数理解析研究所講究録
第 1866 巻 2013 年 105-119

A table of coherent band-Gordian distances between knots

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Abstract

We introduce some criteria for two links, which are related by a coherent band surgery, using the determinant, and the Jones, HOMFLYPT, and Q polynomials. We give a table of coherent band-Gordian distances between two knots with up to seven crossings.

1 Introduction

There are several criterion for two links, which are related by a band surgery or crossing change. In this paper, we introduce further criteria using the determinant, and the Jones, HOMFLYPT, and Q polynomials. A band surgery and a crossing change are local changes in a link diagram as shown in Figure. 1. If we consider oriented links, there are two types for a band surgery according to an orientation; a coherent band surgery (Fig 2) and an incoherent one. In particular, an incoherent band surgery between two knots is called an $H(2)$-move [14] (Figure. 3). Recently, these local moves are studied in connection with an application to the study of DNA site-specific recombination; see [5, 6, 9].

Figure 1: A band surgery and a crossing change.

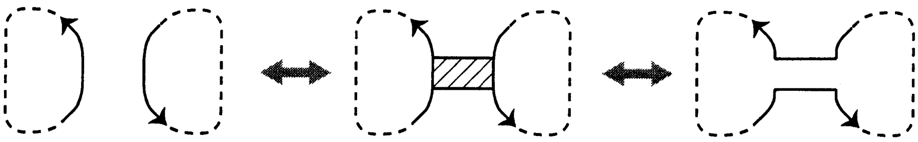

Figure 2: A coherent band surgery.

Given two links L and L', we want to decide whether they are related by a band surgery or a crossing change. The signature and Arf invariant are most useful tools for this problem (Propositions 2.2 and 2.3). There are also several other methods to deal with this problem: for a coherent band surgery, see [19, 21]; for a crossing change, see [30, 32, 35, 40, 41, 42]; for an $H(2)$-move, see [20, 23, 26]; see also [1].

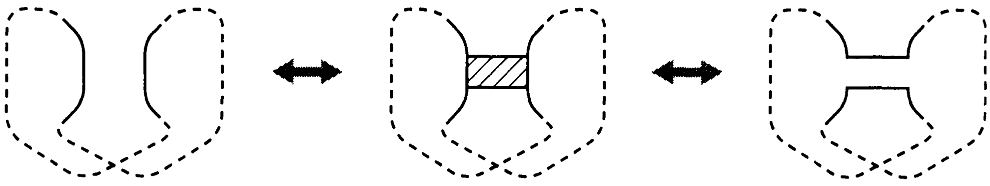

Figure 3: An $H(2)$-move.

Our main results are two criteria: The first one is a condition on the determinant of a link or knot which is obtained from a 2-bridge knot by a coherent band surgery or $H(2)$-move (Theorem 3.2), which is easily obtained by using a condition on the determinant of a knot obtained from a 2-bridge knot by a crossing change due to Hitoshi Murakami [32] (Proposition 3.1).

The second one uses some special values of the polynomial invariants. For the Jones polynomial, we have a criterion on two links which are related by a coherent band surgery [19, Theorem 2.2] (Theorem 4.2). Developing this, we obtain Theorem 4.6. In a similar way, for the HOMFLYPT polynomial we obtain Theorem 5.4 developing Proposition 5.1, and for the Q polynomial Theorem 6.2 developing Proposition 6.1. We give some examples for each of these criteria, which display the efficiency of them. In a forthcoming paper [24] we will make a detailed report on these criteria.

Notation. For knots and links with up to 9 crossings we use Rolfsen notations [38, Appendix C]. For a knot or link L, we denote by $L!$ its mirror image. For an oriented 2-component link with c crossings we use the notations c_n^2 and $c_n^{2\prime}$, where we usually suppose that linking number of c_n^2 is negative and that of $c_n^{2\prime}$ is positive as in Table 2 in [21]; more precisely, c_n^2 denotes an oriented link with negative linking number with diagram as in the table of [38] and $c_n^{2\prime}$ denotes one of the oriented links obtained from c_n^2 by reversing the orientation of one component.

2 Some invariants

The *Conway polynomial* $\nabla(L;z) \in \boldsymbol{Z}[z]$ [4], the *Jones polynomial* $V(L;t) \in \boldsymbol{Z}[t^{\pm 1/2}]$ [17], and the *HOMFLYPT polynomial* $P(L;v,z) \in \boldsymbol{Z}[v^{\pm 1}, z^{\pm 1}]$ [10, 17, 36] are invariants of the isotopy type of an oriented link L, which are defined by the following formulas:

$$\nabla(U;z) = 1; \tag{1}$$
$$\nabla(L_+;z) - \nabla(L_-;z) = z\nabla(L_0;z); \tag{2}$$
$$V(U;t) = 1; \tag{3}$$
$$t^{-1}V(L_+;t) - tV(L_-;t) = \left(t^{1/2} - t^{-1/2}\right) V(L_0;t); \tag{4}$$
$$P(U;v,z) = 1; \tag{5}$$
$$v^{-1}P(L_+;v,z) - vP(L_-;v,z) = zP(L_0;v,z), \tag{6}$$

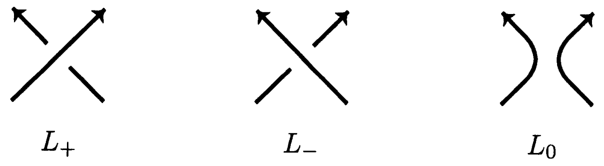

Figure 4: A skein triple.

where U is the unknot and (L_+, L_-, L_0) is a skein triple.

For a skein triple (L_+, L_-, L_0), the link L_+ is obtained from L_- by changing a crossing, and vice versa, and the link L_0 is obtained from L_+ or L_- by a coherent band surgery, and vice versa. Conversely, it is easy to see the following:

Lemma 2.1. *If a c-component link L and a $(c+1)$-component link M are related by a coherent band surgery, then there exist c-component links L_+, L_- and $(c+1)$-component links M_+, M_- such that each of the following is a skein triple: (L_+, L, M), (L, L_-, M), (M_+, M, L), (M, M_-, L).*

For a c-component link L, $i^{c-1}V(L;-1)$ is an integer and the determinant $\det L$ is given by $\det L = |V(L;-1)|$. Putting $t=-1$ in Eq. (4), we obtain

$$-V(L_+;-1) + V(L_-;-1) = 2iV(L_0;-1); \tag{7}$$

Let (L_+, L_-, L_0) be a skein triple. Then Murasugi [34, Lemma 7.1] has shown:

$$|\sigma(L_\pm) - \sigma(L_0)| \leq 1. \tag{8}$$

Since we may consider the link L_+ or L_- as obtained from L_0 by a coherent band surgery, and vice versa, we have the following.

Proposition 2.2. (i) *If two oriented links L and L' are related by a coherent band surgery, then*

$$|\sigma(L) - \sigma(L')| \leq 1. \tag{9}$$

(ii) *If two oriented links L and L' are related by a crossing change, then*

$$|\sigma(L) - \sigma(L')| \leq 2. \tag{10}$$

The *Arf invariant* (or *Robertello invariant*) [37] of a knot K, $\mathrm{Arf}(K)$, is given by

$$\mathrm{Arf}(K) = a_2(K) \in \boldsymbol{Z}_2, \tag{11}$$

where $a_2(K)$ is the coefficient of z^2 of the Conway polynomial of K. Whenever an equality in this paper contains an Arf invariant it is to be understood in the sense of mod 2. We say that an oriented link L is related (in the sense of Robertello [37]) to a knot K if there exists a smooth embedding of a planar surface F in $S^3 \times I$ such that F meets $S^3 \times \{0,1\}$ transversely in K and L, respectively. Let L be a proper link, that is, the sum of the linking numbers of any component of L with all the other components is even. We may define its Arf invariant to be the Arf invariant of any knot K related to it. In particular, we have:

Proposition 2.3. *If a knot K is obtained from a proper 2-component link L by a coherent band surgery, then $\mathrm{Arf}(K) = \mathrm{Arf}(L)$.*

3 Determinant of a link obtained from a 2-bridge knot by a band surgery

For relatively prime integers p, q with $p > q > 0$ and p odd, we let $S_{p,q}$ denote the 2-bridge knot for which the lens space of type (p, q) is the 2-fold branched cover of S^3. More explicitly, let a_1, a_2, $a_3, \dots, a_n$ be positive integers obtained from the continued fraction

$$\frac{p}{q} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}. \tag{12}$$

Then $S_{p,q}$ is isotopic to a 2-bridge knot in Conway's normal form $C(a_1, a_2, a_3, \dots, a_{n-1}, a_n)$ as shown in Figure. 5, where the box containing an integer a or $-a$, $a > 0$, denotes a 2-braid as shown in Figure. 6. Also, $S_{p,-q}$ presents the mirror image of $S_{p,q}$; cf. [25, Sec. 2.1].

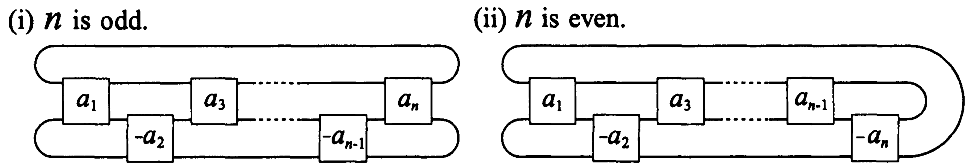

Figure 5: The 2-bridge knot $C(a_1, a_2, a_3, \dots, a_{n-1}, a_n)$.

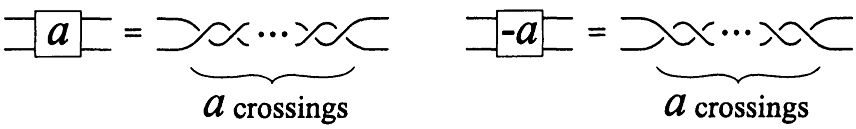

Figure 6: 2-braids.

The following criteria is due to H. Murakami [32, Corollary 2.8].

Proposition 3.1. *Suppose that a knot K is obtained from a 2-bridge knot $S_{p,q}$ by a crossing change. Then there exists an integer s such that:*

$$|\det K - p|/2 \equiv \pm q s^2 \pmod{p}. \tag{13}$$

Using this, we may deduce the following.

Theorem 3.2. *Suppose that a link L is obtained from a 2-bridge knot $S_{p,q}$ by a coherent or incoherent band surgery. Then there exists an integer s such that:*

$$\det L \equiv \pm q s^2 \pmod{p}. \tag{14}$$

Proof. Suppose that L and $S_{p,q}$ are related by a coherent band surgery. Then by Lemma 2.1 there exists a knot K such that $(K, S_{p,q}, L)$ is a skein triple. From Eq. (7) we have

$$-V(K;-1) + V(S_{p,q};-1) = 2iV(L;-1), \tag{15}$$

which implies

$$2\det L = |2iV(L;-1)| = |-V(K;-1) + V(S_{p,q};-1)|. \tag{16}$$

Since K and $S_{p,q}$ are related by a crossing change, by Proposition 3.1 there exists an integer s such that Eq. (13) holds, which implies

$$\det K + p \equiv \det K - p \equiv \pm 2qs^2 \pmod{2p}, \tag{17}$$

Since $\det K = |V(K;-1)|$ and $p = |V(S_{p,q};-1)|$, combining Eqs. (16) and (17), we obtain Eq. (14).

□

By Theorem 3.2 a 2-bridge knot may have some condition on the values of $\det L$, where L is either a 2-component link with $\mathrm{d_{cb}}(S_{p,q}, L) = 1$ or a knot with $\mathrm{d}_2(S_{p,q}, L) = 1$. For 2-bridge knots with up to 8 crossings, Table 1 lists these values; the remaining 2-bridge knots 3_1, 5_2, 6_2, 7_1, 7_2, 7_6, 8_4, 8_6, 8_7, 8_{14} have no such restrictions.

Table 1: Values which $\det L$ does not take with $\mathrm{d_{cb}}(S_{p,q}, L) = 1$ or $\mathrm{d}_2(S_{p,q}, L) = 1$

$S_{p,q}$	$\not\equiv \det L$
$4_1 = S_{5,2}$	1, 4 (mod 5)
$5_1 = S_{5,1}$	2, 3 (mod 5)
$6_1 = S_{9,2}$	3, 6 (mod 9)
$6_3 = S_{13,5}$, $8_1 = S_{13,6}$	1, 3, 4, 9, 10, 12 (mod 13)
$7_3 = S_{13,3}$	2, 5, 6, 7, 8, 11 (mod 13)
$7_4 = S_{15,4}$	2, 3, 7, 8, 12, 13 (mod 15)
$7_5 = S_{17,5}$, $8_2 = S_{17,6}$	1, 2, 4, 8, 9, 13, 15, 16 (mod 17)
$7_7 = S_{21,8}$	1, 4, 5, 16, 17, 20 (mod 21)
$8_3 = S_{17,4}$	3, 5, 6, 7, 10, 11, 12, 14 (mod 17)
$8_8 = S_{25,9}$	2, 3, 5, 7, 8, 10, 12, 13, 15, 17, 18, 20, 22, 23 (mod 25)
$8_9 = S_{25,7}$	1, 4, 5, 6, 9, 10, 11, 14, 15, 16, 19, 20, 21, 24 (mod 25)
$8_{11} = S_{27,10}$	3, 6, 12, 15, 21, 24 (mod 27)
$8_{12} = S_{29,12}$, $8_{13} = S_{29,11}$	1, 4, 5, 6, 7, 9, 13, 16, 20, 22, 23, 24, 25, 28 (mod 29)

Example 3.3. Table 2 shows 2-component links which are not obtained from the 2-bridge knots in Table 1 by a coherent band surgery. The symbol × means that the link in the row is not obtained from the 2-bridge knot in the column by a coherent band surgery. For example, the knot 6_1 and the link 6_1^2 are not related by a coherent band surgery; moreover this implies that $K \in \{6_1, 6_1!\}$ and $L \in \{6_1^2, 6_1^{2\prime}, 6_1^2!, 6_1^{2\prime}!\}$ are not related by a coherent band surgery; cf. [15, Table 2], [16, Table II].

Table 2: Links and 2-bridge knots which are not related by a single coherent band surgery.

L	$\det L$	4_1	5_1	6_1	6_3	7_3	7_5	7_7	8_3	8_8	8_9	8_{12}
			7_4	8_{11}	8_1		8_2					8_{13}
U^2	0											
$2^2_1 = H_-$	2		×			×	×			×		
4^2_1, 7^2_7	4	×			×		×	×			×	×
$3_1\#H_-$, 6^2_1	6	×		×		×			×		×	×
5^2_1, 7^2_8, 8^2_1	8		×			×	×			×		
6^2_2, $4_1\#H_-$, $5_1\#H_-$	10				×				×	×	×	
6^2_3, $3_1\#4^2_1$	12		×	×	×				×	×		
7^2_1, $5_2\#H_-$	14	×			×				×		×	
7^2_3, 7^2_4	16	×			×		×	×			×	×
7^2_2	18		×			×	×			×		
7^2_5	20					×		×	×	×	×	×
	22		×		×			×	×	×		×
7^2_6	24	×		×		×			×		×	×

4 Coherent band-Gordian distance

The following is Proposition 2.3 in [22]:

Proposition 4.1. *If two knots K and K' are related by a sequence of two coherent band surgeries, then they are related by a single $SH(3)$-move, and vice versa. Thus* $\mathrm{d_{cb}}(K, K') = 2\mathrm{sd}_3(K, K')$ *and* $\mathrm{u_{cb}}(K) = 2\mathrm{su}_3(K)$.

The following is Theorem 2.2 in [19].

Theorem 4.2. *If two links L and L' are related by a coherent band surgery,* $\mathrm{d_{cb}}(L, L') = 1$, *then*

$$V(L;\omega)/V(L';\omega) \in \left\{\pm i, -\sqrt{3}^{\pm 1}\right\}. \tag{18}$$

Then we have the following, which is given in [22, Theorem 3.1].

Corollary 4.3. *If two knots K and K' are related by a single $SH(3)$-move,* $\mathrm{sd}_3(K, K') = 1$, *then*

$$V(K;\omega)/V(K';\omega) \in \left\{\pm 1, \pm i\sqrt{3}^{\pm 1}, 3^{\pm 1}\right\}. \tag{19}$$

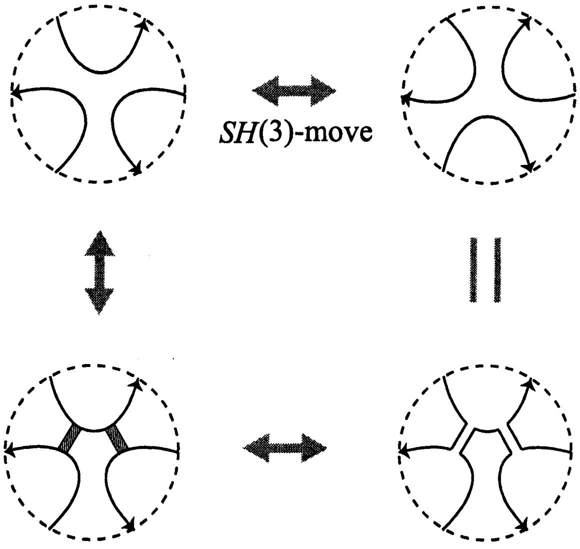

Figure 7: An $SH(3)$-move is correspond to two coherent band surgeries.

Example 4.4. Let $K = 4_1$ and $K' = 3_1!\#3_1$. Then $\mathrm{sd}_3(K, K') > 1$; see [15, Table 1]. Since $\sigma(K) = \sigma(K') = 0$, the signature cannot show $\mathrm{sd}_3(K, K') > 1$. However, since $V(K;\omega) = -1$, $V(K';\omega) = 3$, we can prove by using Corollary 4.3. In Table 3 we list all such pairs of knots with up to 7 crossings.

Table 3: Pairs of knots K and K' with $|\sigma(K) - \sigma(K')| \leq 2$ and $\mathrm{sd}_3(K, K') > 1$.

K	K'	$\sigma(K)$	$\sigma(K')$	$V(K;\omega)$	$V(K';\omega)$
4_1	$3_1!\#3_1$	0	0	-1	3
5_2	$3_1!\#3_1$	2	0	-1	3
7_6	$3_1!\#3_1$	2	0	-1	3
6_2	$3_1\#3_1$	2	4	1	-3
7_2	$3_1\#3_1$	2	4	1	-3
$7_3!$	$3_1\#3_1$	4	4	1	-3

The following is Theorem 5.2 in [24].

Theorem 4.5. *Suppose that a $(c+1)$-component link L' is obtained from a c-component link L by a coherent band surgery. If $V(L';\omega) = \eta i V(L;\omega) = \pm i^c(i\sqrt{3})^\delta$, $\eta = \pm 1$, then $i^c V(L';-1) \equiv \eta i^{c-1} V(L;-1) \pmod{3^{\delta+1}}$.*

Theorem 4.6. *Suppose that two links L and L' are related by a sequence of two coherent band surgeries, $\mathrm{d_{cb}}(L, L') = 2$. Let L be a c-component link. If $V(L;\omega) = -V(L';\omega) = \pm i^{c-1}(i\sqrt{3})^\delta$, then*

$$i^{c-1}V(L;-1) \equiv -i^{c-1}V(L';-1) \pmod{3^{\delta+1}} \tag{20}$$

By Proposition 4.1, we have:

Corollary 4.7. *If two knots K and K' are related by a single $SH(3)$-move,* $\mathrm{sd}_3(K, K') = 1$, *and* $V(K;\omega) = -V(K';\omega) = \pm(i\sqrt{3})^\delta$, *then*

$$V(K;-1) \equiv -V(K';-1) \pmod{3^{\delta+1}} \tag{21}$$

Example 4.8. Let $K = 6_1$ and $K' = 3_1$. Then $\mathrm{sd}_3(K, K') > 1$. Since $\sigma(K) = 0$, $\sigma(K') = 2$, the signature cannot show $\mathrm{sd}_3(K, K') > 1$. However, since $V(K;\omega) = i\sqrt{3}$, $V(K';\omega) = -i\sqrt{3}$, $V(K;-1) = 9$, $V(K';-1) = -3$, we can prove by using Corollary 4.7. In Table 4 we list all such pairs of knots with up to 7 crossings.

Table 4: Pairs of knots K and K' with $|\sigma(K) - \sigma(K')| \leq 2$ and $\mathrm{sd}_3(K, K') > 1$.

K	K'	$\sigma(K)$	$\sigma(K')$	$V(K;\omega)$	$V(K';\omega)$	$V(K;-1)$	$V(K';-1)$
6_1	3_1	0	2	$i\sqrt{3}$	$-i\sqrt{3}$	9	-3
6_1	7_4	0	-2	$i\sqrt{3}$	$-i\sqrt{3}$	9	-15
6_1	7_7	0	0	$i\sqrt{3}$	$-i\sqrt{3}$	9	21
6_1	$3_1!\#4_1$	0	-2	$i\sqrt{3}$	$-i\sqrt{3}$	9	-15
$7_4!$	7_7	2	0	$i\sqrt{3}$	$-i\sqrt{3}$	-15	21
$7_7!$	7_7	0	0	$i\sqrt{3}$	$-i\sqrt{3}$	21	21
$3_1\#4_1$	7_7	2	0	$i\sqrt{3}$	$-i\sqrt{3}$	-15	21

Similarly, we have:

Corollary 4.9. *If two 2-component links L and L' are related by a sequence of two coherent band surgeries,* $\mathrm{d_{cb}}(L, L') = 2$, *and* $V(L;\omega) = -V(L';\omega) = \pm i(i\sqrt{3})^\delta$, *then*

$$V(L;-1)/i \equiv -V(L';-1)/i \pmod{3^{\delta+1}} \tag{22}$$

In Table 4 we list all pairs of 2-component links with up to 6 crossings, which can be shown to have coherent band-Gordian distance > 2 by Corollary 4.9 but cannot be shown by using the signature. Thus by Table 3 in [15] we can conclude they have coherent band-Gordian distance 4.

Table 5: Pairs of links L and L' with $|\sigma(L) - \sigma(L')| \leq 2$ and $\mathrm{d_{cb}}(L, L') = 4$.

L	L'	$\sigma(L)$	$\sigma(L')$	$V(L;\omega)$	$V(L';\omega)$	$V(L;-1)/i$	$V(L';-1)/i$
$3_1\#H_+$	6_3^2	1	3	$-\sqrt{3}$	$\sqrt{3}$	6	-12
$3_1\#H_+$	$6_3^{2'}$	1	-1	$-\sqrt{3}$	$\sqrt{3}$	6	-12
$T_6'!$	6_3^2	1	3	$-\sqrt{3}$	$\sqrt{3}$	6	-12
$T_6'!$	$6_3^{2'}$	1	-1	$-\sqrt{3}$	$\sqrt{3}$	6	-12

5 The HOMFLYPT polynomial

Let $\Sigma_k(L)$ be the k-fold cyclic covering space of S^3 branched over a link L. Lickorish and Millett [27, Theorem 2] have shown:

$$P(L; i, i) = (-2)^{\tau/2}, \tag{23}$$

where $\tau = \dim H_1(\Sigma_3(L); \boldsymbol{Z}_2)$. Putting $v = z = i$ in Eq. (6), we obtain

$$P(L_+; i, i) + P(L_-; i, i) + P(L_0; i, i) = 0, \tag{24}$$

where (L_+, L_-, L_0) is a skein triple. Using this, we have a criterion on the HOMFLYPT polynomials of two links which are related by a crossing change [29, Theorem 1.1] or a coherent band surgery [21, Proposition 2.4].

Proposition 5.1. *If two links L and L' are related by either a crossing change or a coherent band surgery, then*

$$P(L; i, i)/P(L'; i, i) \in \{1, -2^{\pm 1}\}. \tag{25}$$

The Conway polynomial $\nabla(L; z)$ of a c-component link L may be written $\nabla(L; z) = z^{c-1}\varphi(z)$, where $\varphi(z)$ is an integer polynomial in z^2. Then we obtain a symmetric integer polynomial $\tilde{\Delta}_L(t)$ by

$$\tilde{\Delta}_L(t) = \varphi(t^{1/2} - t^{-1/2}), \tag{26}$$

which is called the *Hosokawa polynomial* [12]; cf. [33, pp. 120]. Then Hosokawa and Kinoshita [13] have shown the following; cf. [28, Corollary 9.8]:

Proposition 5.2. *The order of the first homology group of the k-fold cyclic covering space of S^3 branched over a c-component link L, $H_1(\Sigma_k(L); \boldsymbol{Z})$, is given by*

$$k^{c-1} \prod_{j=1}^{k-1} \tilde{\Delta}_L(\xi^j), \tag{27}$$

where ξ is a primitive kth root of unity.

Using Proposition 5.2, we obtain:

Lemma 5.3. *Let L be a c-component link. If $P(L; i, i) = (-2)^h$, then*

$$[\nabla(L; z)/z^{c-1}]_{z^2=-3} \equiv 0 \pmod{2^h}. \tag{28}$$

Using this lemma, we obtain the following.

Theorem 5.4. *Suppose that a $(c+1)$-component link L' is obtained from a c-component link L by a coherent band surgery. If $P(L; i, i) = P(L'; i, i) = (-2)^h$, then*

$$\left[\frac{\nabla(L; z) + z\nabla(L'; z)}{z^{c-1}}\right]_{z^2=-3} \equiv \left[\frac{\nabla(L; z) - z\nabla(L'; z)}{z^{c-1}}\right]_{z^2=-3} \equiv 0 \pmod{2^{h+1}}. \tag{29}$$

6 The Q polynomial

The *Q polynomial* $Q(L; z) \in \boldsymbol{Z}[z^{\pm 1}]$ [3, 11] is an invariant of the isotopy type of an unoriented link L, which is defined by the following formulas:

$$Q(U; z) = 1; \tag{30}$$

$$Q(L_+; z) + Q(L_-; z) = z\left(Q(L_0; z) + Q(L_\infty; z)\right), \tag{31}$$

where U is the unknot and $(L_+, L_-, L_0, L_\infty)$ is an unoriented skein quadruple.

L_+ L_- L_0 L_∞

Figure 8: An unoriented skein quadruple.

Let $\rho(L) = Q\left(L; (\sqrt{5} - 1)/2\right)$. Then Jones [18] has shown

$$\rho(L) = \pm\sqrt{5}^{r} \tag{32}$$

where $r = \dim H_1(\Sigma(L); \boldsymbol{Z}_5)$.

Furthermore, Rong [39] has shown that there are six cases for the ratios among $\rho(L_-)$, $\rho(L_+)$, $\rho(L_0)$, $\rho(L_\infty)$ as in Table 6.

Table 6: The values of the Q polynomials at $z = (\sqrt{5} - 1)/2$.

Cases	$\rho(L_-)/\rho(L_\infty)$	$\rho(L_0)/\rho(L_\infty)$	$\rho(L_+)/\rho(L_\infty)$	$\rho(L_+)/\rho(L_-)$
(a)	1	$\sqrt{5}$	1	1
(b)	$\sqrt{5}$	1	-1	$-\sqrt{5}^{-1}$
(c)	1	-1	-1	-1
(d)	-1	-1	1	-1
(e)	-1	1	$\sqrt{5}$	$-\sqrt{5}$
(f)	$\sqrt{5}^{-1}$	$\sqrt{5}^{-1}$	$\sqrt{5}^{-1}$	1

Using Table 6, we have criteria on the Q polynomials of two links which are related by a crossing change [40, Theorem 4.1] or a band surgery [19, Theorem 3.1].

Proposition 6.1. (i) *If two links L and L' are related by a crossing change, then*

$$\rho(L)/\rho(L') \in \left\{ \pm 1, -\sqrt{5}^{\pm 1} \right\}. \tag{33}$$

(ii) *If two links L and L' are related by a band surgery, then*

$$\rho(L)/\rho(L') \in \left\{ \pm 1, \sqrt{5}^{\pm 1} \right\}. \tag{34}$$

Moreover, using Table 6, we have the following.

Theorem 6.2. *Suppose that two links L and L' are related by either a crossing change or a band surgery and that $\rho(L) = \rho(L') = \pm\sqrt{5}^r$. Then*

$$\det L + \det L' \equiv 0 \ \text{or} \ \det L - \det L' \equiv 0 \pmod{5^{r+1}}. \tag{35}$$

Example 6.3. $\mathrm{d_{cb}}(9_{39}!, 6_2^2) > 1$. Since $\rho(9_{39}!) = \rho(6_2^2) = -\sqrt{5}$, $\det(9_{39}!) = 55$, and $\det(6_2^2) = 10$, the result follows by Theorem 6.2. Note that since $\sigma(9_{39}!) = 2$, $\sigma(6_2^2) = 3$, we cannot use Proposition 2.2.

7 Table of $\mathrm{d_{cb}}(K, K')$

We give a table of coherent band-Gordian distances between two knots (cf: [15, Table 1])

Table 7: Coherent band-Gordian distances between two knots with up to 6 crossings.

	U	3_1	$3_1!$	4_1	5_1	$5_1!$	5_2	$5_2!$	6_1	$6_1!$	6_2	$6_2!$	6_3	$3_1\#3_1$	$3_1!\#3_1!$	$3_1!\#3_1$
U	0	2	2	2	4	4	2	2	2	2	2	2	2	4	4	2
3_1		0	4	2	2	6	2	4	$4^\dagger$	2	2	4	2	2	6	2
$3_1!$			0	2	6	2	4	2	2	$4^\dagger$	4	2	2	6	2	2
4_1				0	4	4	2	2	2	2	2	2	2	4	4	4
5_1					0	8	2	6	4	4	2	6	4	2	8	4
$5_1!$						0	6	2	4	4	6	2	4	8	2	4
5_2							0	4	2	2	2	4	2	2	6	4
$5_2!$								0	2	2	4	2	2	6	2	4
6_1									0	2	2	2	2	4	4	2
$6_1!$										0	2	2	2	4	4	2
6_2											0	4	2	4	6	2
$6_2!$												0	2	6	4	2
6_3													0	4	4	2
$3_1\#3_1$														0	8	4
$3_1!\#3_1!$															0	4
$3_1!\#3_1$																0

†: corrected

Table 8: Coherent band-Gordian distances between two knots with up to 7 crossings.

	7_1	$7_1!$	7_2	$7_2!$	7_3	$7_3!$	7_4	$7_4!$	7_5	$7_5!$	7_6	$7_6!$	7_7	$7_7!$	$3_1\#4_1$	$3_1!\#4_1$
U	6	6	2	2	4	4	2	2	4	4	2	2	2	2	2	2
3_1	4	8	2	4	6	2	4	2	2	6	2	4	2	2	2	4
$3_1!$	8	4	4	2	2	6	2	4	6	2	4	2	2	2	4	2
4_1	6	6	2	2	4	4	2/4	2/4	4	4	2	2	2	2	2	2
5_1	2	10	2	6	8	2	6	2	2	8	2	6	4	4	2	6
$5_1!$	10	2	6	2	2	8	2	6	8	2	6	2	4	4	6	2
5_2	4	8	2	4	6	2	4	2	2	6	2	4	2	2/4	2	4
$5_2!$	8	4	4	2	2	6	2	4	6	2	4	2	2/4	2	4	2
6_1	6	6	2/4	2	4	4	4	2	4	4	2	2	4	2	2	4
$6_1!$	6	6	2	2/4	4	4	2	4	4	4	2	2	2	4	4	2
6_2	4	8	2	4	6	2	4	2/4	2/4	6	2	4	2	2	2/4	4
$6_2!$	8	4	4	2	2	6	2/4	4	6	2/4	4	2	2	2	4	2/4
6_3	6	6	2	2	4	4	2/4	2/4	4	4	2	2	2	2	2	2
$3_1\#3_1$	2	10	4	6	8	4	6	2	2	8	2	6	4	4	2	6
$3_1!\#3_1!$	10	2	6	4	4	8	2	6	8	2	6	2	4	4	6	2
$3_1!\#3_1$	6	6	2/4	2/4	4	4	2/4	2/4	4	4	4	4	2	2	2	2
7_1	0	12	4	8	10	2	8	4	2	10	4	8	6	6	4	8
$7_1!$		0	8	4	2	10	4	8	10	2	8	4	6	6	8	4
7_2			0	4	6	2	4	2	2	6	2	4	2	2/4	2/4	4
$7_2!$				0	2	6	2	4	6	2	4	2	2/4	2	4	2/4
7_3					0	8	2	6	8	2	6	2	4	4	6	2/4
$7_3!$						0	6	2	2	8	2	6	4	4	2/4	6
7_4							0	4	6	2	4	2	2/4	4	4	2
$7_4!$								0	2	6	2	4	4	2/4	2	4
7_5									0	8	2	6	4	4	2	6
$7_5!$										0	6	2	4	4	6	2
7_6											0	4	2	2	2	4
$7_6!$												0	2	2	4	2
7_7													0	4	4	2
$7_7!$														0	2	4
$3_1\#4_1$															0	4
$3_1!\#4_1$																0

The symbol 2/4 means $\mathrm{d_{cb}}(K, K') = 2$ or 4.

Acknowledgements

The first author was partially supported by KAKENHI, Grant-in-Aid for Scientific Research (C) (No. 21540092), Japan Society for the Promotion of Science.

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