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The colored HOMFLY homology and super-A-polynomial

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Abstract

In this article we will discuss the generalizations of the A-polynomial on basis of the colored superpolynomial. The colored superpolynomial is the Poincaré polynomial of the colored HOMFLY homology that categorifies the colored HOMFLY polynomial. In particular for the colored superpolynomial with the symmetric representation, the analogues of the generalized/quantum volume conjecture can be considered. As a result of the study of the colored HOMFLY homology for the $(2,2p+1)$-torus knots and n-twist knots, we find the 2-parameter deformation of the A-polynomial that is named as *super-A-polynomial.*

1 Introduction

Let $\mathcal{C}_K$ be the character variety of the knot K. This variety is defined to be the moduli space of the $SL_2(\mathbb{C})$ holonomy representation ρ of the knot complement $\mathbb{S}^3\backslash K$:

$$\mathrm{Hom}(\pi_1(K);SL_2(\mathbb{C}))/\mathrm{conjugation}. \tag{1}$$

$\mathcal{C}_K$ is the algebraic curve in the variables $(x,y)\in\mathbb{C}^*\times\mathbb{C}^*$ which are given by the eigenvalues of the holonomy matrices for the meridian and longitude elements $\mu,\lambda\in\pi_1(K)$:

$$\rho(\mu)=\begin{pmatrix} x & * \\ 0 & x^{-1}\end{pmatrix},\quad \rho(\lambda)=\begin{pmatrix} y & * \\ 0 & y^{-1}\end{pmatrix}. \tag{2}$$

The defining equation of $\mathcal{C}_K$ is given by the *A-polynomial* $A_K(x, y)$ [8]:

$$\mathcal{C}_K = \{(x, y) \in \mathbb{C}^* \times \mathbb{C}^* | A_K(x, y) = 0\}. \tag{3}$$

As for example, the A-polynomials for $\mathbf{3_1}$ and $\mathbf{4_1}$ are found manifestly from the generator relations of $\pi_1(K)$:

$$A_{\mathbf{3_1}}(x, y) = (y-1)(y+x^3), \tag{4}$$

$$A_{\mathbf{4_1}}(x, y) = (y-1)\big(y^2 + (-x^4 + x^3 + 2x^2 + x - 1)y + 1\big). \tag{5}$$

In the *generalized volume conjecture*, the colored Jones polynomial and A-polynomial are related in the asymptotic limit.

Conjecture 1.1 *([27, 33, 23]) Let $J_n(K; q)$ be an n-colored Jones polynomial. In the asymptotic limit:*

$$q = e^{\hbar} \to 1, \quad n \to \infty, \quad x := q^n \text{ (fixed)}, \tag{6}$$

there exist a saddle point such that the colored Jones polynomial grows exponentially:

$$J_n(K; q = e^{\hbar}) \simeq \exp\left(\frac{1}{\hbar} S_0(x) + \cdots\right). \tag{7}$$

The leading coefficient $S_0(x)$ satisfies the relation:

$$x \frac{\partial S_0(x)}{\partial x} = \log y(x), \quad A_K(x, y(x)) = 0. \tag{8}$$

The quantum version of the volume conjecture (which is also known as *AJ conjecture*) also relates the colored Jones polynomial and A-polynomial via the q-difference equation:

Conjecture 1.2 *([23, 20]) The colored Jones polynomial obeys the q-difference equation:*

$$\hat{A}_K(\hat{x}, \hat{y}; q) J_n(K; q) = 0, \tag{9}$$

$$\hat{x} J_n(K; q) = q^n J_n(K; q), \quad \hat{y} J_n(K; q) = J_{n+1}(K; q), \quad \hat{y}\hat{x} = q\hat{x}\hat{y}. \tag{10}$$

such that the q-difference operator $\hat{A}_K(\hat{x}, \hat{y}; q)$ yields in the limit $q \to 1$ of (6):

$$\hat{A}_K(\hat{x}, \hat{y}; q) \xrightarrow{q \to 1} A_K(x, y). \tag{11}$$

This conjecture is verified directly for some knots via computer talks [20], and the q-difference operator $\hat{A}_K(\hat{x}, \hat{y}; q)$ can be found explicitly.

In recent developments of the string theory, some extensions of the volume conjecture have been studied. In [1], the analogue of the generalized/quantum volume conjecture are also proposed for the colored HOMFLY polynomial:

Conjecture 1.3 *([1]) Let $H_n(K;a,q)$ be a colored HOMFLY polynomial of the r-th completely symmetric representation S^r $(n=r+1)$. In the asymptotic limit:*

$$q=e^{\hbar}\to 1,\quad n\to\infty,\quad a,\ x:=q^n\ (\text{fixed}), \tag{12}$$

there exist a saddle point such that the colored Jones polynomial grows exponentially:

$$H_n(K;a,q=e^{\hbar})\simeq \exp\left(\frac{1}{\hbar}S_0(x;a)+\cdots\right), \tag{13}$$

$$x\frac{\partial S_0(x;a)}{\partial x}=\log y(x),\quad A_K^{Q\text{-def}}(x,y(x);a)=0. \tag{14}$$

The Q-deformed A-polynomial $A_K^{Q\text{-def}}(x,y;a)$ coincides with the augmentation polynomial [35] for the differential graded algebra of the knot contact homology [12].

Conjecture 1.4 *([1]) One finds a q-difference operator $\hat{A}_K^{Q\text{-def}}(\hat{x},\hat{y};a,q)$ that annihilates the colored HOMFLY polynomial:*

$$\hat{A}_K(\hat{x},\hat{y};q)H_n(K;a,q)=0, \tag{15}$$

where $\hat{x}$ and $\hat{y}$ acts on $H_n(K;a,q)$ as (10), and the q-difference operator $\hat{A}_K^{Q\text{-def}}(\hat{x},\hat{y};a,q)$ reduces to the Q-deformed A-polynomial $A_K^{Q\text{-def}}(x,y;a)$ in the limit (12):

$$\hat{A}_K^{Q\text{-def}}(\hat{x},\hat{y};a,q)\ \stackrel{q\to 1}{\longrightarrow}\ A_K^{Q\text{-def}}(x,y;a). \tag{16}$$

In the context of the topological string, this q-difference equation reveals the D-module structure of the open topological string, and such aspect has been studied in $a\to 1$ limit via the topological recursion on the character variety [9, 10, 25, 5].

On basis of these observations for the $(2,2p+1)$-torus knots and n-twist knots, we expect the further extension of the volume conjecture to the *colored superpolynomial.* The colored superpolynomial $\mathcal{P}^R(K;a,q,t)$. is the Poincaré polynomial of the *colored HOMFLY homology* $\mathcal{H}^R_{ijk}(K)$ [24]:

$$\mathcal{P}^R(K;a,q,t)=\sum_{i,j,k}a^iq^jt^k\dim\mathcal{H}^R_{ijk}(K). \tag{17}$$

The triply-graded homology $\mathcal{H}^R_{ijk}(K)$ categorifies the the colored HOMFLY polynomial $H^R(K;a,q)$, therefore these polynomial invariants are related by

$$\mathcal{P}^R(K;a,q,t=-1)=H^R(K;a,q). \tag{18}$$

In particular for the completely symmetric representation $R=S^r$, the colored superpolynomial $\mathcal{P}_n(K;a,q,t):=\mathcal{P}^{S^{n-1}}(K;a,q,t)$, we can study the natural extension of the quantum volume conjecture.

Conjecture 1.5 *([18]) Let $\mathcal{P}_n(K;a,q,t)$ be the colored superpolynomial $\mathcal{P}^R(K;a,q,t)$ for the completely symmetric representation $R = S^r$ $(r = n-1)$. One finds a q-difference equation:*

$$\hat{A}_K^{\mathrm{super}}(\hat{x},\hat{y};a,q,t)\mathcal{P}_n(K;a,q,t) = 0, \tag{19}$$

such that the q-difference operator reduces to the 2-parameter extended A-polynomial $A_K^{\mathrm{super}}(x,y;a,t)$ in the limit:

$$q = e^{\hbar} \to 1, \quad n \to \infty, \quad a,\ t\ , x := q^n \text{ (fixed)}. \tag{20}$$

$A_K^{\mathrm{super}}(x,y;a,t)$ reduces to the Q-deformed A-polynomial in the specialization $t=-1$, and further reduces to the A-polynomial in the specialization $a = 1,\ t = -1$.

The 2-parameter extended A-polynomial $A_K^{\mathrm{super}}(x,y;a,t)$ defined above is named as the *super-A-polynomial*, and this polynomial is also found via asymptotic expansion of the colored superpolynomial:

$$\mathcal{P}_n(K;a,q=e^{\hbar},t) \simeq \exp\left(\frac{1}{\hbar}S_0(x;a,t) + \cdots\right), \tag{21}$$

$$x\frac{\partial S_0(x;a,t)}{\partial x} = \log y(x), \quad A_K^{\mathrm{super}}(x,y(x);a,t) = 0. \tag{22}$$

Conjecture 1.5 is verified for $K = \mathbf{3_1}, \mathbf{4_1}, \mathbf{5_2}, \mathbf{6_1}$ in [18, 34] via the computer talks on basis of the explicit formula for the colored superpolynomial $\mathcal{P}_n(K;a,q,t)$.

The organization of this article is as follows. In section 2, we will survey briefly on the colored superpolynomial [11, 24, 22] and show some computational results obtained in [19]. In section 3, we will discuss about the super-A-polynomial from the explicit formulae of the colored superpolynomial obtained in section 2, and see their properties. In the appendix, we discuss the string-theoretical perturbative invariant associated to the super-A-polynomial which is found via the topological recursion as is discussed in [10, 5] for the A-polynomial.

2 Colored HOMFLY homology and colored superpolynomial

2.1 Categorification of the HOMFLY polymonial

The categorifications of the Jones polynomial and HOMFLY polynomial are established in the celebrated works [28, 30]. In [30] the doubly-graded homology $\mathcal{H}_{kp}^{sl_N}(K)$ is constructed via the matrix factorization, and categorifies the sl_N invariant which is given by the specialization of the HOMFLY polynomial $H(K;a=q^N,q)$. This homology is known as the *sl_N-Khovanov-Rozansky's homology*. As the decategorification of the Khovanov-Rozansky's homology, one finds the sl_N invariant via the q-graded Euler characteristic:*

$$H(K; a = q^N, q) = \sum_{k,p} (-1)^k q^p \mathrm{dim} \mathcal{H}^{sl_N}_{kp}(K). \tag{23}$$

In order to describe the sl_N homologies in the unified manner, the triply-graded homology $\mathcal{H}_{ijk}(K)$ is proposed in [11, 29]. This homology categorifies the HOMFLY polynomial:

$$H(K; a, q) = \sum_{i,j,k} a^i q^j (-1)^k \mathcal{H}_{ijk}(K), \tag{24}$$

and is named as the *HOMFLY homology*. In particular, the Poincarepolynomial of the HOMFLY homology is known as *superpolynomial*:

$$\mathcal{P}(K; a, q, t) = \sum_{i,j,k} a^i q^j t^k \mathrm{dim} \mathcal{H}_{ijk}(K). \tag{25}$$

The properties of the HOMFLY homology are given as its definition in [11], and they are listed as follows:

Properties of the HOMFLY homology

H1. $\mathcal{H}_*$ categorifies the HOMFLY polynomial.

H2. $\mathcal{H}_*$ has finite support: $\mathrm{dim}\mathcal{H}_* < \infty$

H3. The differentials d_N ($N \in \mathbb{Z}$) acts on $\mathcal{H}_*$, and the grading (ijk) is shifted as:†

$$d_N : \mathcal{H}_{ijk} \to \mathcal{H}_{i-1,j+N,k-1} \ (N > 0) \tag{26}$$

$$d_N : \mathcal{H}_{ijk} \to \mathcal{H}_{i-1,j+N,k-3} \ (N \leq 0) \tag{27}$$

Thus the differential d_N are triply-graded of degree: $\mathrm{deg}d_N = (-1, N, -1)$ $(N > 0)$, and $\mathrm{deg}d_N = (-1, N, -3)$ $(N \leq 0)$.

H4. The differentials are anticommuting each other: $d_N \circ d_M = -d_M \circ d_N$ for all $(N, M \in \mathbb{Z})$. Hence for $N = M$, $d_N^2 = 0$.

H5. There exist an involution $\phi : \mathcal{H}_{ij*} \to \mathcal{H}_{i,-j,*}$. The action of ϕ on d_N is $\phi d_N = d_{-N}\phi$.

H6. Taking homology with respect to d_N ($N \geq 2$), one finds the sl_N Khovanov-Rozansky's homology $\mathcal{H}^{sl_N}$ after the amalgamation:

$$\bigoplus_{iN+j=p} (\mathcal{H}_*, d_N)_{ijk} = \mathcal{H}^{sl_N}_{kp}. \tag{28}$$

*In this article, we discuss the *reduced homology* which is the trivial for the unknot.

†Here we obey the convention of grading in [24]. In [11] another degree is assigned for d_N ($N < 0$) such that $\mathrm{deg}d_N = (-1, N, N-1)$. These two gradings are combined into the quadruply-graded homology [22].

H7. Taking homology with respect to $d_{\pm 1}$, one finds the one dimensional sl_1-homology [31]:

$$(\mathcal{H}_*, d_1) = \mathcal{H}_{S,-S,0}\,, \quad (\mathcal{H}_*, d_{-1}) = \mathcal{H}_{S,S,2S}\,. \tag{29}$$

The grading of the surviving generator is specified by the Rasmussen's invariant S [37] (e.g. $S = (p-1)(q-1)/2$ for (p,q)-torus knot).

H8. Taking homology with respect to d_0, one finds the knot Floer homology HFK [36] after a regrading:

$$\bigoplus_{-2i+k=k'} (\mathcal{H}_*, d_0)_{ijk} = HFK_{k'j}. \tag{30}$$

These properties implies that the HOMFLY homology unifies not only the sl_N Khovanov-Rozansky's homology, but also their deformations [21, 26] and knot Floer homology. Conversely speaking, one can determine the structure of the triply-graded homology $\mathcal{H}_{ijk}$ which satisfies the above properties consistently, and the following conjecture is proposed:

Conjecture 2.1 *([11]) There exists a triply-graded homology $\mathcal{H}_{ijk}$ which satisfies the properties* **H1-H8**.

This conjecture is verified for various knots up to 8 crossings, and the superpolynomials (i.e. homological structure of the triply-graded homology) are specified uniquely.

2.2 Colored HOMFLY homology

The categorification of the colored HOMFLY polynomial $H^R(K;a,q)$ is proposed in [24]. The triply-graded homology $\mathcal{H}^R_{ijk}$ which categorifies $H^R(K;a,q)$ is named as the *colored HOMFLY homology*:

$$H^R(K;a,q) = \sum_{ijk} a^i q^j (-1)^k \dim \mathcal{H}^R_{ijk}(K). \tag{31}$$

Here we focus on the colored HOMFLY homology in the completely symmetric representation $R = S^r$. The properties of $\mathcal{H}^{S^r}_{ijk}$ are proposed in [24], and they consists of the natural generalization of the properties **H1-H7** and new aspects are introduced via the colored differentials. Such properties listed as follows:

Properties of the colored HOMFLY homology

C1. $\mathcal{H}^{S^r}_*$ categorifies the colored HOMFLY polynomial $H^{S^r}(K;a,q)$.

C2. $\mathcal{H}_*$ has finite support: $\dim \mathcal{H}_* < \infty$

C3. The differentials $d_N^{S^r}$ $(N \in \mathbb{Z})$ acts on $\mathcal{H}_*$ with degrees:

$$\mathrm{deg} d_N^{S^r} = (-1, N, -1) \;\; (N \geq 1-r), \quad \mathrm{deg} d_N^{S^r} = (-1, N, -3) \;\; (N \leq -r). \tag{32}$$

C4. The differentials are anticommuting each other: $d_N^{S^r} \circ d_M^{S^r} = -d_M^{S^r} \circ d_N^{S^r}$ for all $(N, M \in \mathbb{Z})$. Hence for $N = M$, $(d_N^{S^r})^2 = 0$.

C5. There exist an involution $\phi : \mathcal{H}_{ij*}^{S^r} \to \mathcal{H}_{i,-j,*}^{\Lambda^r}$ where Λ^r denotes the r-th anti-symmetric representation such that $(S^r)^t = \Lambda^r$. The involution acts on $d_N^{S^r}$ as: $\phi d_N^{S^r} = d_{-N}^{\Lambda^r} \phi$.

C6. Taking homology with respect to $d_N^{S^r}$ $(N \geq 2)$, one finds the S^r-colored sl_N homology $\mathcal{H}^{sl_N, S^r}$: $(\mathcal{H}_*^{S^r}, d_N^{S^r})_{ijk} \simeq \mathcal{H}^{sl_N, S^r}$.

C7. Taking homology with respect to $d_1^{S^r}$ and $d_{-r}^{S^r}$, one finds the one dimensional homology:

$$(\mathcal{H}_*^{S^r}, d_1^{S^r}) = \mathcal{H}_{rS,-rS,0}^{S^r}, \quad (\mathcal{H}_*^{S^r}, d_{-r}^{S^r}) = \mathcal{H}_{rS,r^2S,2rS}^{S^r}. \tag{33}$$

C8. Taking homology with respect to $d_{1-k}^{S^r}$ $(1 \leq k \leq r)$, one finds the colored HOMFLY homology of the smaller representation $R = S^k$:

$$(\mathcal{H}_*^{S^r}, d_{1-k}^{S^r}) \simeq \mathcal{H}^{S^k}. \tag{34}$$

This class of differentials $d_{1-k}^{S^r}$ are named as the *vertical colored differentials.*

C9. There exists yet another class of the differential $d_{r \to m}$ which reduces the representation $S^r \to S^m$ $(r > m)$:

$$(\mathcal{H}_*^{S^r}, d_{r \to m}) \simeq \mathcal{H}^{S^m}. \tag{35}$$

For example, the degree of $d_{r \to r-1}$ is $(0, 1, 0)$.

In the statement of the properties **C8** and **C9**, one can identify the generators after a re-grading $(i, j, k) \to (i', j', k')$. Although the re-grading rules are not simply described in the triply-graded homology [24], such rules become manifest when we discuss in the framework of the quadruply-graded homology [22], and larger hidden symmetry of the homology can be found.

Assuming these properties as the axioms of the colored HOMFLY homology, one can also determine the triply-graded homology from the consistency of conditions **C1-C9**. Actually, the explicit expressions of the colored superpolynomial $\mathcal{P}^{S^r}(K; a, q, t)$:

$$\mathcal{P}^{S^r}(K; a, q, t) = \sum_{ijk} a^i q^j t^k \mathrm{dim} \mathcal{H}_{ijk}^{S^r}(K) \tag{36}$$

are determined for the $(2,2p+1)$-torus knots $T^{2,2p+1}$ [17, 19] and the twist knots with n crossings TK_n [34, 19]:

$$\begin{aligned}
\mathcal{P}^{S^r}(T^{2,2p+1};a,q,t) &= a^{pr}q^{-pr}\sum_{0\le k_p\le\cdots\le k_2\le k_1\le k_0=r}\begin{bmatrix} r \\ k_1 \end{bmatrix}_q\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}_q\cdots\begin{bmatrix} k_{p-1} \\ k_p \end{bmatrix}_q \\
&\quad\times q^{(2r+1)\sum_{i=1}^p k_i-\sum_{i=1}^p k_{i-1}k_i}t^{2\sum_{i=1}^p k_i}\prod_{i=1}^{k_1}(1+aq^{i-2}t), \qquad (37)\\
\mathcal{P}^{S^r}(TK_{2n+2};a,q,t) &= \sum_{0\le k_n\le\cdots\le k_2\le k_1\le k_0=r}\begin{bmatrix} r \\ k_1 \end{bmatrix}_q\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}_q\cdots\begin{bmatrix} k_{n-1} \\ k_n \end{bmatrix}_q a^{\sum_{i=1}^n k_i}t^{2\sum_{i=1}^n k_i} \\
&\quad\times q^{\sum_{i=1}^n(k_i^2-k_i)}\prod_{i=1}^{k_1}(1+a^{-1}q^{2-i}t^{-1})(1+a^{-1}q^{1-r-i}t^{-3}), \qquad (38)
\end{aligned}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ denotes the q-binomial coefficient:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, \quad (q;q)_n=\prod_{i=1}^n(1-q^i). \qquad (39)$$

The colored superpolynomial for the twist knots with odd number of crossings can be expressed in the similar manner.

3 Super-A-polynomial

Now we shall discuss about the super-A-polynomial and Conjecture 1.5 based on (37) and (38). From the asymptotic expansion of these expressions for the colored superpolynomials, one finds the super-A-polynomial $A_K^{\mathsf{super}}(x,y;a,t)$ from (7). For $\mathbf{3_1}$ and $\mathbf{4_1}$ knots, the super-A-polynomials are

$$\begin{aligned}
&A_{\mathbf{3_1}}^{\mathsf{super}}(x,y;a,t)=a^2t^4(x-1)x^3+(1+at^3x)y^2 \\
&\quad -a\big(1-t^2x+2t^2(1+at)x^2+at^5x^3+a^2t^6x^4\big)y, \qquad (40)\\
&A_{\mathbf{4_1}}^{\mathsf{super}}(x,y;a,t)=a^2t^5(x-1)^2x^2+at^2x^2(1+at^3x)^2y^3 \\
&\quad +at(x-1)(1+t(1-t)x+2at^3(t+1)x^2-2at^4(t+1)x^3+a^2t^6(1-t)x^4-a^2t^8x^5)y \\
&\quad -(1+at^3x)(1+at(1-t)x+2at^2(t+1)x^2+2a^2t^4(t+1)x^3+a^2t^5(t-1)x^4+a^3t^7x^5)y^2. \\
&\qquad (41)
\end{aligned}$$

Specializing to $a=1$ and $t=-1$, one finds that these expressions reduce to the A-polynomials (4) and (5).

Now our conjecture 1.5 can be verified explicitly. For $\mathbf{3_1}$ and $\mathbf{4_1}$, one finds the quantum super-A-polynomials $\hat{A}^{super}_K(\hat{x},\hat{y};a,q,t)$ manifestly via the computer talks [20]:

$$\hat{A}^{super}_{\mathbf{3_1}}(\hat{x},\hat{y};a,q,t)=\sum_{k=0}^{2}\hat{a}^{(k)}_{\mathbf{3_1}}(\hat{x};a,q,t)\hat{y}^k, \tag{42}$$

$$\hat{a}^{(0)}_{\mathbf{3_1}}=\frac{a^2t^4(\hat{x}-1)\hat{x}^3(1+aqt^3\hat{x}^2)}{q(1+at^3\hat{x})(1+at^3q^{-1}\hat{x}^2)},\quad \hat{a}^{(2)}_{\mathbf{3_1}}=1,$$

$$\hat{a}^{(1)}_{\mathbf{3_1}}=-\frac{a(1+at^3\hat{x}^2)\big(q-q^2t^2\hat{x}+t^2(q^2+q^3+at+aq^2t)\hat{x}^2+aq^2t^5\hat{x}^3+a^2qt^6\hat{x}^4\big)}{q^2(1+at^3\hat{x})(1+at^3q^{-1}\hat{x}^2)}.$$

$$\hat{A}^{super}_{\mathbf{4_1}}(\hat{x},\hat{y};a,q,t)=\sum_{k=0}^{3}\hat{a}^{(k)}_{\mathbf{4_1}}(\hat{x};a,q,t)\hat{y}^k, \tag{43}$$

$$\hat{a}^{(0)}_{\mathbf{4_1}}=\frac{at^3(1-\hat{x})(1-q\hat{x})(1+at^3q^2\hat{x}^2)(1+at^3q^3\hat{x}^2)}{q^3(1+at^3\hat{x})(1+at^3\hat{x}^2)(1+at^3q\hat{x})(1+at^3q^{-1}\hat{x}^2)},\quad \hat{a}^{(3)}_{\mathbf{4_1}}=1,$$

$$\begin{aligned}\hat{a}^{(1)}_{\mathbf{4_1}}=&-\frac{(1-q\hat{x})(1+at^3q^3\hat{x}^2)}{tq^3\hat{x}^2(1+at^3\hat{x})(1+at^3q\hat{x})(1+at^3q^{-1}\hat{x}^2)}\\&\times\Big(1-t(t-1)q\hat{x}+at^3q^{-1}(1+q^3+qt+q^2t)\hat{x}^2\\&\qquad-at^4(q+q^2+t+q^3t)\hat{x}^3-a^2(t-1)t^6q\hat{x}^4-a^2t^8q^2\hat{x}^5\Big),\\ \hat{a}^{(2)}_{\mathbf{4_1}}=&-\frac{(1+at^3q^2\hat{x}^2)}{at^2q^2\hat{x}^2(1+at^3\hat{x}^2)(1+at^3q\hat{x})}\\&\times\Big(1-at(t-1)\hat{x}+at^2(q+q^2+t+q^3t)\hat{x}^2\\&\qquad+a^2t^4(1+q^3+qt+q^2t)\hat{x}^3+a^2(t-1)t^5q^3\hat{x}^4+a^3t^7q^3\hat{x}^5\Big).\end{aligned}$$

These q-difference operators satisfies the following properties:

- $\hat{A}^{super}_K(x,y;a,q=1,t)$ reduces to the super-A-polynomial $A^{super}_K(x,y;a,t)$ up to overall factors.
- Under the specializations $t=-1$ and $a=q^2$, $\hat{A}^{super}_K(\hat{x},\hat{y};a,q,t)$ annihilates the colored HOMFLY polynomial and colored Jones polynomial:

$$\hat{A}^{super}_K(\hat{x},\hat{y};a,q,t=-1)H^{S^{n-1}}(K;a,q)=0, \tag{44}$$
$$\hat{A}^{super}_K(\hat{x},\hat{y};a=q^2,q,t=-1)J_n(K;q)=0. \tag{45}$$

- Under the specialization $x=1$, one finds the

$$A^{super}_K(x=1,y;a,t)=y^k+y^{k-1}\mathcal{P}(K;a,q=1,t), \tag{46}$$

where k is the maximum y-degree of the super-A-polynomial, and $\mathcal{P}(K;a,q,t)$ is the superpolynomial.

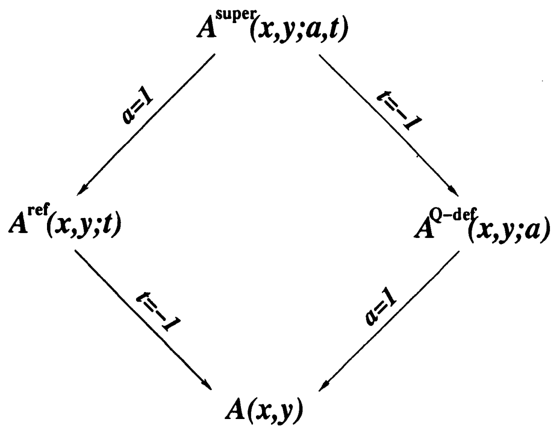

図 1: Hierarchy of A-polynomials

The second property implies an existence of the hierarchical structure of the (a,t)-deformations of the A-polynomial. As a completion of this hierarchy, one also finds the *refined A-polynomial* [17]:

$$A_K^{ref}(x,y;t) := A_K^{super}(x,y;a=1,t), \quad \hat{A}_K^{ref}(\hat{x},\hat{y};t) := \hat{A}_K^{super}(\hat{x},\hat{y};a=q^2,t). \tag{47}$$

In conclusion, the super-A-polynomial reveals the integrable structure behind the colored superpolynomials, and we expect that this polynomial contains a plenty of information for the topology of the knot complement. Via string theoretical interpretation of the A-polynomial [9, 10, 1], the invariants associated to the symplectic structure of the the character variety can be explored. As the first step, we will study such invariant for the super-A-polynomial of $\mathbf{3_1}$ knot in Appendix.

Acknowledgments

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A Topological recursion on the character variety

For the character variety $\mathcal{C}$ whose defining equation is given by the A-polynomial, one finds a class of geometric invariants associated to the symplectic structure of $\mathcal{C}$ via string theoretical interpretations [9, 10, 1] by utilizing the formalism of the *topological recursion.* This formalism is originally developed in the study of the matrix models, and one can calculate such symplectic invariants systematically. In this appendix, we will compute (2|2)-Baker-Akhierzer kernel $\psi^{[2|2]}$ for the (super-) character variety of $\mathbf{3_1}$ knot.‡

A.1 Topological recursion

In the analytic study of the matrix models [3, 2] and topological strings [6, 7], the free energy F_g and correlator $W^{(g,h)}(p_1, \cdots, p_h)$ are studied in detail on basis of the symplectic data of the spectral curve $\mathcal{C}$. In [13, 16], a systematic method for such analysis has been established, and called as the *Eynard-Orantin's topological recursion.* Here we focus on the topological recursion for the genus one curve $\mathcal{C}$ given by the algebraic equation:

$$\mathcal{C}_0 = \{(x, y) \in \mathbb{C}^2 | y^2 = M(x)^2 S(x)\}, \tag{48}$$

$$S(x) = (x - x_1)(x - x_2)(x - x_3)(x - x_4). \tag{49}$$

We denoted a rational function $M(x)$, and the branch points q_i $(i = 1, \cdots, 4)$ of $\mathcal{C}$ which obey $(x(q_i), y(q_i)) = (x_i, 0)$. In the computation of the topological recursion, we mainly use the symplectic data of the smooth model $\mathcal{C}_0$:

$$\mathcal{C}_0 = \{(x, y) \in \mathbb{C}^2 | y_0^2 = S(x)\}, \tag{50}$$

where a compact cycle which encircles around $[q_1, q_2]$ is denoted as $\mathcal{A}$, and its symplectic dual cycle is denoted by $\mathcal{B}$. These cycles obeys $\mathcal{A} \cap \mathcal{A} = 0$, $\mathcal{B} \cap \mathcal{B} = 0$, and $\mathcal{A} \cap \mathcal{B} = 1$. For this genus one curve, the Bergman kernel $B(p_1, p_2)$ is the meromorphic $(1, 1)$-form on ${\mathcal{C}_0}^{\otimes 2} \ni (x(p_1), y_0(p_1); x(p_2), y_0(p_2))$ which obeys the following properties:

$$B(p_1, p_2) \overset{p_1 \to p_2}{\sim} \frac{dp_1 \otimes dp_2}{(p_1 - p_2)^2} + \text{holomorphic}, \tag{51}$$

$$\oint_{\mathcal{A}} B(p_1, \cdot) = 0, \quad \oint_{\mathcal{B}} B(p_1, \cdot) = 2\pi \mathrm{i}\, d\omega_0, \tag{52}$$

$$\oint_{\mathcal{A}} d\omega_0 = 1, \quad \oint_{\mathcal{B}} d\omega_0 = \tau, \tag{53}$$

where in (52) we denoted the integration of the Bergman kernel $B(p_1, p_2)$ with respect to p_2 for an abbreviation, and τ is the period of the elliptic curve $\mathcal{C}_0$.

From these symplectic data of the spectral curve $\mathcal{C}_0$, the recursion relations for the correlators $W^{(g,h)}(p_1, \cdots, p_h)$ are introduced as follows:

Definition A.1 *([16]) The meromorphic* $(1, \cdots, 1)$*-form* $W^{(g,h)}(p_1, \cdots, p_h)$ *on* $\mathcal{C}^{\otimes h}$ *sat-*

‡This point is presented in the problem session of the workshop *ILDT 2013.*

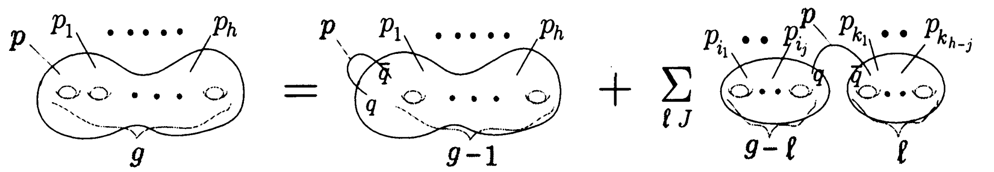

図 2: Diagrammatic representation of the Eynard-Orantin's topological recursion relation (55)

isfies the following recursion relation:

$$W^{(0,1)}(p), \quad W^{(0,2)}(p_1,p_2) = B(p_1,p_2), \tag{54}$$

$$W^{(g,h+1)}(p_0,p_1,\cdots,p_h) = \sum_{q_i:\text{Branch pts.}} \operatorname*{Res}_{q=q_i} \frac{-\frac{1}{2}\int_{\bar{q}}^{q} B(\cdot,p_0)}{(y(q)-y(\bar{q}))} \Big[W^{(g-1,h+2)}(q,\bar{q},p_1,\cdots,p_h)$$
$$+ \sum_{\ell=0}^{g} \sum_{J\subset H} W^{(g-\ell,|J|+1)}(q,p_J) W^{(\ell,|H|-|J|+1)}(\bar{q},p_{H\setminus J}) \Big], \tag{55}$$

where $H = \{1,\cdots,h\}$, $J = \{i_1,\cdots i_j\} \subset H$, and $|H| = h$, $|J| = j$. The points $q,\bar{q} \in \mathcal{C}$ are related by $(x(\bar{q}),y(\bar{q})) = (x(q),-y(q))$. The diagrammatic representation of the recursion (55) is described in Fig.2.

The free energy F_g is defined on basis of the above data of the spectral curve $\mathcal{C}$. For $g = 0,1$, the free energies F_0 and F_1 are defined by

$$\partial_\epsilon F_0 = \oint_B dx\, y_0(x), \quad \epsilon := \frac{1}{2\pi \mathrm{i}} \oint dx\, y_0(x), \quad \tau = \frac{1}{2\pi\mathrm{i}} \partial_\epsilon^2 F_0, \tag{56}$$

$$F_1 = \frac{1}{24} \log\left(\tau_B \prod_i y'(q_i) \right), \tag{57}$$

where τ_B is the Bergmann τ-function which is defined via the relation $\partial \log \tau_B / x = \operatorname*{Res}_{q=q_i} (B(q,\bar{q})/dx(q))$. For $g \geq 2$, the free energy F_g is given by

$$F_g = \frac{1}{2g-2} \operatorname*{Res}_{q=q_i} \Phi(q) W^{(g,1)}(q), \quad \Phi(x) = \int_*^x dx\, y_0(x).] \tag{58}$$

In this way, the free energies and correlators for $\mathcal{C}$ are calculated recursively by solving the equation (55) and (58).

A.2 Non-perturbative partition function and $(n|n)$-kernel

The above definition of the topological recursion gives the perturbative part of the free energy and correlator. When we take into account of the analyticity of these quantities, the non-perturbative effects must be treated carefully. To incorporate the non-perturbative effects systematically, the *non-perturbative partition function* $\mathcal{T}_{g_s}$ is introduced in [14, 15]:

Definition A.2 *([14, 15]) Let $\vartheta := \vartheta\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right](\zeta|\tau)$ be the theta function of the spectral curve C_0 whose period τ is defined in (53):*

$$\vartheta = \sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\pi(n+\mu)^2\tau+2\mathrm{i}\pi(n+\mu)(\zeta+\nu)}, \tag{59}$$

where $\mu, \nu \in \mathbb{C}/\mathbb{Z}$. The theta function satisfies the heat equation

$$D\vartheta = \nabla^{\otimes 2}\vartheta, \quad \nabla = \partial/\partial\zeta, \quad D = 4\pi\mathrm{i}\partial/\partial\tau. \tag{60}$$

Using this theta function and the free energy F_g, the non-perturbative partition function $\mathcal{T}_{g_s}[y_0dx]$ is defined as follows:

$$\begin{aligned}\mathcal{T}_{g_s}[y_0dx] &= \exp\Big(\sum_{h\geq 0} g_s^{2h-2}F_h\Big)\\ &\times\left\{\sum_{r\geq 0}\frac{1}{r!}\sum_{\substack{h_j\geq 0,\ d_j\geq 0\\ 2h_j-2+d_j>0}} g_s^{\sum_j 2h_j-2+d_j}\otimes_{j=1}^{r}\frac{F_{h_j}^{(d_j)}\cdot\nabla^{\otimes d_j}}{(2\mathrm{i}\pi)^{d_j}d_j!}\right\}\vartheta,\end{aligned} \tag{61}$$

where ζ is specialized to ζ_{g_s}:

$$\zeta_{g_s} = \left[\frac{1}{2\pi\mathrm{i}g_s}\oint_B y_0(x)\,dx\right] + \tau\left[\frac{1}{2\pi\mathrm{i}g_s}\oint_A y_0(x)\,dx\right], \tag{62}$$

which is defined modulo $\mathbb{Z}$.

In the case of the $N\times N$ Hermitian 1-matrix model, this non-perturbative partition function is given by summing all possible filling fractions [14], and multi-instanton corrections to the perturbative free energy F_g which is defined by $1/N$ expansion of the partition function are included. The non-perturbative corrections to the correlator is defined to be given by the *Schlesinger transformation* [38] of $\mathcal{T}_{g_s}[y_0dx]$, and such quantity is named as the *Baker-Akhierzer's $(n|n)$-kernel* $\psi_{g_s}^{[n|n]}(p_1,o_1;\cdots;p_n,o_n)$ (abbreviated as $(n|n)$-BA kernel). The $(n|n)$-BA-kernel is defined as follows:

Definition A.3 *([4]) Let $dS_{o,p}$ be the Abelian differential of the third kind on the genus one curve C_0 which is given by the Bergman kernel as:*

$$dS_{o,p}(q) = \int_o^p B(\cdot,q). \tag{63}$$

Shifting 1-form y_0dx by $y_0dx + \sum_{i=1}^n dS_{o_i,p_i}$ in $\mathcal{T}_{g_s}[y_0dx]$, one finds the Baker-Akhierzer's $(n|n)$-kernel $\psi_{g_s}^{[n|n]}(p_1,o_1;\cdots,p_n,o_n)$:

$$\psi_{g_s}^{[n|n]}(p_1,o_1;\cdots;p_n,o_n) = \frac{\mathcal{T}_{g_s}[y_0dx\to y_0dx+\sum_{i=1}^n dS_{o_i,p_i}]}{\mathcal{T}_{g_s}[y_0dx]}. \tag{64}$$

This definition is formal, and $\psi_{g_s}^{[n|n]}$ can be described more explicitly in terms of the data of $\mathcal{C}$. For details of the above definitions, see [4, 5].

To evaluate the above quantities explicitly, we have to take care about the dependence on the choice of the symplectic basis $(\mathcal{A}, \mathcal{B})$ of the spectral curve $\mathcal{C}_0$. Since the free energy, correlator, and $(n|n)$-BA kernel are sensitive to the choice of the symplectic basis, one finds a shift of these quantities under the change of $(\mathcal{A}, \mathcal{B})$. Therefore, we have to specify the choice of symplectic basis when we study the topological recursion. The shift of the topological recursion can be treated simply as follows. When one changes the symplectic basis $(\mathcal{A}, \mathcal{B})$ by

$$(\mathcal{A}^\kappa, \mathcal{B}^\kappa) = (\mathcal{A} - \kappa\mathcal{B}^\kappa, \mathcal{B} - \tau\mathcal{A}), \tag{65}$$

the Bergman kernel is shifted as:

$$B_\kappa(p_1, p_2) = B(p_1, p_2) + 2i\pi d\omega_0(p_1) \cdot \kappa \cdot d\omega_0(p_2), \tag{66}$$

and the correlators $W^{(g,h)}$ are also shifted to $W_\kappa^{(g,h)}$ by adopting this shifted Bergman kernel $B_\kappa(p_1, p_2)$ to the topological recursion (55) [7, 5].

In [5], the asymptotic expansion of (2|2)-BA kernel in $g_s \to 0$ is found explicitly by using the κ-shifted correlator $W_\kappa^{(g,h)}$ as follows:

$$\psi_{g_s}^{[2|2]}(p_1, o_1; p_2, o_2) \sim \exp\left[\sum_k g_s^k G_k(p_1, o_1; p_2, o_2)\right], \tag{67}$$

$$G_1 = G_{1|\kappa}^{1,(0)} + G_{3|\kappa}^{0,(0)} + T_{2,\bullet|\kappa} G_{1|\kappa}^{0,(2)} + T_{1,\bullet|\kappa} G_{2|\kappa}^{0,(1)} + (T_{3,\bullet|\kappa} - T_{3|\kappa}) G_{0|\kappa}^{0,(3)}, \tag{68}$$

$$G_2 = G_{2|\kappa}^{1,(0)} + G_{4|\kappa}^{0,(0)} + T_{2,\bullet|\kappa} G_{2|\kappa}^{0,(2)} + \frac{1}{2}(T_{4,\bullet|\kappa} + T_{2,\bullet|\kappa}^2)(G_{1|\kappa}^{0,(2)})^2 + \cdots, \tag{69}$$

$$\cdots.$$

where $G_{n|\kappa}^{h,(d)}$ and $T_{d|\kappa}$ are given by

$$G_{n|\kappa}^{h,(d)}(p_1, o_1; p_2, o_2) = \frac{1}{n!}\frac{1}{(2\pi i)^d d!} \underbrace{\int_{o_1,o_2}^{p_1,p_2} \cdots \int_{o_1,o_2}^{p_1,p_2}}_{n \text{ times}} \underbrace{\oint_{\mathcal{B}_\kappa} \cdots \oint_{\mathcal{B}_\kappa}}_{d \text{ times}} W_\kappa^{(n+d,h)}, \tag{70}$$

$$T_{d|\kappa} = \sum_{d'=0}^{[d/2]} \frac{d!(-1)^{d'}(2i\pi)^{d'}}{2^{d'} d'!} \kappa^{\otimes d'} \otimes \frac{\nabla^{\otimes(d-2d')}\vartheta}{\vartheta}. \tag{71}$$

The notation $T_{d,\bullet|\kappa}$ means the evaluation of $T_{d|\kappa}$ for $\zeta = \zeta_{g_s} + \int_{o_1,o_2}^{p_1,p_2} d\omega_0$. In (69), we have described only the terms which are relevent in the evaluation on the spectral curve with a reciprocal symmetry: $\iota_* B_\kappa = -B_\kappa$.

A.3 Quantum volume conjecture and (2|2)-BA kernel

Choosing the spectral curve $\mathcal{C}$ as the character variety $\mathcal{C}_K$ of the knot K that is defined by the A-polynomial $A_K(x,y)$:

$$\mathcal{C}_K = \{(x,y) \in \mathbb{C}^* \times \mathbb{C}^* | A_K(x,y) = 0\}, \tag{72}$$

one can evaluate the (2|2)-BA kernel as the invariant associated to the symplectic structure of the character variety. In particular for $K = \mathbf{4_1}$, $\mathcal{C}_{\mathbf{4_1}}$ is genus one curve, and we can evaluate (2|2)-BA kernel explicitly using the formulae (67)-(71). Furthermore, since the character variety has reciprocal symmetry that is invariant under the involution ι:

$$x(\iota(p)) = 1/x(p), \quad y(\iota(p)) = 1/y(p), \tag{73}$$

we can consider the (2|2)-BA kernel $\psi_{g_s}^{[2|2]}(p,o;\iota(p),\iota(o))$ invariant under the reciprocal symmetry (73).

On the other hand, in the quantum volume conjecture [23, 20], it is proposed that the colored Jones polynomial satisfies a q-difference equation: $\hat{A}_K(\hat{x},\hat{y};q)J_n(K;q) = 0$, and the asymptotic solution of the q-difference equation can be found iteratively solving the differential equation:

$$\hat{A}_K(\hat{x},\hat{y};q=e^{2\hbar})\mathcal{J}_K(x;\hbar) = 0, \tag{74}$$

$$\hat{x}J_K(x;\hbar) = x\mathcal{J}_K(x;\hbar), \quad \hat{y}\mathcal{J}_K(x;\hbar) = e^{\hbar\partial_x}J_K(x;\hbar), \tag{75}$$

$$\mathcal{J}_K(x;\hbar) \sim \hbar^{\delta/2}\exp\left(\sum_{k\geq -1}\hbar^k j_k(x)\right). \tag{76}$$

On basis of the observation for the figure eight knot and $m009$, the following conjecture is proposed in [10, 5]:§

Conjecture A.1 *([5]) Choose κ as $\kappa(\tau) = -\frac{1}{2i\pi}\frac{E_2(\tau)}{3\pi^2}$. There exist a choice of the end-point o and $\mu,\nu \in \mathbb{C}/\mathbb{Z}$ for the theta function $\vartheta\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right](\zeta|\tau)$ such that*

$$\mathcal{J}_K(x(p),2\hbar) = (\psi_\hbar^{[2|2]}(p,o;\iota(p),\iota(o)))^{1/2}. \tag{77}$$

At each level of the asymptotic expansion, (77) implies

$$2j_k(x(p)) = G_k(p,o;\iota(p),\iota(o)). \tag{78}$$

§Physically this conjecture is proposed in [10]. But in [10] only the perturbative part is discussed, and the non-perturbative part is introduced as an *ad hoc* regularization. This *ad hoc* point is resolved in [5] by the completion of the non-perturbative completion, and here we use the statement of the conjecture proposed in [5].

A.4 (2|2)-BA kernel for the super-A-polynomial of $\mathbf{3_1}$

Now we discuss the invariants associated to the super-A-polynomial. The super-A-polynomial for $\mathbf{3_1}$ knot is

$$\begin{aligned} A^{\mathrm{super}}_{\mathbf{3_1}}(x,y;a,t) &= a^2t^4(x-1)x^3+(1+at^3x)y^2 \\ &\quad -a\big(1-t^2x+2t^2(1+at)x^2+at^5x^3+a^2t^6x^4\big)y, \end{aligned} \tag{79}$$

and the super-character variety $\mathcal{C}^{\mathrm{super}}_K=\{(x,y)\in\mathbb{C}^*\times\mathbb{C}^*|A^{\mathrm{super}}_K(x,y;a,t)=0\}$ is genus one curve in this case. Furthermore, $A^{\mathrm{super}}_{\mathbf{3_1}}$ has the deformed reciprocal symmetry:

$$x(\iota(p))=-\frac{1}{at^3}x(p)^{-1},\quad y(\iota(p))=\frac{1}{t^2}y(p)^{-1}. \tag{80}$$

To discuss the topological recursion for the super-character variety as the genus one curve, we rewrite the super-A-polynomial (79) [32]:

$$\mathcal{C}^{\mathrm{super}}_{\mathbf{3_1}}=\{(x,y)\in\mathbb{C}^2|y^2=M(X)^2S(x)\}, \tag{81}$$

$$M(x)=\frac{1}{x\sqrt{S(x)}}\log\frac{A(x)+\sqrt{S(x)}}{A(x)-\sqrt{S(x)}}, \tag{82}$$

$$A(x)=\frac{1-t^2x+2t^2x^2+2at^3x^2+at^5x^3+a^2t^6x^4}{at^3(1+at^3x^2)}, \tag{83}$$

$$S(x)=a^{-2}t^{-6}(1-2t^2x+4t^2x^2+2at^3x^2+t^4x^2+2at^5x^3+a^2t^6x^4). \tag{84}$$

In terms of this coordinate, the smooth model is given by

$$\mathcal{C}^{\mathrm{super}}_{\mathbf{3_1}0}=\{(x,y_0)\in\mathbb{C}^2|y_0^2=S(x)\}. \tag{85}$$

The smooth model $\mathcal{C}^{\mathrm{super}}_{\mathbf{3_1}0}$ can be rewritten in the Weierstrass form:

$$\mathrm{y}^2=4\mathrm{x}^3-g_2\mathrm{x}-g_3, \tag{86}$$

$$g_2=\frac{16+16at+8t^2+16a^2t^2+16at^3+t^4}{12a^4t^8}, \tag{87}$$

$$g_3=\frac{(4+8at+t^2)(-16+8at-8t^2+8a^2t^2-16at^3-t^4)}{216a^6t^{12}}. \tag{88}$$

The value of the theta functions of $\mathcal{C}^{\mathrm{super}}_{\mathbf{3_1}0}$ are found explicitly via the relations: $E_4=\vartheta_2^8+\theta_4^8+\theta_2^4\theta_4^4=3(2\varpi_A)^4g_2/4\pi^4$, $E_6=-\theta_2^{12}-3\theta_2^8\theta_4^4/2+3\theta_2^4\theta_4^8/2+\theta_4^{12}=27(2\varpi_A)^6g_3/8\pi^6$ where $2\varpi_A:=\oint_A dz$ with $(x,y)=(\wp(z/2\varpi_A|\tau),\wp'(z/2\varpi_A|\tau))$. The coefficient $T_{2d|\kappa(\tau)}$ for $\kappa(\tau)=-\frac{1}{2i\pi}\frac{E_2(\tau)}{3\pi^2}$ is given by [5]

$$T_{2d|\kappa(\tau)}=D_{2d+1/2}\circ\cdots\circ D_{2d-3/2}\circ D_{1/2}\theta_i(\tau)/\theta_i(\tau),\quad D_kf:=Df+\frac{2}{3\pi^2}kE_2. \tag{89}$$

The choice of μ,ν corresponds to the choice of $i=2,3,4$ for the Jacobi's theta functions $\theta_i(\tau)$ with $\theta_2^4+\theta_4^4=\theta_3^4$. For $\mathbf{4_1}$ knot, the appropriate choice of μ,ν is found for one of the solutions for θ_i of the above relations between $E_a(\tau)$ and $\theta_i(\tau)$ [5]. In the case of super-character variety of $\mathbf{3_1}$, we choose the appropriate values of $T_{2|\kappa(\tau)}$ and $T_{4|\kappa(\tau)}$ as:

$$T_{2|\kappa(\tau)}/(2\varpi_A)^2 = -\frac{4+8at+t^2}{12a^2t^4}, \tag{90}$$
$$T_{4|\kappa(\tau)}/(2\varpi_A)^4 = -\frac{-16+32at-8t^2+32a^2t^2-16at^3-t^4}{48a^4t^8}. \tag{91}$$

where these are unique solution that is given by the rational function of a, t.

From these geometric data of the super-character variety $\mathcal{C}^{\text{super}}_{\mathbf{3_1}}$, one can calculate $G^{h,(d)}_{n|\kappa}(p, o; \iota(p), \iota(o))$ which appears in G_k term. Here we consider the (2|2)-BA kernel which is invariant under the deformed reciprocal symmetry (80). The terms G_{-1} and G_0 are treated independently, and they are given as the Abel map and discriminant of $\mathcal{C}^{\text{super}}_K$ as follows [6, 10]:¶

$$G_{-1} = \int_{o,\iota(o)}^{p,\iota(p)} \frac{dx}{x}\log y(x), \quad G_0 = \frac{1}{2}\log\frac{\gamma}{\sqrt{S(x)/x^2}}, \tag{92}$$

where $\gamma = \sqrt{-(3+4at+t^2)}/2at$. The next order G_1 (68) is found via topological recursion (55):

$$\begin{aligned}
G^{1(0)}_{1|\kappa(\tau)} =& \big(-16-16t^2-24at^3-11t^4-18at^5-2t^6-8a^2t^6-4at^7-2a^2t^8+(48t^2+24t^4\\
&+32at^5+27t^6-16a^2t^6+24at^7+6t^8-16a^2t^8+12at^9-16a^3t^9+6a^2t^{10})x+(-128t^2\\
&-80at^3-208t^4-336at^5-80t^6-152a^2t^6-207at^7-33t^8-178a^2t^8-20at^9-72a^3t^9\\
&-6t^{10}+40a^2t^{10}-12at^{11}+30a^3t^{11}-6a^2t^{12})x^2+(-224at^5-112t^6-128a^2t^6-560at^7\\
&-24t^8-768a^2t^8-302at^9-288a^3t^9+9t^{10}-664a^2t^{10}-16at^{11}-544a^3t^{11}+2t^{12}\\
&-52a^2t^{12}-160a^4t^{12}+4at^{13}-28a^3t^{13}+2a^2t^{14})x^3+(128at^5+80a^2t^6+208at^7\\
&+336a^2t^8+80at^9+152a^3t^9+207a^2t^{10}+33at^{11}+178a^3t^{11}+20a^2t^{12}+72a^4t^{12}+6at^{13}\\
&-40a^3t^{13}+12a^2t^{14}-30a^4t^{14}+6a^3t^{15})x^4+(48a^2t^8+24a^2t^{10}+32a^3t^{11}+27a^2t^{12}\\
&-16a^4t^{12}+24a^3t^{13}+6a^2t^{14}-16a^4t^{14}+12a^3t^{15}-16a^5t^{15}+6a^4t^{16})x^5+(16a^3t^9\\
&+16a^3t^{11}+24a^4t^{12}+11a^3t^{13}+18a^4t^{14}+2a^3t^{15}+8a^5t^{15}+4a^4t^{16}+2a^5t^{17})x^6\big)\\
&/\big(12(1+at)(1+t^2+at^3)(16+8t^2+16at^3+t^4)\\
&\times(1-2t^2x+(4t^2+2at^3+t^4)x^2+2at^5x^3+a^2t^6x^4)^{3/2}\big),
\end{aligned} \tag{93}$$

$$\begin{aligned}
G^{0(0)}_{3|\kappa(\tau)} =& \big(1+(3t^2+6at^3)x+(-12t^2-15at^3-9t^4-12at^5)x^2+(4t^4-10at^5+5t^6\\
&-20a^2t^6+6at^7)x^3+(12at^5+15a^2t^6+9at^7+12a^2t^8)x^4+(3a^2t^8+6a^3t^9)x^5-a^3t^9x^6\big)\\
&/\big(12(1+at)(1-2t^2x+(4t^2+2at^3+t^4)x^2+2at^5x^3+a^2t^6x^4)^{3/2}\big),
\end{aligned} \tag{94}$$

¶In [10] the Bergman kernel is calculated in the variable $w = (x + x^{-1})/2$ that is invaraint under the involution ι. In this case, we use the variable $w = \frac{1}{2}(x - 1/(at^3x))$ to evaluate G_0 term.

$$G^{0,(1)}_{2|\kappa(\tau)} = 0, \quad G^{0,(3)}_{0|\kappa(\tau)} = 0, \tag{95}$$

$$\begin{aligned} G^{0,(2)}_{1|\kappa(\tau)} \cdot (2\varpi_A)^2 = &\big(a^2t^4(-4 - 3t^2 - 4at^3 + (4t^2 + 5t^4 + 6at^5)x + (4at^3 + 3at^5 + 4a^2t^6)x^2\big) \\ &/((1 + at)(16 + 8t^2 + 16at^3 + t^4)(1 - 2t^2x + (4t^2 + 2at^3 + t^4)x^2 + 2at^5x^3 + a^2t^6x^4)^{1/2}\big). \end{aligned} \tag{96}$$

Applying these $G^{h,(d)}_{n|\kappa(\tau)}$ and (90) to (68), one finds the G_1 term:

$$\begin{aligned} G_1(x) = &\Big(1 + 2at + 2t^2 + 4at^3 + 2a^2t^4 + \big(3t^2 - 2t^4 - 4at^5 - 2a^2t^6\big)x \\ &+\big(- 16t^2 - 11at^3 - 21t^4 + 2a^2t^4 - 28at^5 - 2t^6 - 8a^2t^6 - 4at^7 + 2a^3t^7 - 2a^2t^8\big)x^2 \\ &+\big(- 30at^5 - 7t^6 - 24a^2t^6 - 48at^7 + 2t^8 - 68a^2t^8 + 4at^9 - 28a^3t^9 + 2a^2t^{10}\big)x^3 \\ &+\big(16at^5 + 11a^2t^6 + 21at^7 - 2a^3t^7 + 28a^2t^8 + 2at^9 + 8a^3t^9 + 4a^2t^{10} - 2a^4t^{10} + 2a^3t^{11}\big)x^4 \\ &+\big(3a^2t^8 - 2a^2t^{10} - 4a^3t^{11} - 2a^4t^{12}\big)x^5 + \big(- a^3t^9 - 2a^4t^{10} - 2a^3t^{11} - 4a^4t^{12} - 2a^5t^{13}\big)x^6\Big) \\ &/\Big(48(1 + at)(1 + t^2 + at^3)(1 - 2t^2x + 4t^2x^2 + 2at^3x^2 + t^4x^2 + 2at^5x^3 + a^2t^6x^4)^{3/2}\Big) \\ = &\frac{-1}{48(at+1)} + \frac{1 - 5x^2 + x^4}{16(1 - x + x^2)^2} + \mathcal{O}(a - 1, t + 1). \end{aligned} \tag{97}$$

The above result and the solution of the q-difference equation (21) with fixed a variable behave differently even in the augmentation limit $t = -1$. This discrepancy would be related with the subtlety of the q-dependence of a and x in the asymptotic expansion. In terms of the mirror symmetry proposed in [1], the open string amplitude may be identified with (2|2)-BA kernel after the pull back by the mirror map. This point will be studied in the future works.

A.4.1 G_2 term for the augmentation variety

The G_2 is also computed via the above data. Here we note the result for $t = -1$ specialization:

$$\begin{aligned} G^{0,(0)}_{4|\kappa(\tau)} = &-\big(a - (8 + 10a - 8a^2)x + (64 - 101a + 58a^2)x^2 + (-16 - 76a + 82a^2 - 40a^3)x^3 \\ &+(-160 + 375a - 60a^2 - 65a^3)x^4 + (-104 + 810a - 1524a^2 + 644a^3 + 48a^4)x^5 + (224 \\ &-1043a + 1154a^2 + 306a^3 - 500a^4)x^6 + (-104a + 810a^2 - 1524a^3 + 644a^4 + 48a^5)x^7 \\ &+(-160a^2 + 375a^3 - 60a^4 - 65a^5)x^8 + (-16a^3 - 76a^4 + 82a^5 - 40a^6)x^9 + (64a^4 - 101a^5 \\ &+58a^6)x^{10} + (-8a^5 + 10a^6 - 8a^7)x^{11} + a^7x^{12}\big) \\ &/\big(128(-1 + a)^2(1 - 2x + 5x^2 - 2ax^2 - 2ax^3 + a^2x^4)^3\big), \end{aligned} \tag{98}$$

$$G^{0,(2)}_{2|\kappa(\tau)}\cdot(2\varpi_A)^2 = a^2\big(125+57a-192a^2+64a^3+(-100-1248a+1644a^2-512a^3)x$$
$$+(2650-2382a+272a^3)x^2+(3300-20508a+29220a^2-15948a^3+3072a^4)x^3$$
$$+(-14775+57821a-72418a^2+37086a^3-6560a^4-128a^5)x^4+(3300a-20508a^2$$
$$+29220a^3-15948a^4+3072a^5)x^5+(2650a^2-2382a^3+272a^5)x^6+(-100a^3-1248a^4$$
$$+1644a^5-512a^6)x^7+(125a^4+57a^5-192a^6+64a^7)x^8\big)$$
$$/\big(96(-1+a)^2(-25+16a)^2(1-2x+5x^2-2ax^2-2ax^3+a^2x^4)^2\big), \quad (99)$$

$$G^{0,(4)}_{0|\kappa(\tau)}\cdot(2\varpi_A)^4 = -\frac{(a^4(-1120+1983a-1200a^2+256a^3)}{8(-1+a)^2(-25+16a)^3}, \quad (100)$$

$$G^{1,(0)}_{2|\kappa(\tau)} = \big(3850+1186a-11437a^2+8008a^3-16a^4-1280a^5+256a^6+(-49300$$
$$+103052a-122582a^2+128516a^3-92528a^4+34304a^5-4864a^6)x+(128150$$
$$-72366a-223255a^2+258862a^3-75964a^4-9728a^5+6720a^6-512a^7)x^2$$
$$+(-1218200+4566948a-8455640a^2+9515930a^3-6630236a^4+2759120a^5$$
$$-625920a^6+59648a^7)x^3+(3323350-12843994a+22690091a^2-22999486a^3$$
$$+13992613a^4-4981960a^5+941328a^6-70656a^7-256a^8)x^4+(-628500-3175260a$$
$$+15266330a^2-24549808a^3+19927324a^4-8353480a^5+1355680a^6+141056a^7$$
$$-54784a^8)x^5+(-333750+4605350a-7420845a^2-4329006a^3+22532762a^4$$
$$-25674524a^5+14219576a^6-3962368a^7+441728a^8+1024a^9)x^6+(-628500a$$
$$-3175260a^2+15266330a^3-24549808a^4+19927324a^5-8353480a^6+1355680a^7$$
$$+141056a^8-54784a^9)x^7+(3323350a^2-12843994a^3+22690091a^4-22999486a^5$$
$$+13992613a^6-4981960a^7+941328a^8-70656a^9-256a^{10})x^8+(-1218200a^3$$
$$+4566948a^4-8455640a^5+9515930a^6-6630236a^7+2759120a^8-625920a^9$$
$$+59648a^{10})x^9+(128150a^4-72366a^5-223255a^6+258862a^7-75964a^8-9728a^9$$
$$+6720a^{10}-512a^{11})x^{10}+(-49300a^5+103052a^6-122582a^7+128516a^8-92528a^9$$
$$+34304a^{10}-4864a^{11})x^{11}+(3850a^6+1186a^7-11437a^8+8008a^9-16a^{10}-1280a^{11}$$
$$+256a^{12})x^{12}\big)$$
$$/\big(1152(-2+a)^2(-1+a)^2(-25+16a)^2(1-2x+5x^2-2ax^2-2ax^3+a^2x^4)^3\big), \quad (101)$$

$$G^{1,(2)}_{0|\kappa(\tau)}\cdot(2\varpi_A)^2$$
$$= -\frac{a^2(-3575-2759a+23055a^2-29532a^3+16568a^4-4512a^5+512a^6)}{48(-2+a)^2(-1+a)^2(-25+16a)^3}. \quad (102)$$

Applying the above results to (69), one finds the G_2 term:

$$\begin{aligned}
G_2 = \big(&2+6a-6a^2+a^3+(-4-60a+60a^2-14a^3)x+(-194+910a-1038a^2\\
&+455a^3-70a^4)x^2+(-696+1988a-3396a^2+3000a^3-1238a^4+192a^5)x^3\\
&+(2846x^4-8254ax^4+10940a^2-7367a^3+2410a^4-305a^5)x^4+(-1156-196a\\
&+3260a^2-2942a^3+172a^4+676a^5-192a^6)x^5+(610-790a+6442a^2-15635a^3\\
&+15598a^4-6934a^5+1132a^6)x^6+(-1156a-196a^2+3260a^3-2942a^4+172a^5\\
&+676a^6-192a^7)x^7+(2846a^2-8254a^3+10940a^4-7367a^5+2410a^6-305a^7)x^8\\
&+(-696a^3+1988a^4-3396a^5+3000a^6-1238a^7+192a^8)x^9+(-194a^4+910a^5\\
&-1038a^6+455a^7-70a^8)x^{10}+(-4a^5-60a^6+60a^7-14a^8)x^{11}+(2a^6+6a^7\\
&-6a^8+a^9)x^{12}\big)\\
/\big(&48(-2+a)^2(-1+a)^2(1-2x+5x^2-2ax^2-2ax^3+a^2x^4)^3\big). \qquad (103)
\end{aligned}$$

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