (10) and (11), for all $S \subseteq M$

$$C(S) = |S| + |S|c_1^S + (1 + c_2^S) \sum_{\bar{t}\in S} p_2^{\bar{t}}. \tag{20}$$

From this and (17), we get (18). □

If we define new games $C_\alpha(S) \equiv c_\alpha^S \sum_{\bar{t}\in S} p_\alpha^{\bar{t}}$ for $\alpha = 1, 2$ then (18) becomes $\sum_{j=1}^{r} \lambda_j C_2(S_j) \geq C_2(M)$ if $p_1^{\bar{t}} > \max_{S:\bar{t}\in S}\{p_*^S\}$. (19) becomes $\sum_{j=1}^{r} \lambda_j C_1(S_j) \geq C_1(M)$ if $p_1^{\bar{t}} < \min_{S:\bar{t}\in S}\{p_*^S\}$. The existence of the core $\mathcal{C}(M, C)$ depends on the existence of the core for (M, C_α) for $\alpha = 1, 2$.

Example 4.2. Let $M = \{\bar{1}, \bar{2}\}$. Then (18) becomes to

$$(c_2^{\bar{1}} - c_2^{\bar{2}})(p_2^{\bar{1}} - p_2^{\bar{2}}) \geq 0, \text{ if } p_1^{\bar{1}} > \max(p_*^{\bar{1}}, p_*^{\overline{12}}), p_1^{\bar{2}} > \max(p_*^{\bar{2}}, p_*^{\overline{12}}), \tag{21}$$

and (19) becomes to

$$(c_1^{\bar{1}} - c_1^{\bar{2}})(p_1^{\bar{1}} - p_1^{\bar{2}}) \geq 0, \text{ if } p_1^{\bar{1}} < \min(p_*^{\bar{1}}, p_*^{\overline{12}}), p_1^{\bar{2}} < \min(p_*^{\bar{2}}, p_*^{\overline{12}}), \tag{22}$$

4.2. Merger of Gluss's search problems

In this subsection we review the model in [Gluss 1961] and tries to consider mergers. First we state the model in [Gluss 1961]. The set of nodes is $N = \{0, 1, \ldots, n\}$ and the set of edges is $E = \{(i, i+1) : 0 \leq i \leq n-1\}$, where a seeker starts at the node 0. A priori probabilities are $p_i = \frac{2i}{n(n+1)}, 1 \leq i \leq n$. The examination cost is $g > 0$ for every node except for the node 0. The strategies for the seeker are restricted to two kinds of permutations. The first is, for $1 \leq r \leq n$,

$$\sigma^r(k) = \begin{cases} r + k - 1, & \text{if } 1 \leq k \leq n + 1 - r; \\ n - k + 1, & \text{if } n + 2 - r \leq k \leq n. \end{cases} \tag{23}$$

The second is, for $1 \leq u \leq n-2$,

$$\sigma_u(k) = \begin{cases} k, & \text{if } 1 \leq k \leq u; \\ n - k + u + 1, & \text{if } u + 1 \leq k \leq n. \end{cases} \tag{24}$$

He found optimal strategies in the restricted strategies as follows:

$$\begin{aligned} g \leq \frac{4}{n} &\Longrightarrow \sigma^1 \text{ is optimal.} \\ 2(n-1) \leq g &\Longrightarrow \sigma^n \text{ is optimal.} \\ \frac{4}{n} \leq g \leq 2(n-1) &\Longrightarrow \sigma^r \text{ is optimal,} \end{aligned} \tag{25}$$

where

$$r \equiv r(g) = \left[\frac{(n+2)g + 4}{g + 4}\right].$$

Now, we consider the merger of m seekers, and let $g^{\bar{t}}$ be the examination cost for every nodes for the seeker $\bar{t} \in M$. The expected costs under optimal strategies for the seeker $\bar{t}$

become

$$C(\bar{t}) = \min_{1\le r\le n} f(\sigma^r;\Gamma^{\bar{t}})$$

$$= \begin{cases} f(\sigma^1;\Gamma^{\bar{t}}) = \frac{2n+1}{3}(1+g^{\bar{t}}), & \text{if } g^{\bar{t}} \le \frac{4}{n}; \\ f(\sigma^n;\Gamma^{\bar{t}}) = \frac{4n-1}{3} + \frac{n+2}{3}g^{\bar{t}}, & \text{if } 2(n-1) \le g^{\bar{t}}; \\ f(\sigma^r;\Gamma^{\bar{t}}) = \frac{2n+1}{3} + \frac{2r(r-1)}{n+1} - \frac{2r(r-1)(2r-1)}{3n(n+1)} & \\ \quad +(\frac{2n-3r+4}{3} + \frac{r(r-1)(3n-r+2)}{3n(n+1)})g^{\bar{t}}, & \text{if } \frac{4r-4}{n+2-r} \le g^{\bar{t}} \le \frac{4r}{n+1-r}, 2 \le r \le n. \end{cases} \tag{26}$$

Example 4.3. When $n = 5$, the optimal strategies are

$$\begin{aligned} &[12345], & r = 1, \quad & 0 \le g < 0.8, \\ &[23451], & r = 2, \quad & 0.8 \le g < 2, \\ &[34521], & r = 3, \quad & 2 \le g < 4, \\ &[45321], & r = 4, \quad & 4 \le g < 8, \\ &[54321], & r = 5, \quad & 8 \le g. \end{aligned} \tag{27}$$

The expected cost $C(\bar{t})$ is as in Figure 4.2,

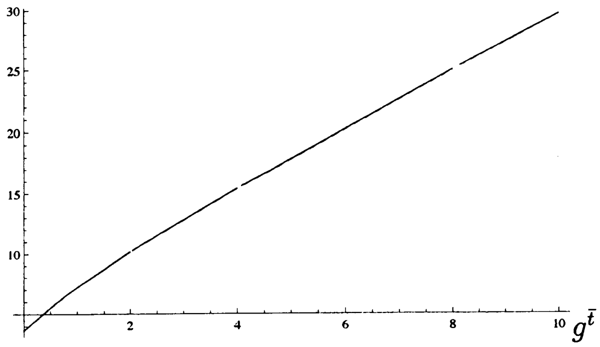

Figure 4.2: Expected cost, $n = 5$

Let c(S) be the amount obtained by replacing $g^{\bar{t}}$ by g^S in (26) where

$$g^S = \frac{1}{|S|}\sum_{\bar{i}\in S} g^{\bar{i}}. \tag{28}$$

Then the cost for the coalition S is $C(S) = |S|c(S)$ since $C(S,\bar{t}) = c(S)$ for all $\bar{t} \in S$. From (27), we see

$$g^{S\cup T} = \frac{1}{|S|+|T|}(|S|g^S + |T|g^T), \tag{29}$$

for $S, T \subseteq N$ such that $S \cap T = \emptyset$. From (29), it is easy to see that if $r(g^S) = r(g^T)$ then $r(g^{S \cup T}) = r(g^S)$. This implies $C(S) + C(T) = C(S \cup T)$.

5. Comments

(i) It is difficult to find an exact solution for a search problem on a finite graph with traveling and examination costs. An alternative way is to give a heuristic solution by simulation, and to calculate a heuristic cost function.

(ii) Another way to approach this search problem is to consider special cases.

References

[1] S. Alpern and S.Gal (2003), *The theory of search games and rendezvous.* Kluwer's INTERNATIONAL SERIES. esp. pp.95-97.

[2] P.Borm, H.Hamers and R.Hendrickx (2001), Operations research games: a survey. *TOP*, pp.139-199.

[3] B. Gluss (1961), Approximately optimal one-dimensional search policies in which search costs vary through time. *Naval Research Logistics*, Vol. 8, pp. 277-283.

[4] K. Kikuta (1990), A one-dimensional search with traveling cost. *Journal of Operations Research Society of Japan*, Vol.33, pp. 262-276.

[5] K.Kikuta (1991), An exactly optimal strategy for a search problem with traveling cost. *Journal of Operations Research Society of Japan*, Vol.34, pp.436-448.

[6] K.Kikuta (2012), On an operations research game related to a search problem. (In Japanese) RIMS Kokyuroku **1802**,pp.153-158.

[7] Lössner, U. and Wegener, I. (1982), Discrete sequential search with positive switch cost. *Math.Oper.Res.* 7: 426-440.

[8] G.Owen (1975), *Game theory*, Third ed.

[9] W. Ruckle (1983), *Geometric games and their applications*, Pitman Research Notes in Mathematics 82, Boston.