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Quantum representation and dual Garside structure

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1 Introduction

It is widely known that the quantum group $U_q(\mathfrak{g})$, the quantum enveloping algebra of a lie algebra $\mathfrak{g}$, gives rise to a representation of the braid group B_n called a *quantum representation.* For an $U_q(\mathfrak{g})$-module V, one gets a linear representation $B_n \to \mathrm{GL}(V^{\otimes n})$ using a *universal R-matrix.* Such braid representations, especially for the case $\mathfrak{g}$ is a simple lie algebra such as $\mathfrak{sl}_2$, have gathered much attentions since they produce topological invariants of knots, links and 3-manifolds called *quantum invariants.*

Although quantum invariants have been actively studied, the quantum representation themselves are still mysterious. In this paper we illustrate a new point of view in the study of quantum braid representations. We show that "generic" quantum representations nicely behaves with respect to the *dual Garside structure* of the braid groups. This suggests that quantum representations have various nice properties than we first expected.

The dual Garside structure is a combinatorial structure of braid groups which dates back to Garside's solution of words and conjugacy problem for the braid groups [G]. The dual Garside structure introduces a normal form of braids called a *(dual Garside) normal form*, and we have a nice length function called the *dual Garside length* which can be computed quite effectively.

A relationship between a linear representation of the braid groups and dual Garside structure was inspired by author's previous works [I1, IW], which established a connection between *Homological* representations of the braid groups and the dual Garside length.

In this paper, we restrict our attention to the simplest case, $\mathfrak{g} = \mathfrak{sl}_2$ and we omit the proof of the main theorem. The proof of our main Theorem, Theorem 4.2, consists of several (tricky) calculations of the action of B_n, with a help of dual Garside structures. Details will be included in [I2]. We will treat the case where $\mathfrak{g}$ is a general lie algebra in [I3].

2 Dual Garside structure of the braid groups

In this section we summarize basic facts on the dual Garside structure of the braid groups. For details, see Birman-Ko-Lee [BKL]. [BGG, Section 1] provides a good overview of Garside structures emphasizing the role of normal forms. See [DDKM] for general and categorical treatments of Garside theory.

2.1 Dual Garside structure and normal forms

For $1 \le i < j \le n$, let $a_{i,j}$ be the braid

$$a_{i,j} = (\sigma_{i+1} \cdots \sigma_{j-2}\sigma_{j-1})^{-1}\sigma_i(\sigma_{i+1} \cdots \sigma_{j-2}\sigma_{j-1}).$$

The generating set $\Sigma^* = \{a_{i,j} \,|\, 1 \le i < j \le n\}$ was introduced in [BKL]. An element of Σ^* is called the *dual Garside generators*, or *band generators*, or *Birman-Ko-Lee generators.* The dual braid monoid B_n^{+*} is a submonoid of B_n generated by Σ^*. An element of B_n^{+*} is called a *dual-positive braid.* The braid $\delta = a_{1,2}a_{2,3} \cdots a_{n-1,n}$ is called the *dual Garside element.*

Let $\preccurlyeq$ be the suffix ordering with respect to the dual Garside generators Σ^*: $\beta_1 \preccurlyeq \beta_2$ if and only if $\beta_2\beta_1^{-1} \in B_n^{+*}$. This defines a lattice ordering on B_n, that is, for $s,t \in B_n$, there exists a unique least common multiple $s \vee t$ and a unique greatest common divisor $s \wedge t$. A dual-positive braid x is called a *dual-simple* if $x \preccurlyeq \delta$. The set of dual-simple element is denoted by $[1,\delta]$. Instead of Σ^*, we will often use $[1,\delta]$ as a generator of B_n.

A (right-greedy, dual Garside) *normal form* of a braid $\beta \in B_n$ is a decomposition of β as a product of dual simple elements of the form

$$\beta = x_r \cdots x_1\delta^p$$

that is defined by

1. p is the maximal integer that satisfies $\delta^p \preccurlyeq \beta$.
2. For $i = 1, \ldots, r$, $x_i = (x_r \cdots x_i) \wedge \delta$.

We will denote the normal form of β by $N(\beta)$. The normal form has the following remarkable property.

Proposition 2.1. *$x_r \cdots x_1\delta^p$ is a normal form if and only if $x_1 \neq \delta$ and $x_{i+1}x_i \wedge \delta = x_i$ for each i (in other words, $x_{i+1}x_i$ is a normal form for each i).*

This proposition leads to an effictive way of computing a normal form. Moreover, the normal forms induces a bi-automatic structure of the braid groups. See [ECHLPT, Deh].

The *supremum* $\sup(\beta)$ and the *infimum* $\inf(\beta)$ of β are integers defined by

$$\begin{cases} \sup(\beta) = \min\{m \in \mathbb{Z} \mid \beta \preccurlyeq_{\Sigma^*} \delta^m\} \\ \inf(\beta) = \max\{M \in \mathbb{Z} \mid \delta^M \preccurlyeq_{\Sigma^*} \beta\} \end{cases}$$

These values are closely related to the normal form of β. If $N(\beta) = x_r \cdots x_1\delta^p$ then

$$\sup(\beta) = p + r, \text{ and } \inf(\beta) = p.$$

The *dual Garside length* $l = l_{\Sigma^*}$ is the length function of B_n with respect to the dual-simple elements $[1,\delta]$. It is known that

$$l_{\Sigma^*}(\beta) = \max\{0, \sup_{\Sigma^*}(\beta)\} - \min\{\inf_{\Sigma^*}(\beta), 0\}.$$

Thus one can efficiently compute the dual Garside length by computing the normal forms.

2.2 Diagrammatic expression of dual Garside structures

Here we explain a convenient expression of dual simple elements using convex polytopes.

Let $D_n = \{z \in \mathbb{C} \mid |z| \leq 1\}$ be the n-punctured disc. We put the puncture points $p_1, \ldots, p_n$ on the circle $|z| = \frac{1}{2}$, as shown in the right of Figure 1. Then there is a one-to-one correspondence between the set of disjoint collections of convex polygons in D_n whose vertices are puncture points, and the set of dual-simple elements.

This correspondence is given as follows: First assume that a convex polygon P is connected. Let $p_{m_1}, \ldots, p_{m_k}$ $(1 \leq m_1 < m_2 < \cdots < m_k \leq n)$ be the vertices of P. We define a braid x_P

$$x_P = a_{m_1,m_2} a_{m_2,m_3} \cdots a_{m_{k-1},m_k}.$$

For a disjoint collection of convex polygons $\mathbb{P} = \{P_1, \ldots, P_M\}$, we define

$$x_{\mathbb{P}} = x_{P_1} x_{P_2} \cdots x_{P_M}.$$

Then it is seen that $x_{\mathbb{P}}$ is a dual-simple element. Conversely, every dual-simple element can be expressed in such a way. For $x \in [1, \delta]$, we will write the corresponding convex polygons by P_x.

This correspondence can be easily understood by using geometric interpretation of the braid groups. As is well-known, the braid group B_n is identified with the mapping class group of n-punctured disc D_n.

For $1 \leq i < j \leq n$, let e_{ij} be the line segment that connects the i-th and the j-th punctures. As an element of mapping class group, the band-generator $a_{i,j}$ corresponds to the left-handed half-Dehn twist along e_{ij}: $a_{i,j}$ interchanges the position of punctures p_i and p_j by rotating the small disc neighborhood of e_{ij} in a clockwise direction (see Figure 1).

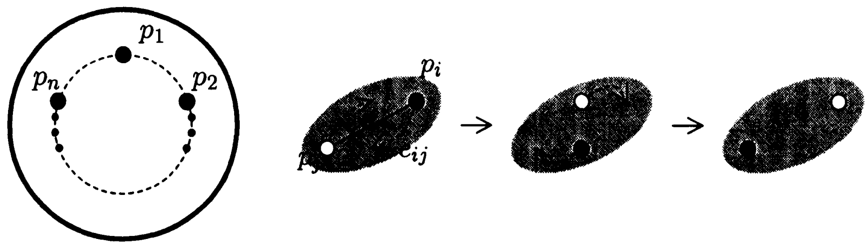

Figure 1: n-punctured disc D_n and action of a_{ij}

By generalizing this move of punctures, for a collection of convex polygons we associate a dance of the puncture points. Each puncture which belongs to some polygon P moves to the position of the adjacent vertex, in the clockwise direction along the boundary of P, see Figure 2. In particular, the dual Garside element δ acts on D_n as rotation of disc by $(2\pi/n)$.

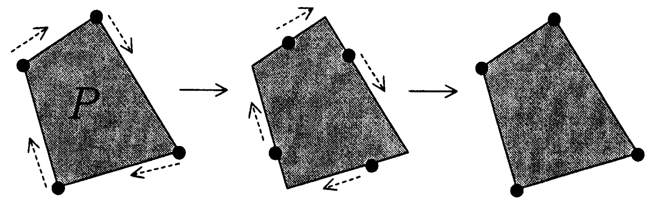

Figure 2: Polygon expression of dual-simple elements: an action on D_n

3 Generic quantum $\mathfrak{sl}_2$ representation

In this section, we review a construction of generic quantum $\mathfrak{sl}_2$-representation following Jackson-Kerler [JK]. For basics of $U_q(\mathfrak{sl}_2)$ we refer [Kas]. (Here we remark that to make correspondence between the dual Garside structure and quantum representations simple, we slightly modified the sign convention: the variable s in this paper corresponds to s^{-1} in [JK, I1].)

We define the q-numbers, q-fractionals, and q-binomial coefficients as

$$[n]_q! = [n]_q[n-1]_q\cdots[2]_q[1]_q,$$

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad \begin{bmatrix} n \\ j \end{bmatrix}_q = \frac{[n]_q!}{[n-j]_q![j]_q!}.$$

Let $\mathbb{C}[[\hbar]]$ be the algebra of the complex formal power series in one variable $\hbar$. A quantum enveloping algebra $U_\hbar(\mathfrak{sl}_2)$ is a topological Hopf algebra over $\mathbb{C}[[\hbar]]$ generated by H, E, F, with relations

$$\begin{cases} [H,E] = 2E, \quad [H,F] = -2F, \\ [E,F] = \frac{e^{\hbar H} - e^{-\hbar H}}{e^{\hbar} - e^{-\hbar}} \end{cases} \tag{3.1}$$

The coproduct $\Delta : U_\hbar(\mathfrak{sl}_2) \to U_\hbar(\mathfrak{sl}_2)\widetilde{\otimes} U_\hbar(\mathfrak{sl}_2)$ (here $\widetilde{\otimes}$ denotes the topological tensor product, the $\hbar$-adic completion of $U_\hbar(\mathfrak{sl}_2) \otimes U_\hbar(\mathfrak{sl}_2)$), and the antipode S are given by

$$\begin{cases} \Delta(E) = E \otimes e^{\hbar H} + 1 \otimes E, \\ \Delta(F) = F \otimes 1 + e^{-\hbar H} \otimes F, \\ \Delta(H) = H \otimes 1 + 1 \otimes H, \\ S(E) = -Ee^{-\hbar H}, \quad S(F) = -e^{\hbar H}F_i, \quad S(H) = -H. \end{cases} \tag{3.2}$$

$U_\hbar(\mathfrak{sl}_2)$ is a quasi-triangular topological Hopf algebra. Namely, there exists an element $\mathcal{R} \in U_\hbar(\mathfrak{sl}_n) \widetilde{\otimes} U_\hbar(\mathfrak{sl}_n)$ called a *universal R-matrix* that satisfies the properties

$$\begin{cases} \mathcal{R}\Delta(x) = \Delta^{op}(x)\mathcal{R}, \\ (\Delta \otimes \mathsf{id})\mathcal{R} = \mathcal{R}_{12}\mathcal{R}_{23}, \\ (\mathsf{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}. \end{cases}$$

where Δ^{op} denotes the opposite of Δ. These properties show that $\mathcal{R}$ satisfies the Yang-Baxter equation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}.$$

The universal R-matrix $\mathcal{R}$ for $U_\hbar(\mathfrak{sl}_2)$ is given by

$$\mathcal{R} = e^{\frac{\hbar}{2}(H\otimes H)}\left(\sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}}\frac{(q-q^{-1})^n}{[n]_q!}E^n\otimes F^n\right), \tag{3.3}$$

where we put $q = e^{\hbar}$.

For $\lambda \in \mathbb{C}^*$, let V_λ be the Verma module with highest weight λ, which is a topologically free $U_\hbar(\mathfrak{sl}_2)$-module generated by a highest weight vector v_0 that satisfies

$$Hv_0 = \lambda v_0, \text{ and } Ev_0 = 0.$$

If λ is not an integer ("generic"), then as a $\mathbb{C}[[\hbar]]$-module, the Verma module V_λ is freely generated by $\{v_i\}_{i=0,1,\dots}$ and the action of $U_\hbar(\mathfrak{sl}_2)$ is given by

$$\begin{cases} Hv_i = (\lambda - 2i)v_i \\ Ev_i = v_{i-1} \\ Fv_i = [i+1]_q\frac{e^{\hbar(\lambda-i)}-e^{-\hbar(\lambda-i)}}{e^{\hbar}-e^{-\hbar}}v_{i+1}. \end{cases} \tag{3.4}$$

Now we treat all generic Verma modules at one time by regarding a weight λ as a variable instead of treating as a complex parameter. To this end, we regard $U_\hbar(\mathfrak{sl}_2)$ as a topological Hopf algebra over the coefficient ring $\mathbb{C}[\lambda][[\hbar]]$, the polynomial ring with coefficients in $\mathbb{C}[[\hbar]]$. We regard the formula (3.4) as a definition of a $U_\hbar(\mathfrak{sl}_2)$-module $V_\hbar$: namely, as a topological $\mathbb{C}[\lambda][[\hbar]]$-module, $V_\hbar$ is a $\mathbb{C}[\lambda][[\hbar]]$-module freely generated by $\{v_0, v_1, \dots,\}$ and $U_\hbar(\mathfrak{sl}_2)$ acts on $V_\hbar$ by the formula (3.4). We call $V_\hbar$ *generic Verma module.*

Let $\mathbb{L} = \mathbb{C}[q^{\pm1}, s^{\pm1}]$ be the ring of two-variable Laurent polynomial, and we regard $\mathbb{L}$ as a subring of $\mathbb{C}[\lambda][[\hbar]]$ via the injective homomorphism $i_\hbar : \mathbb{L} \to \mathbb{C}[\lambda][[\hbar]]$ defined by $i_\hbar(q) = e^{\hbar}$, $i_\hbar(s) = e^{\hbar\lambda}$.

Let $V_{\mathbb{L}} \subset V_\hbar$ be the free $\mathbb{L}$-module generated by basis vectors $\{v_0, v_1, \dots,\}$ of $V_\hbar$, and let $R = e^{-\frac{\hbar}{2}\circ\lambda^2}\circ\mathcal{R}\circ T : V_{\mathbb{L}}^{\otimes 2} \to V_{\mathbb{L}}^{\otimes 2}$.

Then by (3.3), the action of R is written as

$$R(v_i\otimes v_j) = s^{(i+j)}\sum_{n=0}^{i} F_{i,j,n}(q)\prod_{k=0}^{n-1}(s^{-1}q^{-k-j} - sq^{k+j})v_{j+n}\otimes v_{i-n}, \tag{3.5}$$

where $F_{i,j,n}(q) = q^{2(i-n)(j+n)}q^{\frac{n(n-1)}{2}}\begin{bmatrix} n+j \\ j \end{bmatrix}_q$.

Similarly, the action of R^{-1} is written as

$$R^{-1}(v_i\otimes v_j) = s^{-(i+j)}\sum_{n=0}^{j}(-1)^n F'_{i,j,n}(q)\prod_{k=0}^{n-1}(sq^{k+i} - s^{-1}q^{-k-i})v_{j-n}\otimes v_{i+n},$$

where $F'_{i,j,n}(q) = q^{-2ij}q^{\frac{-n(n-1)}{2}} \begin{bmatrix} n+i \\ i \end{bmatrix}_q$.

Thus $R(V_{\mathbb{L}}^{\otimes 2}) = V_{\mathbb{L}}^{\otimes 2}$, so we get a (infinite dimensional) linear representation

$$\rho : B_n \to \mathrm{GL}(V_{\mathbb{L}}^{\otimes n}), \quad \rho(\sigma_i) = \mathsf{id}^{\otimes(i-1)} \otimes R \otimes \mathsf{id}^{\otimes(n-i-1)}$$

which we call a *generic quantum $\mathfrak{sl}_2$-representation.*

To deduce finite dimensional reprsentation, we take a weight decomposition of ρ. For $m \geq 0$, let $V_{n,m} = \{v \in V_{\mathbb{L}}^{\otimes n} \mid e^{\hbar H} v = s^{-n}q^{-2m}v\}$ be the weight space corresponding to the weight $s^{-n}q^{-2m}$. ($e^{\hbar H}$ is often denoted by K in a literature). It is directly checked that $V_{\mathbb{L}}^{\otimes n}$, as a $\mathbb{C}B_n$-module, decomposes as $V_{\mathbb{L}}^{\otimes n} = \bigoplus_{m=0}^{\infty} V_{n,m}$.

The set $\{v_{k_1} \otimes \cdots \otimes v_{k_n} \mid k_i \geq 0,\ k_1 + \cdots + k_n = m\}$ forms a basis of $V_{n,m}$. To relate the representation $V_{n,m}$ and the dual Garside structure, we use the following slightly modified basis of $V_{n,m}$, obtained by shifting the degree of the variable s. For $\mathbf{k} = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}_{\geq 0}^n$ we define

$$|\mathbf{k}| = \sum_{i=1}^{n} k_i, \text{ and } w_{\mathbf{k}} = s^{-\sum_{i=1}^{n} ik_i} v_{k_1} \otimes v_{k_2} \otimes \cdots \otimes v_{k_n} \in V^{\otimes n}.$$

Then the set

$$\mathcal{B} = \mathcal{B}(m) = \{w_{\mathbf{k}} \mid |\mathbf{k}| = m\}$$

form a basis of $V_{n,m}$. The cardinal of $\mathcal{B}(m)$ is $\binom{n+m-1}{m}$. By using this basis $\mathcal{B}(m)$, we express the braid group representation $V_{n,m}$ as an explicit matrix

$$\rho_{m,n} : B_n \to \mathrm{GL}\left(\binom{n+m-1}{m}; \mathbb{L}\right).$$

We call this representation a *generic quantum $\mathfrak{sl}_2$ representation.*

4 Main Theorem

From now on, we fix $m > 1$, and we put $V = V_{n,m}$ and $\mathcal{B} = \mathcal{B}(m)$. By abuse of notation, we may often identify the basis vector $w_{\mathbf{k}} \in \mathcal{B}$ and its corresponding sequence of integers $\mathbf{k} = (k_1, \ldots, k_n)$. To make notation simple, for $\beta \in B_n$ and $w \in V$, we will write $\beta(w)$ to imply $\rho_{m,n}(\beta)(w)$.

4.1 Statement of Main theorem

For monomials $s^i q^j$ and $s^{i'} q^{j'}$ of $\mathbb{L} = \mathbb{Z}[s^{\pm 1}, q^{\pm 1}]$, we define the lexicographical ordering $\leq_{s,q}$ by

$$s^i q^j \leq_{s,q} s^{i'} q^{j'} \text{ if } i < i', \text{ or if } i = i' \text{ and } j \leq j'.$$

For $a \in \mathbb{L}$, we will concentrate our attention to the $<_{s,q}$-maximal monomial. We denote the maximal and the minimum degree of the variable s in a by $M_s(a)$ and $m_s(a)$, respectively, and we write the $<_{s,q}$-maximal monomial in a by $s^{M_s(a)} q^{N_q(a)} = s^M q^N$. The *sign* $\varepsilon(a) \in \{\pm 1\}$ is defined as the sign of the coefficient of the $<_{s,q}$-maximal monomial $s^M q^N$ in a.

For $i = 1, \dots, n$, let $\mathbf{k}_i \in \mathcal{B}$ be a basis vector

$$\mathbf{k}(i) = (0, \dots, 0, \overset{i}{m}, 0, \dots, 0).$$

and define $w \in V$ by

$$w = \sum_{i=1}^{n} w_{\mathbf{k}(i)}.$$

w and $w_{\mathbf{k}(i)}$ plays an important role in computations in quantum representations.

For $v = \sum_{w_{\mathbf{k}} \in \mathcal{B}} a_{\mathbf{k}}(s, q) w_{\mathbf{k}} \in V$, we define

$$M_s(v) = \max\{M_s(a_{\mathbf{k}}) \mid \mathbf{k} \in \mathcal{B}\}.$$

By looking at the $<_{s,q}$-maximal monomials of $a_{\mathbf{k}}$, we assign a graph $\Gamma(v)$ in D_n in the following manner:

The vertices of $\Gamma(v)$ is a subset of the puncture points of D_n. The i-th puncture p_i is a vertex of $\Gamma(v)$ if and only if

(V) $M_s(a_{\mathbf{k}(i)}) = M_s(v)$

holds.

Now assume that for $1 \leq i < j \leq n$, both p_i and p_j are vertices of $\Gamma(v)$. For $e = 0, \dots, m$, let us put

$$\mathbf{k}(e; i, j) = (0, \dots, 0, \overset{i}{\check{e}}, 0, \dots, 0, \overset{j}{m \check{-} e}, 0, \dots, 0) \in \mathcal{B}.$$

We connect two vertices p_i and p_j by an edge if and only if

(E) The $<_{s,q}$-maximal monomial part of $a_{\mathbf{k}(e;i,j)}$ is

$$(-1)^e \varepsilon(a_{\mathbf{k}(0)}) \cdot c \cdot s^{M_s(v)} q^{N_q(a_{\mathbf{k}(0)})} q^{2em - e^2 - e}.$$

where $c > 0$ is the absolute value of the coefficient of the $<_{s,q}$-maximal monomial.

holds.

Finally we assign a graph $\Gamma(x)$ for each dual simple element x.

Definition 4.1. For a dual simple element $x \in [1, \delta]$ we define the graph $\Gamma(x)$ by $\Gamma(x) = \Gamma(x(w))$.

At first glance, the definition of the graph Γ seems to be artificial. Here we explain the background motivation of the definition of Γ.

Let us rewrite a formula of the R-action on $V \otimes V$ in terms of our modified (s-degree shifted) basis $\{w_{i,j} = s^{i+2j} v_i \otimes v_j\}$ of $V \otimes V$, and concentrate our attention to the $<_{s,q}$-maximal monomials. Then the $<_{s,q}$-maximal monomial is given by

$$\begin{aligned} R(w_{i,j}) &= \sum_{n=0}^{i} s^{2i-n} q^{2(i-n)(j+n)} q^{\frac{n(n-1)}{2}} \begin{bmatrix} n+i \\ i \end{bmatrix}_q \prod_{k=0}^{n-1} (sq^{-k-j} - s^{-1} q^{k+j}) w_{j+n,i-n} \\ &= \sum_{n=0}^{i} ((-1)^n s^{2i} q^{2ij+2ni-n^2-n} + \cdots) w_{j+n,i-n} \end{aligned}$$

This formula says that $M_s(R(w_{i,j})) = 2i \leq 2m$. Since we are interested in the case s-degree maximal part, let us consider the case $i=m$ and $j=0$. Then we get

$$R(w_{m,0}) = \sum_{n=0}^{m}((-1)^m s^{2m} q^{2nm-n^2-n} + \cdots)w_{n,m-n}$$

This shows that the graph $\Gamma(a_{i,i+1}(w_{\mathbf{k}(i)}))$ coincides with the convex polygon $e_{i,i+1}$ (an edge connecting p_i and p_{i+1}).

More generally by using the above formula of R, one can check that for $i \leq k < j$, $\Gamma(a_{i,j}(w_{\mathbf{k}(k)}))$ coincides with the convex polygon $e_{i,j}$ (an edge connecting p_i and p_j). Thus, the graph Γ was defined so that it captures the behaviour of the $<_{s,q}$-maximal part of $a_{i,j}(w)$ or $a_{i,j}(w_{\mathbf{k}(k)})$.

For a general dual simple element x, like $x = a_{i,j}$ case, its graph $\Gamma(x)$ is closely related to the corresponding convex polygon P_x although the relations are more complicated (especially when P_x is not connected): Figure 3 shows several examples of the graph $\Gamma(x)$. As $\Gamma(a_{1,4}a_{2,3})$ suggests, not all edges of $\Gamma(x)$ is contained in the corresponding convex polygon P_x. It is checked that $\Gamma(x)$ does not depend on m, and for $x, y \in [1, \delta]$, $\Gamma(x) \neq \Gamma(y)$ if $x \neq y$.

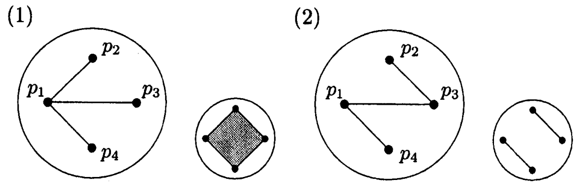

Figure 3: (1) $\Gamma(a_{1,2}a_{2,3}a_{3,4}), P_{a_{1,2}a_{2,3}a_{3,4}}$ and (2) $\Gamma(a_{1,4}a_{2,3}), P_{a_{1,4}a_{2,3}}$

Now we are ready to state the main theorem.

Theorem 4.2 (Dual Garside normal form and generic quantum $\mathfrak{sl}_2$-representation). *Let $N(\beta) = x_r \cdots x_1 \delta^p$ be the normal form of $\beta \in B_n$. Then*

1. $M_s(\beta w) = 2m \sup(\beta)$.
2. $m_s(\beta w) = 2m \inf(\beta)$.
3. $\Gamma(\beta w) = \Gamma(x_r)$.

This theorem shows that, the maximal $<_{s,q}$-maximal part of a generic quantum $\mathfrak{sl}_2$ representation nicely reflects the dual Garside normal form. In particular, one can compute the normal from of the braid β by looking at the single vector $\beta(w)$.

Recall that the variable s in a generic quantum representation $\rho_{m,n}$ comes from the weight of the Verma module. Thus we may view the maximal s-degree part of $\beta(w)$ as "highest weight" parts. Thus or main theorem suggests that there is an unexpected relationship between representation theory of lie algebras and quantum groups (highest weight vectors), and the dual Garside structures.

4.2 Several consequences of main theorem

We close the paper by presenting several consequences of our main theorem.

First we observe that as a corollary of our main theorem, we provide an alternative, algebraic proof of the main results in [I1]. For an $N \times N$-matrix of $\mathbb{L}$ coefficient $A = (a_{ij})$, we denote the maximal and the minimal degree of s in A, $\max_{i,j} M_s(a_{ij})$ and $\min_{i,j} m_s(a_{ij})$, by $M_s(A)$ and $m_s(A)$, respectively.

Corollary 4.3 (Dual Garside length formula [I1]). *Let $\beta \in B_n$.*

1. $M_s(\rho_{m,n}(\beta)) = 2m\sup(\beta)$.
2. $m_s(\rho_{m,n}(\beta)) = -2m\inf(\beta)$.
3. $l(\beta) = 2m\left(\max\{0, M_s(\rho_{m,n}(\beta))\} - \min\{0, m_s(\rho_{m,n}(\beta))\}\right)$.

Our argument provides a remarkable restriction for an image of generic quantum representation $\rho_{n,m}$.

Theorem 4.4 (Image of quantum representation). *Let $A \in GL(\binom{n+m-1}{m}; \mathbb{L})$.*

1. *If A lies in the image of the generic quantum representation $\rho_{n,m}$, then for $1 \leq i \leq n$, $\Gamma(A_i) = \Gamma(x)$ for some $x \in [1, \delta]$. Here A_i denotes the low of A that corresponds to the basis vector $\mathbf{k}(i)$.*
2. *There is an effective algorithm to determine whether A lies in the image of the generic quantum representation $\rho_{n,m}$ or not.*

We also remark that Theorem 4.2 gives a new, quantum-group theoretical proof of the faithfulness of the Lawrence-Krammer-Bigelow representation and its natural generalizations called *Lawrence's representation* $L_{n,m}$,

$$L_{n,m} : B_n \to \mathrm{GL}\left(\binom{n+m-2}{m}; \mathbb{Z}[\mathsf{q}^{\pm 1}, \mathsf{t}^{\pm 1}]\right).$$

Lawrence's representations are obtained by considering the action of the braid groups on the homology group (of local system coefficients) of the configuration space of m-points in n-punctured disc D_n. For details, see [I1, Law]. $L_{n,1}$ is identical with the reduced Burau representation, and $L_{n,2}$ is called the *Lawrence-Krammer-Bigelow representation.* It is known that generic quantum representation decomposes as $\rho_{n,m} = \bigoplus_{i=0}^{m} L_{n,i}$.

Theorem 4.5 (Dual Garside length formula [I1]). *For $\beta \in B_n$,*

1. $M_s(L_{n,m}(\beta)) = m\sup(\beta)$.
2. $m_s(L_{n,m}(\beta)) = -m\inf(\beta)$.
3. $l(\beta) = m\left(\max\{0, M_s(L_{n,m}(\beta))\} - \min\{0, m_s(L_{n,m}(\beta))\}\right)$.

In particular, $L_{n,m}$ is faithful.

It is already known that $L_{n,m}$ is faithful ([I2] for details). However, the known proof of the faithfulness for $m > 2$ is based on a topological argument due to Bigelow [Big]. Our quantum-representation proof gives a purely algebraic proof.

References

[Big] S. Bigelow, *Braid groups are linear*, J. Amer. Math. Soc. **14**, (2000), 471–486.

[BGG] J. Birman, V. Gebhardt, and J. González-Meneses *Conjugacy in Garside groups. I. Cyclings, powers and rigidity,* Groups Geom. Dyn. **1** (2007), 221–279.

[BKL] J. Birman, K.H. Ko, and S.J. Lee, *A new approach to the word problem in the braid groups,* Adv. Math. **139** (1998), 322–353.

[Deh] P. Dehornoy, *Groupes de Garside,* Ann. Sci. Ec. Norm. Sup., **35** (2002) 267–306.

[DDKM] F. Digne, P. Dehornoy, E. Godelle, D. Krammer, and J. Michel *Garside Theory,* Draft of book, available at `http://www.math.unicaen.fr/~garside/Garside.html`

[ECHLPT] D. Epstein, J. Cannon, D. Holt, S. Levy, M. Paterson, and W. Thurston *Word processing in groups,* Jones and Bartlett Publisher, Boston, MA, 1992

[G] F. Garside, The braid group and other groups, Quart. J. Math **20** (1969) 235–254

[I1] T. Ito, *Reading the dual Garside length of braids from homological and quantum representations,* arXiv:1205.5245

[I2] T. Ito, *Quantum representation of braid groups and dual Garside structure I: $\mathfrak{sl}_2$ case,* In preparation.

[I3] T. Ito, *Quantum representation of braid groups and dual Garside structure II: Generic $U_q(\mathfrak{g})$ representations,* In preparation.

[IW] T. Ito and B. Wiest, *Lawrence-Krammer-Bigelow representation and dual Garside length of braids,* arXiv:1201.0957v1

[JK] C. Jackson and T. Kerler, *The Lawrence-Krammer-Bigelow representations of the braid groups via $U_q(\mathfrak{sl}2$,* Adv. Math, **228**, (2011), 1689–1717.

[Kas] C. Kassel, *Quantum groups*, Graduate Texts in Mathematics **155**. Springer-Verlag, New York, 1995.

[Law] R. Lawrence, *Homological representations of the Hecke algebra*, Comm. Math. Phys. **135**, (1990), 141–191.

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