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Hyperbolic knots with left-orderable, non–L–space surgeries

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1 Introduction

We say that a nontrivial group G is *left-orderable* if there exists a strict total ordering $<$ on its elements such that $g<h$ implies $fg<fh$ for all elements $f, g, h\in G$. A typical example of a left-orderable group is the infinite cyclic group $\mathbb{Z}$. The left-orderability of fundamental groups of 3–manifolds has been studied by Boyer, Rolfsen and Wiest [3]. In particular, they prove that the fundamental group of a P^{2}–irreducible 3–manifold is left-orderable if and only if it has an epimorphism to a left-orderable group [3, Theorem 1.1(1)]. Since the infinite cyclic group $\mathbb{Z}$ is left-orderable, a P^{2}–irreducible 3–manifold with first Betti number $b_{1}\geq 1$ has a left-orderable fundamental group. One obstruction for G being left-orderable is an existence of torsion elements in G. Thus, for instance, lens spaces, more generally, spherical 3–manifolds cannot have left-orderable fundamental groups. It is interesting to characterize rational homology 3–spheres whose fundamental groups are left-orderable. Examples suggest that there exists a correspondence between 3–manifolds whose fundamental groups are left-orderable and L–spaces which appear in the Heegaard Floer homology theory [28, 29]. Recall that a rational homology 3–sphere Y is called an *L–space* if the rank of its Heegaard Floer homology $\widehat{HF}(Y)$ coincides with $|H_{1}(Y;\mathbb{Z})|$. Following [2, 1.1], for homogeneity, we use $\mathbb{Z}_{2}$–coefficients for $\widehat{HF}(Y)$.

The following conjecture is formulated by Boyer, Gordon and Watson [2].

Conjecture 1.1 *An irreducible rational homology 3–sphere is an L–space if and only if its fundamental group is not left-orderable.*

In [2] the conjecture is verified for geometric, non-hyperbolic 3–manifolds and the 2–fold branched covers of non-splitting alternating links. See also [1, 6, 15, 18, 32] for related results.

A useful way to construct rational homology 3–spheres is Dehn surgery on knots in the 3–sphere S^{3}. For any knot K in S^{3} the exterior $E(K)=S^{3}-\mathrm{int}N(K)$ has the left-orderable fundamental group, and the longitudinal surgery (i.e. 0–surgery) on K yields a 3–manifold with left-orderable fundamental group; see [12, Corollary 8.3] and [3, Theorem 1.1]. On the other hand, the result $K(r)$ of r–Dehn surgery may not have such a fundamental group if $r \neq 0$; see Examples 1.5 and 1.7. A Dehn surgery is said to be *left-orderable* if the resulting manifold of the surgery has the left-orderable fundamental group, and a Dehn surgery is called an *L–space surgery* if the resulting manifold of the surgery is an L–space.

Define the set of left-orderable surgeries on K as

$$\mathcal{S}_{LO}(K) = \{r \in \mathbb{Q} \mid \pi_1(K(r)) \text{ is left-orderable}\}.$$

Similarly define the set of L–space surgeries on K as

$$\mathcal{S}_L(K) = \{r \in \mathbb{Q} \mid K(r) \text{ is an } L\text{–space}\}.$$

In this setting, Conjecture 1.1, together with the cabling conjecture [13], suggests:

Conjecture 1.2 *Let K be a knot in S^3 which is not a cable of a nontrivial knot. Then $\mathcal{S}_{LO}(K) \cup \mathcal{S}_L(K) = \mathbb{Q}$ and $\mathcal{S}_{LO}(K) \cap \mathcal{S}_L(K) = \emptyset$.*

Remark 1.3 *The cabling conjecture [13] asserts that if $K(r)$ is reducible for a nontrivial knot K, then K is cabled and r is a cabling slope. Let us show that there exists a cable knot K for which $\mathcal{S}_{LO}(K) \cup \mathcal{S}_L(K) \neq \mathbb{Q}$. For instance, let K be a (p,q) cable of a non-fibered knot k $(q > 0)$. Then $K(pq) = k(\frac{p}{q}) \sharp L(q,p)$ [14, Corollary 7.3]. Since $\pi_1(K(pq))$ has a torsion, $pq \notin \mathcal{S}_{LO}(K)$. Furthermore, since k is a non-fibered knot, $k(\frac{p}{q})$ is not an L–space [26, 27], and hence $K(pq) = k(\frac{p}{q}) \sharp L(q,p)$ is not an L–space neither; see [34, 8.1(5)] ([29]). It follows that $pq \notin \mathcal{S}_{LO}(K) \cup \mathcal{S}_L(K)$.*

For the trivial knot and nontrivial torus knots, Examples 1.4 and 1.5 describe $\mathcal{S}_{LO}(K)$ and $\mathcal{S}_L(K)$ explicitly. Note that these knots satisfy Conjecture 1.2.

Example 1.4 (trivial knot) *Let K be the trivial knot in S^3. Then $\mathcal{S}_{LO}(K) = \{0\}$ and $\mathcal{S}_L(K) = \mathbb{Q} - \{0\}$.*

Example 1.5 (torus knots) *For a nontrivial torus knot $T_{p,q}$ $(p > q \geq 2)$, the argument in the proof of [8, Theorem 1.4] shows that $\mathcal{S}_{LO}(T_{p,q}) = (-\infty, pq-p-q) \cap \mathbb{Q}$ and $\mathcal{S}_L(T_{p,q}) = [pq - p - q, \infty) \cap \mathbb{Q}$.*

Example 1.6 (figure-eight knot) *Let K be the figure-eight knot. Following [30, 31], $\mathcal{S}_L(K) = \emptyset$. Thus it is expected that $\mathcal{S}_{LO}(K) = \mathbb{Q}$. Boyer, Gordon and Watson [2] show that $\mathcal{S}_{LO}(K) \supset (-4,4) \cap \mathbb{Q}$, and Clay, Lidman and Watson [6] improve that $\mathcal{S}_{LO}(K) \supset [-4,4] \cap \mathbb{Q}$. Furthermore, [11] implies that $\mathcal{S}_{LO}(K) \supset \mathbb{Z}$.*

Example 1.7 (pretzel knot $P(-2,3,7)$) *Let K be a pretzel knot $P(-2,3,7)$. Then since the genus of $P(-2,3,7)$ is 5, [31, Proposition 9.6] ([17, Lemma 2.13]) implies that $\mathcal{S}_L(K) = [9,\infty) \cap \mathbb{Q}$. Hence it is expected that $\mathcal{S}_{LO}(K) = (-\infty, 9) \cap \mathbb{Q}$. While Clay and Watson [9, Theorem 28] prove that $\mathcal{S}_{LO}(K) \subset (-\infty, 17] \cap \mathbb{Q}$.*

For further related results, see [7, 16, 21, 35, 37].

In the present note, we will focus on left-orderable, non–L–space surgeries on knots in S^3. We will introduce a "periodic construction" (Theorem 2.1) which enables us to provide infinitely many hyperbolic knots having left-orderable, non–L–space surgeries from a given knot with left-orderable surgeries. See Theorem 2.1 for the precise statement.

In Sections 3, we will give some examples illustrating how the periodic construction works. In Section 4 we will apply the "periodic construction" with the help of Proposition 4.1 in [8] to demonstrate the following result.

Theorem 1.8 *There exist infinitely many hyperbolic knots K each of which enjoys the following properties.*

(1) *$K(r)$ is a hyperbolic 3–manifold for all $r \in \mathbb{Q}$.*

(2) $\mathcal{S}_{LO}(K) = \mathbb{Q}$.

(3) $\mathcal{S}_L(K) = \emptyset$.

2 Periodic constructions

The construction of knots in Theorem 1.8 is based on the following theorem. For a subset $\mathcal{S} \subset \mathbb{Q}$ and a positive integer p, we denote by $p\mathcal{S}$ the subset $\{pr \mid r \in \mathcal{S}\} \subset \mathbb{Q}$. Note that if $\mathcal{S} = \mathbb{Q}$, then $p\mathcal{S} = \mathbb{Q}$.

Theorem 2.1 (periodic construction) *Let $\overline{K}$ be a knot in S^3 and $\overline{C}$ an unknotted circle which is disjoint from $\overline{K}$. If $\overline{K}$ is a fibered knot, $\overline{C}$ satisfies the inequality $|\overline{S} \cap \overline{C}| > lk(\overline{K}, \overline{C})$ for any fiber surface (i.e. minimal genus Seifert surface) $\overline{S}$. Let p be an integer such that $p \geq 2$ and $(p, lk(\overline{K}, \overline{C})) = 1$. Take the p–fold cyclic branched cover of S^3 branched along $\overline{C}$ to obtain a periodic knot $K_{\overline{C}}^{\frac{p}{}}$ which is the preimage of $\overline{K}$. Then $K_{\overline{C}}^{\frac{p}{}}$ enjoys the following properties:*

(1) $\mathcal{S}_{LO}(K_{\overline{C}}^{\frac{p}{}}(s) \supset p\mathcal{S}_{LO}(\overline{K})$.

(2) $\mathcal{S}_L(K_{\overline{C}}^{\frac{p}{}}(s)) = \emptyset$.

If $\overline{K}$ is a trivial knot, then $\mathcal{S}_{LO}(\overline{K}) = \{0\}$ and hence $p\mathcal{S}_{LO}(\overline{K}) = \{0\}$. So we will apply Theorem 2.1 in the case where $\overline{K}$ is nontrivial.

The first assertion follows from the "inheritance" property of left-orderability: The fundamental groups of 3–manifolds obtained by Dehn surgeries on a periodic knot K inherit the left-orderability from those of 3–manifolds obtained by Dehn surgeries on the factor knot $\overline{K}$.

Theorem 2.2 *Let K be a nontrivial knot in S^3 with cyclic period p, and let $\overline{K}$ be its factor knot. Then $\mathcal{S}_{LO}(K) \supset p\mathcal{S}_{LO}(\overline{K})$.*

The second assertion in Theorem 2.1 follows from the next result whose proof is based on Ni's result [26, 27].

Theorem 2.3 *Let K be a periodic knot in S^3 with the axis C, and let $\overline{K}$ be its factor knot with the branch circle $\overline{C}$. Suppose that K has an L–space surgery. Then $E(\overline{K})$ has a fibering over the circle with a fiber surface $\overline{S}$ such that $|\overline{S} \cap \overline{C}|$ equals the algebraic intersection number between $\overline{S}$ and $\overline{C}$, i.e. the linking number $lk(\overline{K}, \overline{C})$.*

In particular, we have:

Corollary 2.4 *Let K be a periodic knot with the factor knot $\overline{K}$. If $\overline{K}$ is not fibered, then $\mathcal{S}_L(K) = \emptyset$.*

As Ni [26, 27] proves, the fiberedness of K is necessary for K having an L–space surgery. On the other hand, the periodicity of K itself also puts strong restrictions on 3–manifolds obtained by Dehn surgeries on K. For instance, if a periodic knot K with period $p > 2$ has a finite surgery, which is also an L–space surgery, then K is a torus knot or a cable of a torus knot [23, Proposition 5.6]. So we would like to ask:

Question 2.5 *Let K be a knot in S^3 with cyclic period $p > 2$ other than a torus knot, a cable of a torus knot. Then does K admit an L–space surgery?*

For proofs of the above results, see [24].

Remark 2.6 *We denote the genus of a knot k in S^3 by $g(k)$. For $\overline{K}$ and $K^{\frac{p}{C}}$, we have $g(K^{\frac{p}{C}}) \geq pg(\overline{K})$ [25, Theorem 3.2].*

When we apply Theorem 2.1 to a given nontrivial (not necessarily hyperbolic) knot $\overline{K}$, there are infinitely many choices for $\overline{C}$, and we can expect that in most cases, $K^{\frac{p}{C}}$ are hyperbolic knots and $K^{\frac{p}{C}}(s)$ are hyperbolic 3–manifolds. In fact, we can prove the following. See [24] for the proof.

Theorem 2.7 *For a given nontrivial knot $\overline{K}$ in S^3, we have the following.*

(1) *There are infinitely many unknotted circles $\overline{C}$ such that $\overline{K} \cup \overline{C}$ is a hyperbolic link.*

(2) *If $\overline{K} \cup \overline{C}$ is a hyperbolic link and $p > 2$, then $K^p_{\overline{C}}$ is a hyperbolic knot, and $K^p_{\overline{C}}(r)$ is a hyperbolic 3–manifold for all $r \in \mathbb{Q}$.*

(3) *Assume that $p > 2$ and $\overline{C_i}$ $(i = 1, 2)$ is an unknotted circle such that $lk(\overline{K}, \overline{C_i})$ and p are relatively prime, and $\overline{K} \cup \overline{C_i}$ is a hyperbolic link. If $K^p_{\overline{C_1}}$ and $K^p_{\overline{C_2}}$ are isotopic in S^3, then $\overline{K} \cup \overline{C_1}$ and $\overline{K} \cup \overline{C_2}$ are isotopic.*

3 Examples

In this section, we present two examples illustrating how the periodic construction works according as the initial knot $\overline{K}$ is fibered or not fibered.

First we apply Theorem 2.1 in the case where $\overline{K}$ is not fibered. In such a case we can choose $\overline{C}$ arbitrarily with $lk(\overline{K}, \overline{C}) \neq 0$ to obtain a knot $K^p_{\overline{C}}$ having properties (1) and (2) in Theorem 2.1.

Let T_n $(n \neq 0, \pm 1)$ be a twist knot illustrated in Figure 3.1.

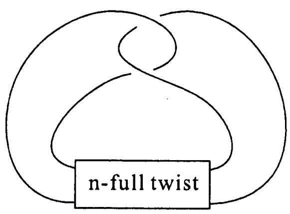

図 3.1: A twist knot T_n

Then T_n is a hyperbolic knot, and since the Alexander polynomial of T_n is not monic, it is not fibered [4, 8.16 Proposition]. Suppose that $n > 1$. Then it follows from [37, 16] that $\pi_1(T_n(r))$ is left-orderable for $r \in (-4n, 4)$. Furthermore, it is known by [35] that $\pi_1(T_n(4))$ is left-orderable. Hence $\mathcal{S}_{LO}(T_n(r)) \supset (-4n, 4] \cap \mathbb{Q}$.

Example 3.1 *Let us take a 2–component link $T_2 \cup \overline{C}$ as in Figure 3.2; $lk(T_2, \overline{C}) = 1$. Let p be any integer with $p > 2$. Take the p–fold cyclic branched cover of S^3 branched along $\overline{C}$ to obtain a knot $K^p_{2,\overline{C}}$ which is the preimage of T_2. Then $K^p_{2,\overline{C}}$ enjoys the following properties:*

(1) $K^p_{2,\overline{C}}$ *is a hyperbolic knot in* S^3.

(2) $K^p_{2,\overline{C}}(r)$ *is a hyperbolic* 3*-manifold for all* $r \in \mathbb{Q}$.

(3) $\mathcal{S}_{LO}(K^p_{2,\overline{C}}) \supset (-8p, 4p] \cap \mathbb{Q}$.

(4) $\mathcal{S}_L(K^p_{2,\overline{C}}) = \emptyset$.

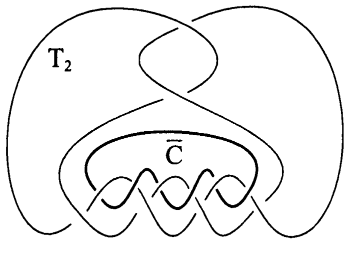

図 3.2: The twist knot T_2 and an axis $\overline{C}$

Proof. Assertions (1) and (2) follow from Theorem 2.7(2) once we show that $T_2 \cup \overline{C}$ is a hyperbolic link. Since $T_2 \cup \overline{C}$ is a non-split prime alternating link [22, Theorem 1], it is either a torus link or a hyperbolic link [22, Corollary 2]. The former cannot happen, because T_2 is not a torus knot. Hence $T_2 \cup \overline{C}$ is a hyperbolic link as desired. Since T_2 is not fibered and $\pi_1(T_2(r))$ is left-orderable for $r \in (-8, 4]$, assertions (3) and (4) follow from Theorem 2.1. □(Example 3.1)

Next we apply Theorem 2.1 to the trefoil knot $T_{-3,2}$, which is a fibered knot. As described in Example 1.5, $\mathcal{S}_{LO}(T_{3,2}) = (-\infty, 1) \cap \mathbb{Q}$. Since $T_{3,2}(r)$ is orientation reversingly diffeomorphic to $T_{-3,2}(-r)$, we see that $\mathcal{S}_{LO}(T_{-3,2}) = (-1, \infty) \cap \mathbb{Q}$.

Example 3.2 *Let us take a* 2*-component link* $T_{-3,2} \cup \overline{C}$ *as in Figure 3.3;* $lk(T_{-3,2}, \overline{C}) = 1$. *Let* p *be any integer with* $p > 2$. *Take the* p*-fold cyclic branched cover of* S^3 *branched along the trivial knot* $\overline{C}$ *to obtain a knot* $K^p_{-3,2,\overline{C}}$ *which is the preimage of* $T_{-3,2}$. *Then* $K^p_{-3,2,\overline{C}}$ *enjoys the following properties:*

(1) $K^p_{-3,2,\overline{C}}$ *is a hyperbolic knot in* S^3.

(2) $K^p_{-3,2,\overline{C}}(r)$ *is a hyperbolic* 3*-manifold for all* $r \in \mathbb{Q}$.

(3) $\mathcal{S}_{LO}(K^p_{-3,2,\overline{C}}) \supset (-p, \infty) \cap \mathbb{Q}$.

(4) $\mathcal{S}_L(K^p_{-3,2,\overline{C}}) = \emptyset$.

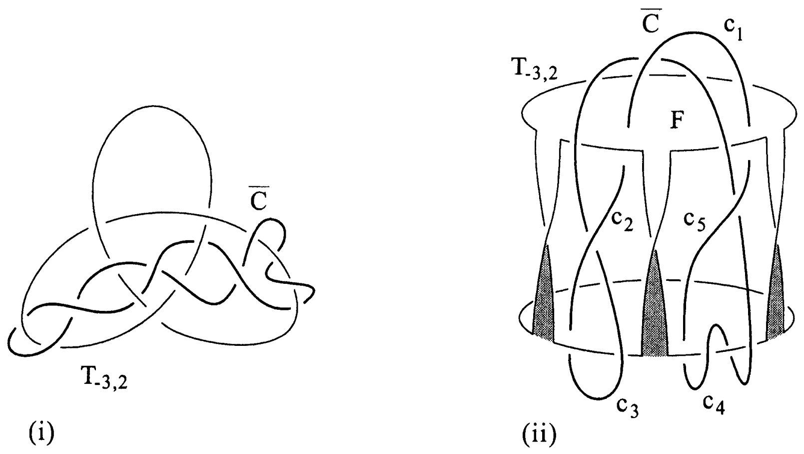

図 3.3: The trefoil knot $T_{-3,2}$ and an unknotted circle $\overline{C}$

Proof of Example 3.2. Assertions (1) and (2) follow from Theorem 2.7(2) once we see that $T_{-3,2} \cup \overline{C}$ is a hyperbolic link. Since as illustrated in Figure 3.3(i) $T_{-3,2} \cup \overline{C}$ is a non-split prime alternating link [22, Theorem 1], it is either a torus link or a hyperbolic link [22, Corollary 2]. If we have the former case, then $T_{-3,2}$ is isotopic to $\overline{C}$ which is a trivial knot, a contradiction. Hence $T_{-3,2} \cup \overline{C}$ is a hyperbolic link as desired.

To see (3) and (4), we apply Theorem 2.1. Since $T_{-3,2}$ is fibered, we need to show that for any fiber surface $\overline{S}$ of $E(T_{-3,2})$, $|\overline{S} \cap \overline{C}|$ is strictly bigger than the algebraic intersection number between $\overline{S}$ and $\overline{C}$, i.e. $lk(T_{-3,2}, \overline{C})$.

In Figure 3.3(ii), we give a minimal genus Seifert surface F of $T_{-3,2}$, which is a once-punctured torus with $\partial F = T_{-3,2}$. Put $\overline{S} = F \cap E(T_{-3,2})$. Then by [10, Lemma 5.1] $\overline{S}$ is a fiber surface of $E(T_{-3,2})$. We see that $|\overline{S} \cap \overline{C}| = 5$ and the algebraic intersection number between $\overline{S}$ and $\overline{C}$ is one. Assume for a contradiction that we have another fiber surface $\overline{S}'$ of $E(T_{-3,2})$ such that $|\overline{S}' \cap \overline{C}| < |\overline{S} \cap \overline{C}|$. Since $\overline{S}$ and $\overline{S}'$ are fiber surfaces of $E(T_{-3,2})$, they are isotopic; see [10, Lemma 5.1], [36]. This then implies that we can isotope $\overline{C}$ to $\overline{C}'$ in $E(T_{-3,2})$ so that $|\overline{S} \cap \overline{C}'| < |\overline{S} \cap \overline{C}|$.

Claim 3.3 *There exists a smooth map φ from a semi-disk D into $E(T_{-3,2})$ such that $\varphi^{-1}(\overline{C})$ is an arc $c \subset \partial D$ and $\varphi^{-1}(\overline{S})$ is the arc $\alpha = \overline{\partial D - c}$.*

Proof of Claim 3.3. Let $\Phi : S^1 \times [0,1] \to E(T_{-3,2})$ be a smooth map giving an isotopy between $\overline{C}(= \Phi(S^1 \times \{0\}))$ to $\overline{C}'(= \Phi(S^1 \times \{1\}))$. We may assume Φ is transverse to $\overline{S}$. Furthermore, the essentiality of $\overline{S}$ in $E(T_{-3,2})$ enables us to modify Φ to eliminate the circle components as usual. Since $|\overline{S} \cap \overline{C}'| < |\overline{S} \cap \overline{C}| = 5$ and the algebraic intersection number between $\overline{S}$ and $\overline{C}'$ coincides with the algebraic intersection number between $\overline{S}$ and $\overline{C}$, we have $|\overline{S}\cap\overline{C}'| = 1$ or 3. Thus $\Phi^{-1}(\overline{S})$ consists of three properly embedded arcs α, α' and β, where $\partial\alpha \subset S^1\times\{0\}$, $\partial\alpha' \subset S^1\times\{0\}$, and β connects $S^1\times\{0\}$ and $S^1\times\{1\}$ (Figure 3.4(i), (ii)), consists of four properly embedded arcs α, β, β' and β'', where $\partial\alpha \subset S^1 \times \{0\}$, and each of β, β', β'' connects $S^1 \times \{0\}$ and $S^1 \times \{1\}$ (Figure 3.4(iii)), or consists of four properly embedded arcs α, α', β and γ, where $\partial\alpha \subset S^1 \times \{0\}$, $\partial\alpha' \subset S^1 \times \{0\}$, β connects $S^1 \times \{0\}$ and $S^1 \times \{1\}$, and $\partial\gamma \subset S^1 \times \{1\}$ (Figure 3.4(iv), (v)). In either case there is a semi-disk D cobounded by α and an arc $c \subset S^1 \times \{0\}$.

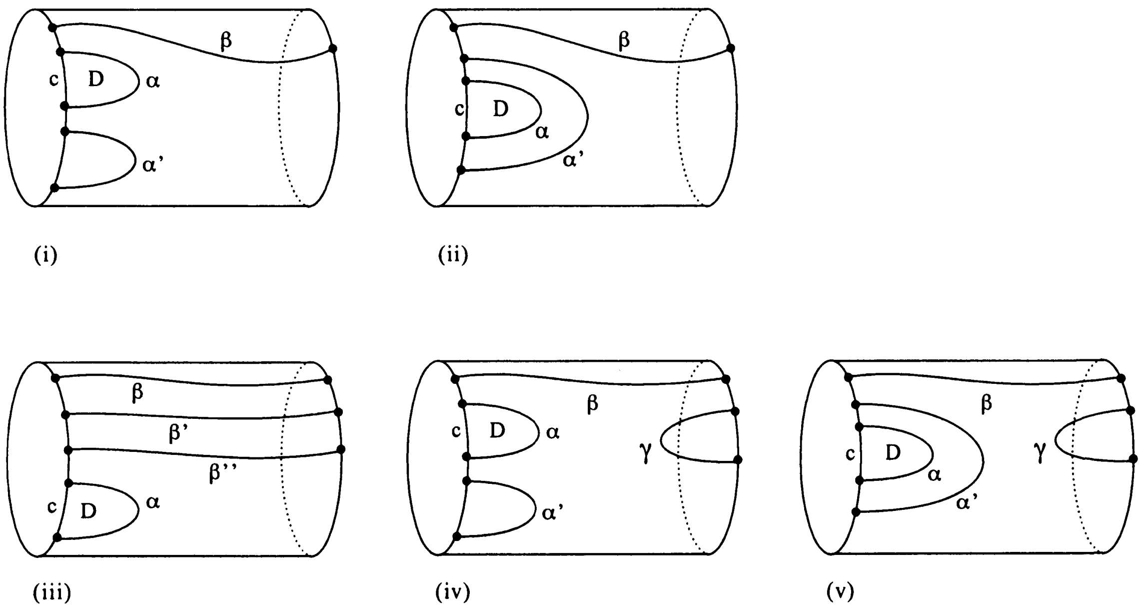

図 3.4: $\Phi^{-1}(\overline{S})$ in $S^1 \times [0,1]$

Putting $\varphi = \Phi|_D : D \longrightarrow E(T_{-3,2})$, we obtain a desired smooth map. □(Claim 3.3)

Cut open $E(T_{-3,2})$ along $\overline{S}$ to obtain a product 3–manifold $\overline{S} \times [0,1]$. The circle $\overline{C}$ is cut into five arcs c_1, c_2, c_3, c_4 and c_5 as in Figure 3.3(ii). Note that $\partial c_1 \subset \overline{S} \times \{0\}$, $\partial c_3 \subset \overline{S} \times \{1\}$, and each of c_2, c_4, c_5 connects $\overline{S} \times \{0\}$ and $\overline{S} \times \{1\}$. Moreover, we see that c_1 and c_3 are linking once relative their boundaries.

On the other hand, since c is either c_1 or c_3, Claim 3.3 shows that c_1 and c_3 are unlinked relative their boundaries. This contradiction shows that for any fiber surface $\overline{S}$, $|\overline{S}\cap\overline{C}| = 5$ and $|\overline{S} \cap \overline{C}| > lk(T_{-3,2}, \overline{C})$.

Since $\pi_1(T_{-3,2}(r))$ is left-orderable if $r \in (-1, \infty)$, the conclusions (3) and (4) follow fromTheorem 2.1. This completes the proof of Example 3.2. □(Example 3.2)

For any fibered knot K in S^3 a minimal genus Seifert surface in $E(K)$ is a fiber surface and vice versa, and furthermore, any fiber surface is unique up to isotopy. See [10, Lemma 5.1], [36].

4 Proofs of Theorems 1.8.

The goal of this section is to prove Theorems 1.8.

Proof of Theorem 1.8. Let us consider the connected sum $T_{-3,2} \sharp T_{3,2}$. We recall the following well-known general fact.

Claim 4.1 *Let $K_1, \dots, K_n$ be nontrivial knots. Then $(K_1 \sharp \cdots \sharp K_n)(r)$ is irreducible for all $r \in \mathbb{Q}$.*

Proof of Claim 4.1. First we note that the exterior $E(K_1 \sharp \cdots \sharp K_n)$ is a union of a composing space C_n (i.e. [disk with n – holes] $\times S^1$) and $E(K_1), \dots, E(K_n)$. Hence for any $r \in \mathbb{Q}$, $(K_1 \sharp \cdots \sharp K_n)(r)$ is a union of $C_n \cup V$ and $E(K_1), \dots, E(K_n)$, where V is a filled solid torus. Note that $C_n \cup V$ has a Seifert fibration over the disk with $(n-1)$–holes with at most one exceptional fiber, and hence it is irreducible and boundary-irreducible. Then since $C_n \cup V$ and $E(K_i)$ $(1 \leq i \leq n)$ are irreducible and boundary-irreducible, $(K_1 \sharp \cdots \sharp K_n)(r)$ is also irreducible. □(Claim 4.1)

Let us regard $T_{-3,2} \sharp T_{3,2}$ as a satellite knot with the companion knot $T_{3,2}$ and the pattern knot $T_{-3,2}$. Since $\pi_1(T_{-3,2}(r))$ is left-orderable if $r > -1$ [8], and $(T_{-3,2} \sharp T_{3,2})(r)$ is irreducible for all $r \in \mathbb{Q}$ (Claim 4.1), Proposition 4.1 in [8] shows that $\pi_1((T_{-3,2} \sharp T_{3,2})(r))$ is also left-orderable if $r > -1$. Using the amphicheirality of $T_{-3,2} \sharp T_{3,2}$, we see that $\pi_1((T_{-3,2} \sharp T_{3,2})(r))$ is left-orderable also when $r < 1$. Therefore it is left-orderable for all $r \in \mathbb{Q}$. Note that $T_{-3,2} \sharp T_{3,2}$ is a fibered knot.

Before we apply Theorem 2.1, for ease of handling, take the connected sum $(T_{-3,2} \sharp T_{3,2}) \sharp T_2$. The Alexander polynomial of $(T_{-3,2} \sharp T_{3,2}) \sharp T_2$ is $(t^2 - t + 1)^2(2t^2 - 5t + 2)$, which is not monic, and hence $(T_{-3,2} \sharp T_{3,2}) \sharp T_2$ is not fibered. We regard $(T_{-3,2} \sharp T_{3,2}) \sharp T_2$ as a satellite knot with the companion knot T_2 and the pattern knot $T_{-3,2} \sharp T_{3,2}$. As we observe above, $\pi_1((T_{-3,2} \sharp T_{3,2})(r))$ is left-orderable for all $r \in \mathbb{Q}$. Moreover by Claim 4.1 $(T_{-3,2} \sharp T_{3,2} \sharp T_2)(r)$ is irreducible for all $r \in \mathbb{Q}$. We apply [8, Proposition 4.1] again to conclude that $(T_{-3,2} \sharp T_{3,2} \sharp T_2)(r)$ has the left-orderable fundamental group for all $r \in \mathbb{Q}$.

To obtain hyperbolic knots with this property, we will apply the periodic construction (Theorem 2.1). Let us put $\overline{K} = T_{-3,2} \sharp T_{3,2} \sharp T_2$ and take an unknotted circle $\overline{C}$ as in Figure 4.1; $lk(\overline{K}, \overline{C}) = 1$.

Since $\overline{K} \cup \overline{C}$ is a non-split prime alternating link [22, Theorem 1], it is either a torus link or a hyperbolic link [22, Corollary 2]. The former cannot happen, because $\overline{K}$ is not a

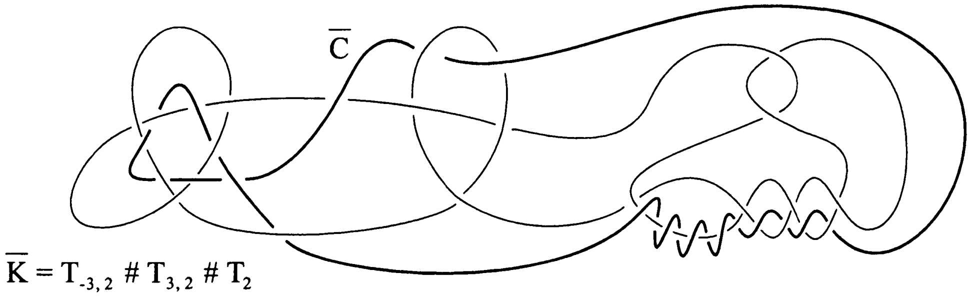

図 4.1: $\overline{K} \cup \overline{C}$

torus knot. Hence $\overline{K} \cup \overline{C}$ is a hyperbolic link. Let $p > 2$ be any integer. Take the p–fold cyclic branched cover of S^3 branched along $\overline{C}$ to obtain a periodic knot $K_{\overline{C}}^{p}$ which is the preimage of $\overline{K}$.

It follows from Theorem 2.1 and Theorem 2.7(2) that $K_{\overline{C}}^{p}$ is a hyperbolic knot and enjoys the properties (1), (2) and (3) in Theorem 1.8. By changing p, we obtain infinitely many such knots. For instance, see Remark 2.6. □(Theorem 1.8)

Remark 4.2 (1) *By Theorem 2.7 there are infinitely many unknotted circles for* $\overline{K} = T_{-3,2} \sharp T_{3,2} \sharp T_2$*, and for each unknotted circle* $\overline{C}$ *we obtain infinitely many hyperbolic knots* $K_{\overline{C}}^{p}$*, where* p *and* $lk(\overline{K}, \overline{C})$ *are relatively prime.*

(2) *Recall that any knot* K *obtained by the "periodic construction", for instance a knot obtained in the proof of Theorem 1.8, is not fibered and every nontrivial surgery on* K *is a left-orderable, non–L–space surgery. So we can apply Theorem 2.1 again to the knot* K *and an arbitrarily chosen unknotted circle to obtain yet further infinitely many non-fibered knots* K' *each of which has the (same) factor knot* K*. Then* r*–surgery on* K' *is also a left-orderable, non–L–space surgery for all* $r \in \mathbb{Q}$*. We can apply this procedure repeatedly arbitrarily many times.*

(3) *Let* K *be the knot* 10_{99} *in Rolfsen's knot table [33]. Recently Clay [5] uses an epimorphism from* $E(K)$ *to* $E(T_{3,2})$ *which preserves the peripheral subgroup [20] to show that every nontrivial surgery on* K *is left-orderable surgery. Since* K *has no cyclic period [19, Appendix F], this example cannot be explained by the periodic construction.*

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