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Cluster Algebra and Complex Volume

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1 Introduction

The volume conjecture [13, 15] indicates an intimate relationship between quantum invariants of knots and complex volume of knot complements. Generally quantum invariants of knots are constructed combinatorically based on R-matrix, and it is expected that complex volume can be formulated based on knot diagram. Indeed in [24, 3, 2, 1] constructed was the Neumann–Zagier potential function [20] based on the R-matrices of the Kashaev invariant and the colored Jones polynomial at root of unity.

Our purpose in this article is to study complex volume of knot complements from viewpoint of cluster algebra. See our works [9, 10] for detail.

2 Cluster Algebra and 3-Dimensional Hyperbolic Geometry

2.1 Cluster Algebra

We follow a definition of cluster algebras in [6, 7]. Let $(\mathbb{P}, \oplus, \cdot)$ be a semifield endowed an auxiliary addition $\oplus$, which is commutative, associative, and distributive with respect to the group multiplication $\cdot$ in $\mathbb{P}$. Let $\mathbb{QP}$ denote the quotient field of the group ring $\mathbb{ZP}$ of $\mathbb{P}$. Fix $N \in \mathbb{Z}_{>0}$.

Definition 2.1. A seed is a triple $(\boldsymbol{x}, \boldsymbol{\varepsilon}, \mathbf{B})$, where

- a cluster $\boldsymbol{x} = (x_1, \dots, x_N)$ is an N-tuple of N algebraically independent variables with coefficients in $\mathbb{QP}$,
- a coefficient tuple $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_N)$ is an N-tuple of elements in $\mathbb{P}$,
- an exchange matrix $\mathbf{B} = (b_{ij})$ is an $N \times N$ skew symmetric integer matrix.

We call x_i a cluster variable, and ε_i a coefficient.

Definition 2.2. Let $(\boldsymbol{x},\boldsymbol{\varepsilon},\mathbf{B})$ be a seed. For each $k = 1,\dots,N$, we define the mutation of $(\boldsymbol{x},\boldsymbol{\varepsilon},\mathbf{B})$ by μ_k as

$$\mu_k(\boldsymbol{x},\boldsymbol{\varepsilon},\mathbf{B}) = (\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{\varepsilon}},\widetilde{\mathbf{B}}),$$

where

- the cluster $\widetilde{\boldsymbol{x}} = (\widetilde{x}_1,\dots,\widetilde{x}_N)$ is

$$\widetilde{x}_i = \begin{cases} x_i, & \text{for } i \neq k, \\ \dfrac{\varepsilon_k}{1\oplus\varepsilon_k}\cdot\dfrac{1}{x_k}\displaystyle\prod_{j:b_{jk}>0} x_j^{\,b_{jk}} + \dfrac{1}{1\oplus\varepsilon_k}\cdot\dfrac{1}{x_k}\displaystyle\prod_{j:b_{jk}<0} x_j^{\,-b_{jk}}, & \text{for } i = k, \end{cases} \tag{1}$$

- the coefficient tuple $\widetilde{\boldsymbol{\varepsilon}} = (\widetilde{\varepsilon}_1,\dots,\widetilde{\varepsilon}_N)$ is

$$\widetilde{\varepsilon}_i = \begin{cases} \varepsilon_k^{-1}, & \text{for } i = k, \\ \varepsilon_i\left(\dfrac{\varepsilon_k}{1\oplus\varepsilon_k}\right)^{b_{ki}}, & \text{for } i \neq k,\ b_{ki} \geq 0, \\ \varepsilon_i\,(1\oplus\varepsilon_k)^{-b_{ki}}, & \text{for } i \neq k,\ b_{ki} \leq 0, \end{cases} \tag{2}$$

- the exchange matrix $\widetilde{\mathbf{B}} = (\widetilde{b}_{ij})$ is

$$\widetilde{b}_{ij} = \begin{cases} -b_{ij}, & \text{for } i = k \text{ or } j = k, \\ b_{ij} + \dfrac{1}{2}\left(|b_{ik}|\,b_{kj} + b_{ik}\,|b_{kj}|\right), & \text{otherwise.} \end{cases} \tag{3}$$

Note that the resulted triplet $(\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{\varepsilon}},\widetilde{\mathbf{B}})$ is again a seed.

By starting from an initial seed $(\boldsymbol{x},\boldsymbol{\varepsilon},\mathbf{B})$, we iterate mutations and collect all obtained seeds. The cluster algebra $\mathscr{A}(\boldsymbol{x},\boldsymbol{\varepsilon},\mathbf{B})$ is the $\mathbb{ZP}$-subalgebra of the rational function field $\mathbb{QP}(\boldsymbol{x})$ generated by all the cluster variables. We use the following.

Proposition 2.3 ([7]). *Let $\boldsymbol{y}$ be an N-tuple $\boldsymbol{y} = (y_1,\dots,y_N)$, defined by use of cluster $\boldsymbol{x}$ and coefficient $\boldsymbol{\varepsilon}$ as*

$$y_j = \varepsilon_j \prod_k x_k^{\,b_{kj}}. \tag{4}$$

Then we have a mutation,

$$\mu_k(\boldsymbol{y},\mathbf{B}) = (\widetilde{\boldsymbol{y}},\widetilde{\mathbf{B}}), \tag{5}$$

where

- *$\widetilde{\boldsymbol{y}} = (\widetilde{y}_1,\dots,\widetilde{y}_N)$ is analogous to (2),*

$$\widetilde{y}_i = \begin{cases} y_k^{-1}, & \textit{for } i = k, \\ y_i\left(\dfrac{y_k}{1+y_k}\right)^{b_{ki}}, & \textit{for } i \neq k,\ b_{ki} \geq 0, \\ y_i\,(1+y_k)^{-b_{ki}}, & \textit{for } i \neq k,\ b_{ki} \leq 0, \end{cases} \tag{6}$$

- $\widetilde{\mathbf{B}} = (\widetilde{b}_{ij})$ *is* (3).

For the later use, we introduce the permutation acting on seeds.

Definition 2.4. For $i,j \in \{1,\dots,N\}$ and $i \neq j$, let $s_{i,j}$ be a permutation of subscripts i and j in seeds. For example permutated cluster $s_{i,j}(\boldsymbol{x})$ is defined by

$$s_{i,j}(\cdots,x_i,\cdots,x_j,\cdots) = (\cdots,x_j,\cdots,x_i,\cdots).$$

Actions on $\boldsymbol{\varepsilon}$ and $\mathbf{B}$ are defined in the same manner. They induce an action on $\boldsymbol{y}$, and $s_{i,j}(\boldsymbol{y})$ has a same form.

2.2 Hyperbolic Geometry

A fundamental object in three-dimensional hyperbolic geometry is an ideal hyperbolic tetrahedron $\triangle$ in Fig. 1 [23]. The tetrahedron is parameterized by a modulus $z \in \mathbb{C}$, and each dihedral angle is given as in the figure. We mean z' and z'' for given modulus z by

$$z' = 1 - \frac{1}{z}, \qquad\qquad z'' = \frac{1}{1-z}. \tag{7}$$

The cross section by the horosphere at each vertex is similar to the triangle in $\mathbb{C}$ with vertices 0, 1, and z. We have assigned a vertex ordering following [27], which is crucial in computing the complex volume of tetrahedra modulo π^2. See Fig. 1.

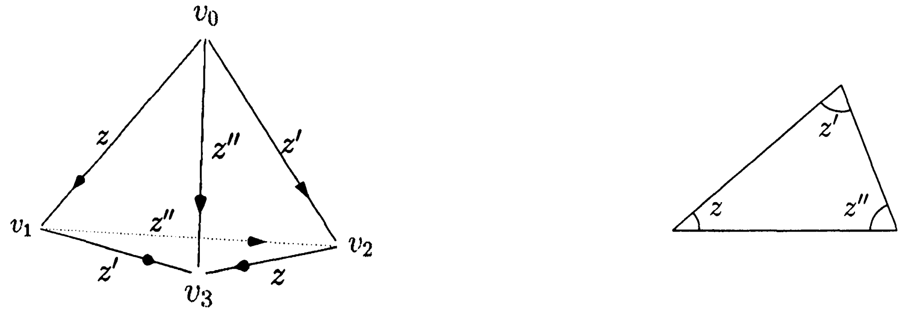

Figure 1: An ideal hyperbolic tetrahedron $\triangle$ with modulus z. Dihedral angles are given by z, $z' = 1 - 1/z$, and $z'' = 1/(1-z)$. Each v_a denotes a vertex ordering. We give an orientation to an edge from v_a to v_b ($a < b$).

The hyperbolic volume of an ideal tetrahedron $\triangle$ with modulus z is given by the Bloch–Wigner function

$$D(z) = \Im \mathrm{Li}_2(z) + \arg(1-z)\log|z|, \tag{8}$$

where $\mathrm{Li}_2(z)$ is the dilogarithm function,

$$\mathrm{Li}_2(z) = -\int_0^z \log(1-s)\frac{\mathrm{d}s}{s}.$$

See, *e.g.*, [26].

A set of ideal tetrahedra $\{\triangle_\nu\}$ is glued together to construct a cusped hyperbolic manifold M. A modulus z_ν of each ideal tetrahedron $\triangle_\nu$ is determined from both gluing conditions around each edge and a completeness condition [23, 20, 18]. Then the hyperbolic volume of M is given by

$$\mathrm{Vol}(M) = \sum_{v} D(z_v). \tag{9}$$

The complex volume, $\mathrm{Vol}(M) + \mathrm{i}\,\mathrm{CS}(M)$, is given in terms of an extended Rogers dilogarithm function

$$\mathrm{L}([z;p,q]) = \mathrm{Li}_2(z) + \frac{1}{2}\log z \log(1-z) + \frac{\pi \mathrm{i}}{2}(q \log z + p \log(1-z)) - \frac{\pi^2}{6}, \tag{10}$$

where $p, q \in \mathbb{Z}$. To compute the complex volume, we need an additional structure to the moduli of ideal tetrahedra:

Definition 2.5 ([19]). A flattening of an ideal tetrahedron $\triangle$ is

$$(w_0, w_1, w_2) = (\log z + p\pi \mathrm{i}, -\log(1-z) + q\pi \mathrm{i}, \log(1-z) - \log z - (p+q)\pi \mathrm{i}), \tag{11}$$

where z is the modulus of $\triangle$ and $p, q \in \mathbb{Z}$. We use $[z;p,q]$ to denote the flattening of $\triangle$.

In [19], the extended pre-Bloch group is defined as the free abelian group on flattenings subject to a five-term relation, and shown is that the flattening gives the complex volume.

Proposition 2.6 ([19]). *The complex volume of M is*

$$\mathrm{i}\,(\mathrm{Vol}(M) + \mathrm{i}\,\mathrm{CS}(M)) = \sum_{v} \mathrm{sign}(v)\, \mathrm{L}([z_v; p_v, q_v]), \tag{12}$$

where $[z_v; p_v, q_v]$ and $\mathrm{sign}(v) = \pm 1$ respectively denote a flattening and a vertex ordering of a tetrahedron $\triangle_v$.

For a tetrahedron $\triangle$ in Fig. 1, let c_{ab} be a complex number assigned to an edge connecting vertices v_a and v_b. Zickert clarified that the flattening $(z;p,q)$ of $\triangle$ is given by c_{ab} as follows.

Proposition 2.7 ([27]). *When we have*

$$\frac{c_{03}\,c_{12}}{c_{02}\,c_{13}} = \pm z, \qquad \frac{c_{01}\,c_{23}}{c_{03}\,c_{12}} = \pm\left(1 - \frac{1}{z}\right), \qquad \frac{c_{02}\,c_{13}}{c_{01}\,c_{23}} o = \pm \frac{1}{1-z}, \tag{13}$$

the flattening $(z;p,q)$ is given by

$$\begin{aligned} \log z + p\pi \mathrm{i} &= \log c_{03} + \log c_{12} - \log c_{02} - \log c_{13}, \\ -\log(1-z) + q\pi \mathrm{i} &= \log c_{02} + \log c_{13} - \log c_{01} - \log c_{23}. \end{aligned} \tag{14}$$

Remark 2.8. In gluing tetrahedra to construct M, identical edges have the same complex numbers.

2.3 Interrelationship

Correspondence between the cluster algebra and the hyperbolic geometry can be seen in a simple example (see also [17]). We study a triangulation of surface and its flip as in

Fig. 2. Triangulation is related to quiver where the number of edges in a triangulation is the same as the fixed number N in the cluster algebra, and flip can be regarded as mutation, as depicted in the figure. Note that the exchange matrix $\mathbf{B} = (b_{ij})$ of quiver is

$$b_{ij} = \#\{\text{arrows from } i \text{ to } j\} - \#\{\text{arrows from } j \text{ to } i\}.$$

By definition (4), the mutation $\mu_3(\boldsymbol{y}, \mathbf{B}) = (\widetilde{\boldsymbol{y}}, \widetilde{\mathbf{B}})$, is explicitly written as

$$\begin{aligned} \widetilde{y}_1 &= y_1(1+y_3), \\ \widetilde{y}_2 &= y_2(1+y_3{}^{-1})^{-1}, \\ \widetilde{y}_3 &= y_3{}^{-1}, \\ \widetilde{y}_4 &= y_4(1+y_3{}^{-1})^{-1}, \\ \widetilde{y}_5 &= y_5(1+y_3). \end{aligned} \tag{15}$$

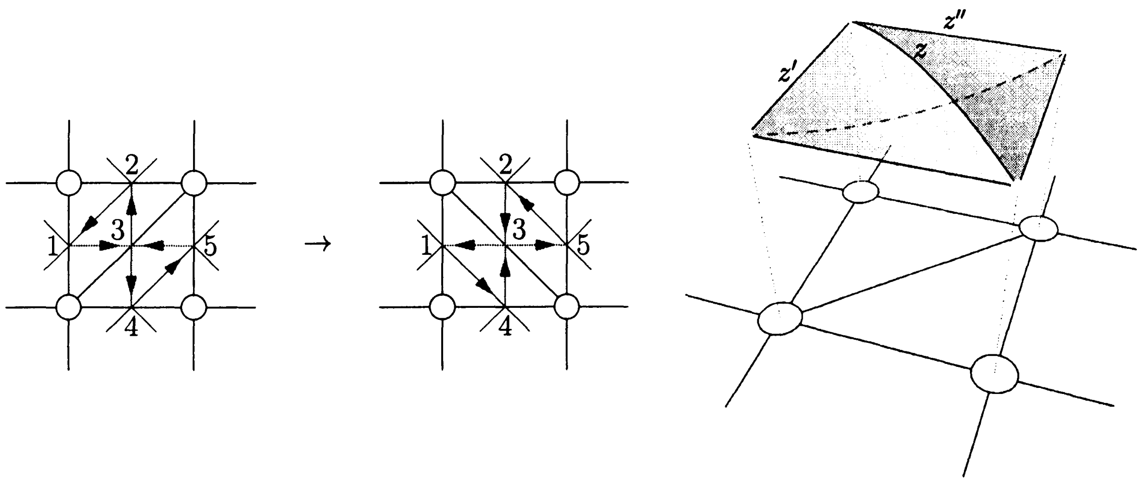

Figure 2: [Left] Triangulation of a punctured surface. Associated quiver is depicted in red. [Right] Flip and attachment of pleated tetrahedron.

On the other hand, we may regard a flip in Fig. 2 as an attachment of ideal tetrahedron Δ with modulus z whose faces are pleated. When we denote z_k as a dihedral angle on edge k, dihedral angle $\widetilde{z}_k$ after attaching Δ is given by

$$\begin{aligned} \widetilde{z}_1 &= z_1 z', \\ \widetilde{z}_2 &= z_2 z'', \\ \widetilde{z}_3 &= z, \\ \widetilde{z}_4 &= z_4 z'', \\ \widetilde{z}_5 &= z_5 z', \end{aligned} \tag{16}$$

with a hyperbolic gluing condition

$$z_3 z = 1.$$

Comparing (15) with (16), we observe that the cluster y-variable is related to dihedral angle by

$$y_k = -z_k,$$

and especially a modulus of ideal tetrahedron $\triangle$ is given by

$$z = -\frac{1}{y_3}, \tag{17}$$

where a subscript "3" is a direction of mutation.

3 Braid Relation

3.1 R-operator

We set the exchange matrix $\mathbf{B}$ as

$$\mathbf{B} = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix}. \tag{18}$$

By regarding the matrix element as

$$b_{ij} = \#\{\text{arrows from } i \text{ to } j\} - \#\{\text{arrows from } j \text{ to } i\}, \tag{19}$$

exchange matrix $\mathbf{B}$ corresponds to quiver, which is dual to triangulated surface (see, *e.g.*, [5]). In our case (18), we have the quiver and the triangulated surface depicted in Fig. 3.

Figure 3: Quiver and triangulated surface

Definition 3.1 ([10]). We define the R-operator by

$$\mathsf{R} = s_{3,5}\, s_{2,5}\, s_{3,6}\, \mu_4\, \mu_2\, \mu_6\, \mu_4. \tag{20}$$

Note that the inverse of the R-operator is given by

$$\mathsf{R}^{-1} = s_{3,6}\, s_{2,5}\, s_{3,5}\, \mu_4\, \mu_5\, \mu_3\, \mu_4. \tag{21}$$

The permutations are included in the R-operator so that the exchange matrix **B** (18) is invariant under R. We use a trivial semi-field [7], and we set all cluster coefficients to be 1. Explicitly we have

$$\mathsf{R}^{\pm 1}(\boldsymbol{x}, \mathbf{B}) = (\mathsf{R}^{\pm 1}(\boldsymbol{x}), \mathbf{B}), \tag{22}$$

where

$$\mathsf{R}(\boldsymbol{x}) = \begin{pmatrix} x_1 \\ x_5 \\ \frac{x_1 x_3 x_5 + x_3 x_4 x_5 + x_1 x_2 x_6}{x_2 x_4} \\ \frac{x_1 x_3 x_4 x_5 + x_3 x_4^2 x_5 + x_1\; x_3 x_5 x_7 + x_3 x_4 x_5 x_7 + x_1 x_2 x_6 x_7}{x_2 x_4 x_6} \\ \frac{x_3 x_4 x_5 + x_3 x_5 x_7 + x_2 x_6 x_7}{x_4 x_6} \\ x_3 \\ x_7 \end{pmatrix}^{\mathsf{T}},$$

$$\mathsf{R}^{-1}(\boldsymbol{x}) = \begin{pmatrix} x_1 \\ \frac{x_1 x_3 x_5 + x_1 x_2 x_6 + x_2 x_4 x_6}{x_3 x_4} \\ x_6 \\ \frac{x_1 x_2 x_4 x_6 + x_2 x_4^2 x_6 + x_1 x_3 x_5 x_7 + x_1 x_2 x_6 x_7 + x_2 x_4 x_6 x_7}{x_3 x_4 x_5} \\ x_2 \\ \frac{x_2 x_4 x_6 + x_3 x_5 x_7 + x_2 x_6 x_7}{x_4 x_5} \\ x_7 \end{pmatrix}^{\mathsf{T}}. \tag{23}$$

Correspondingly, actions of the R-operator, (20) and (21), on the y-variable are respectively given as follows:

$$\mathsf{R}(\boldsymbol{y}) = \begin{pmatrix} y_1 (1 + y_2 + y_2 y_4) \\ \frac{y_2 y_4 y_5 y_6}{1 + y_2 + y_6 + y_2 y_6 + y_2 y_4 y_6} \\ \frac{1 + y_2 + y_6 + y_2 y_6 + y_2 y_4 y_6}{y_2 y_4} \\ \frac{y_4}{(1 + y_2 + y_2 y_4)(1 + y_6 + y_4 y_6)} \\ \frac{1 + y_2 + y_6 + y_2 y_6 + y_2 y_4 y_6}{y_4 y_6} \\ \frac{y_2 y_3 y_4 y_6}{1 + y_2 + y_6 + y_2 y_6 + y_2 y_4 y_6} \\ (1 + y_6 + y_4 y_6)\, y_7 \end{pmatrix}^{\mathsf{T}}, \quad \mathsf{R}^{-1}(\boldsymbol{y}) = \begin{pmatrix} \frac{y_1 y_3 y_4}{1 + y_4 + y_3 y_4} \\ \frac{y_5}{1 + y_4 + y_3 y_4 + y_4 y_5 + y_3 y_4 y_5} \\ (1 + y_4 + y_3 y_4 + y_4 y_5 + y_3 y_4 y_5)\, y_6 \\ \frac{(1 + y_4 + y_3 y_4)(1 + y_4 + y_4 y_5)}{y_3 y_4 y_5} \\ y_2 (1 + y_4 + y_3 y_4 + y_4 y_5 + y_3 y_4 y_5) \\ \frac{y_3}{1 + y_4 + y_3 y_4 + y_4 y_5 + y_3 y_4 y_5} \\ \frac{y_4 y_5 y_7}{1 + y_4 + y_4 y_5} \end{pmatrix}^{\mathsf{T}}. \tag{24}$$

During the workshop, R. Kashaev kindly informed us that an essentially same action with (24) was studied in [4].

It should be noted that the R-operator (20) can be written as

$$\mathsf{R} = s_{2,5}\, s_{3,6}\, \mu_2\, \mu_6\, \mu_4\, \mu_2\, \mu_6. \tag{25}$$

3.2 Braid Relation

We generalize the quiver in Fig. 3 to that in Fig. 4. Therein also given is the triangulated surface, and an exchange matrix **B** is given by the rule (19) as a generalization of (18).

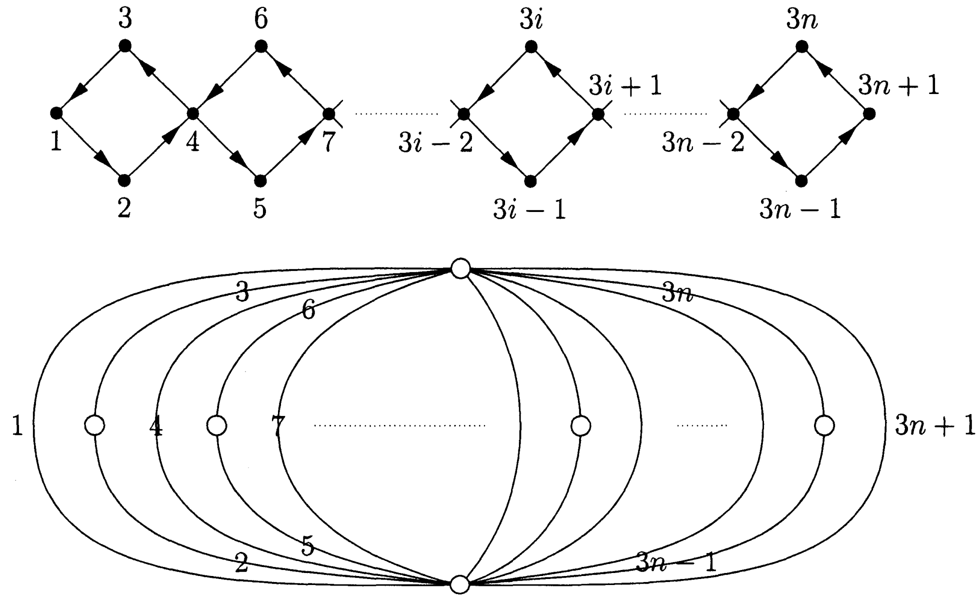

Figure 4: Quiver and triangulated surface.

Definition 3.2. By use of (20), we define the R-operator $\overset{i}{\mathsf{R}}$ for $i = 1, \ldots, n-1$ associated with the quiver in Fig. 4 by

$$\overset{i}{\mathsf{R}} = s_{3i,3i+2}\, s_{3i-1,3i+2}\, s_{3i,3i+3}\, \mu_{3i+1}\, \mu_{3i-1}\, \mu_{3i+3}\, \mu_{3i+1}. \tag{26}$$

Note that

$$\overset{i}{\mathsf{R}}{}^{-1} = s_{3i,3i+3}\, s_{3i-1,3i+2}\, s_{3i,3i+2}\, \mu_{3i+1}\, \mu_{3i+2}\, \mu_{3i}\, \mu_{3i+1}. \tag{27}$$

The exchange matrix associated to Fig. 4 is invariant under the action of the R-operators $\overset{i}{\mathsf{R}}{}^{\pm 1}$. The explicit forms of the actions on the cluster variable $\boldsymbol{x} = (x_1, x_2, \ldots, x_{3n+1})$ and the y-variable $\boldsymbol{y} = (y_1, y_2, \ldots, y_{3n+1})$ are as follows.

$$\overset{i}{\mathsf{R}}{}^{\pm 1}(\boldsymbol{x}) = \big(x_1, \ldots, x_{3i-3}, \mathsf{R}^{\pm 1}(x_{3i-2}, \ldots, x_{3i+4}), x_{3i+5}, \ldots, x_{3n+1}\big), \tag{28}$$

$$\overset{i}{\mathsf{R}}{}^{\pm 1}(\boldsymbol{y}) = \big(y_1, \ldots, y_{3i-3}, \mathsf{R}^{\pm 1}(y_{3i-2}, \ldots, y_{3i+4}), y_{3i+5}, \ldots, y_{3n+1}\big), \tag{29}$$

where $\mathsf{R}^{\pm 1}(x_1, \ldots, x_7)$ and $\mathsf{R}^{\pm 1}(y_1, \ldots, y_7)$ are defined in (23) and (24) respectively.

Theorem 3.3 ([10]). *The* R*-operator satisfies the braid relation, namely we have*

$$\overset{i}{\mathsf{R}}\overset{i+1}{\mathsf{R}}\overset{i}{\mathsf{R}} = \overset{i+1}{\mathsf{R}}\overset{i}{\mathsf{R}}\overset{i+1}{\mathsf{R}}, \quad \textit{for } i = 1, 2, \ldots, n-2, \tag{30}$$

$$\overset{i}{\mathsf{R}}\overset{j}{\mathsf{R}} = \overset{j}{\mathsf{R}}\overset{i}{\mathsf{R}}, \quad \textit{for } |i-j| > 1. \tag{31}$$

3.3 Octahedron

Based on 3-dimensional interpretation of the cluster mutation given in Section 2.3, we can see that the R-operator (20) is realized as an octahedron in Fig. 5, which is composed of four tetrahedra $\{\Delta_N, \Delta_S, \Delta_W, \Delta_E\}$. The four tetrahedra originate from four mutations in the R-operator, (20) and (21). In octahedra, the cluster variables x_k and $\tilde{x}_k$ defined by

$$\tilde{\boldsymbol{x}} = \mathsf{R}^{\pm 1}(\boldsymbol{x}),$$

are assigned to edges, and we have used

$$x_c = \frac{x_2 x_6 + x_3 x_5}{x_4}. \tag{32}$$

Note that we have fixed vertex ordering for our convention, and that edges with the same complex parameters (*e.g.*, two pairs of edges v_0–v_2, v_1–v_3) are identical.

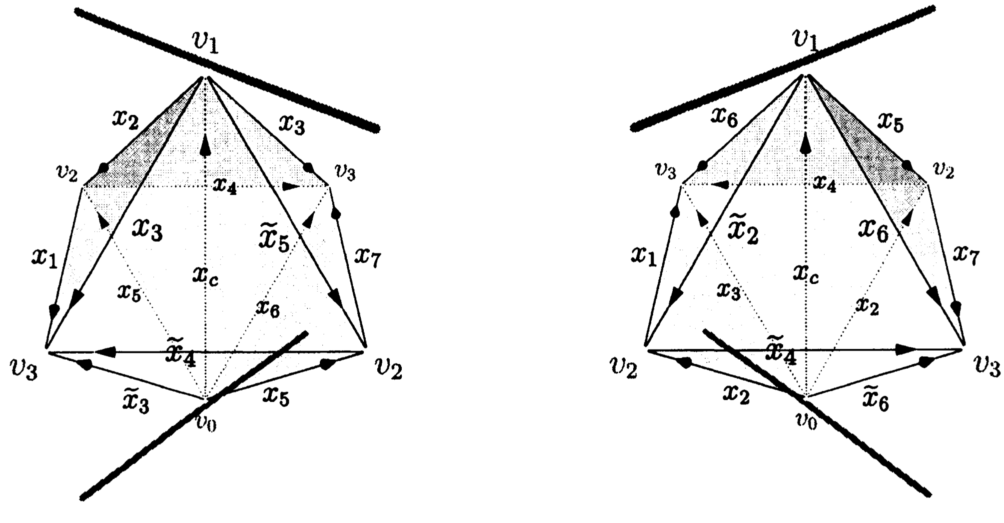

Figure 5: Octahedron for R (left) and R^{-1} (right)

As the R-operator satisfies the braid relation (Theorem 3.3), we can interpret that each octahedron is assigned to every crossing of knot diagram as in Fig. 6. This reminds a fact [22] that octahedron was assigned to the Kashaev R-matrix [12] (see also, [8, 1, 3, 24]). Note that another expression (25) of the same R-operator corresponds to a decomposition of octahedron into five tetrahedra, which was used in studies of the colored Jones R-matrix at root of unity [22, 2].

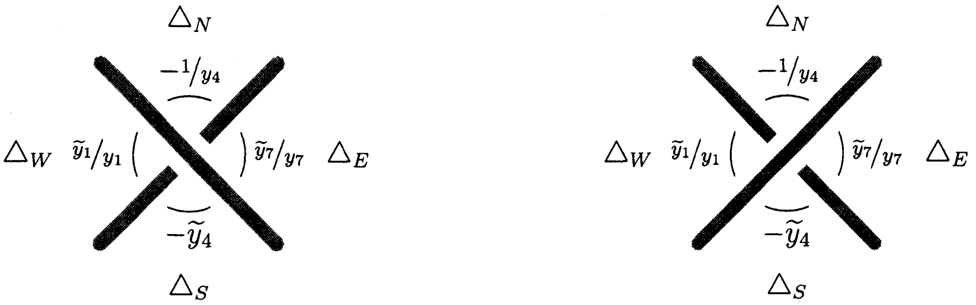

Figure 6: Dihedral angle at crossings, R (left) and R^{-1} (right).

Taking into account of the vertex ordering of tetrahedra, we can determine moduli of each tetrahedron from (13) as in Table 1. From these results, we define dilogarithm functions for every crossing by

$$\mathrm{L}([\mathsf{R}^{\pm 1}]) = \sum_{t \in \{N,S,W,E\}} \mathrm{sign}(\triangle_t)\, \mathrm{L}\big([z_{\triangle_t}; p_{\triangle_t}, q_{\triangle_t}]\big). \tag{33}$$

Here integers $p_{\triangle_t}$ and $q_{\triangle_t}$ are given from (14) by use of Table 1. For instance, $p_{\triangle_E}$ and $q_{\triangle_E}$ in the operator R are given as

$$p_{\triangle_E} \pi \mathrm{i} = -\log\left(\frac{\tilde{x}_5 x_6}{x_3 x_5}\right) + \log(\tilde{x}_5) + \log(x_6) - \log(x_3) - \log(x_5),$$

$$q_{\triangle_E} \pi \mathrm{i} = -\log\left(-\frac{x_3 x_5}{x_c x_7}\right) + \log(x_3) + \log(x_5) - \log(x_c) - \log(x_7).$$

		R			R^{-1}		
$\triangle$	Volume	sign($\triangle$)	$z_\triangle$	$\frac{1}{1-z_\triangle}$	sign($\triangle$)	$z_\triangle$	$\frac{1}{1-z_\triangle}$
$\triangle_N$	$D\left(-\frac{1}{y_4}\right)$	$-$	$-\frac{x_2 x_6}{x_3 x_5}$	$\frac{x_3 x_5}{x_4 x_c}$	$+$	$-\frac{x_3 x_5}{x_2 x_6}$	$\frac{x_2 x_6}{x_4 x_c}$
$\triangle_S$	$D(-\tilde{y}_4)$	$-$	$-\frac{\tilde{x}_3 \tilde{x}_5}{x_3 x_5}$	$\frac{x_3 x_5}{\tilde{x}_4 x_c}$	$+$	$-\frac{\tilde{x}_2 \tilde{x}_6}{x_2 x_6}$	$\frac{x_2 x_6}{x_c \tilde{x}_4}$
$\triangle_W$	$D\left(\frac{\tilde{y}_1}{y_1}\right)$	$+$	$\frac{x_2 \tilde{x}_3}{x_3 x_5}$	$-\frac{x_3 x_5}{x_1 x_c}$	$-$	$\frac{\tilde{x}_2 x_3}{x_2 x_6}$	$-\frac{x_2 x_6}{x_1 x_c}$
$\triangle_E$	$D\left(\frac{\tilde{y}_7}{y_7}\right)$	$+$	$\frac{\tilde{x}_5 x_6}{x_3 x_5}$	$-\frac{x_3 x_5}{x_c x_7}$	$-$	$\frac{x_5 \tilde{x}_6}{x_2 x_6}$	$-\frac{x_2 x_6}{x_c x_7}$

Table 1: Moduli of four tetrahedra assigned to operators R and R^{-1}. Sgn "+" (resp. "−") means that vertex ordering of tetrahedron is same (resp. inverse) with Fig. 1.

It should be remarked that, to identify the R-operator with a hyperbolic octahedron, we need a consistency condition around a central edge labeled by x_c in Fig. 5. This condition is automatically satisfied by

$$y_1 y_4 y_7 = \tilde{y}_1 \tilde{y}_4 \tilde{y}_7,$$

where $\tilde{\boldsymbol{y}} = \mathsf{R}^{\pm 1}(\boldsymbol{y})$ (24). In Fig. 6 denoted are dihedral angles around central axis assigned to each crossing.

3.4 Gluing Octahedra

Theorem 3.4. *Let knot K have a braid group presentation* $\sigma_{k_1}^{\varepsilon_1}\sigma_{k_2}^{\varepsilon_2}\cdots\sigma_{k_m}^{\varepsilon_m}$, *where* $\varepsilon_j = \pm 1$ *and*

$$\mathscr{B}_n = \left\langle \sigma_1, \sigma_2, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{c} \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j|>1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } i = 1,2,\dots,n-2 \end{array} \right\rangle.$$

We define a cluster pattern for $\boldsymbol{x}[j] = (x[j]_1, \dots, x[j]_{3n+1})$ *by*

$$\boldsymbol{x}[1] \xrightarrow[\mathsf{R}^{\varepsilon_1}]{k_1} \boldsymbol{x}[2] \xrightarrow[\mathsf{R}^{\varepsilon_2}]{k_2} \cdots \xrightarrow[\mathsf{R}^{\varepsilon_m}]{k_m} \boldsymbol{x}[m+1], \tag{34}$$

with the exchange matrix associated to Fig. 4. We assume that the initial cluster variable $\boldsymbol{x}[1]$ *satisfies*

$$\boldsymbol{x}[1] = \boldsymbol{x}[m+1]. \tag{35}$$

Then there exist an algebraic solution of (35) *such that the complex volume of K is given by*

$$\mathrm{i}\left(\mathrm{Vol}(S^3 \setminus K) + \mathrm{i}\,\mathrm{CS}(S^3 \setminus K)\right) = \sum_{j=1}^{m} \mathrm{L}([\overset{k_j}{\mathsf{R}^{\varepsilon_j}}]). \tag{36}$$

We study a case of trefoil 3_1. The braid group presentation is σ_1^{3}, and its cluster pattern is

$$\boldsymbol{x}[1] \xrightarrow{\overset{1}{\mathsf{R}}} \boldsymbol{x}[2] \xrightarrow{\overset{1}{\mathsf{R}}} \boldsymbol{x}[3] \xrightarrow{\overset{1}{\mathsf{R}}} \boldsymbol{x}[4].$$

We solve $\boldsymbol{x}[1] = \boldsymbol{x}[4]$ by choosing an initial cluster variable as

$$\boldsymbol{x}[1] = (1, x, -x, 1, -x^2, 1, 1),$$

and get $x = \frac{1\pm\mathrm{i}\sqrt{15}}{4}$. We check numerically that (36) gives $-8.22467\cdots \simeq -\frac{5}{6}\pi^2$. It agrees with the Chern-Simons invariant of 3_1, which is also given from asymptotic limit of the Kashaev invariant [14, 25, 11].

4 2-Bridge Knots

Computation from the cluster pattern based on the R-operator (20) is much involved, and it can be much simplified when we know a simple triangulation of hyperbolic manifolds, such as once-punctured torus bundle and 2-bridge knot complements. In this section, we employ a canonical triangulation of 2-bridge knot complements studied in [21].

Let $K_{q/p}$ be a hyperbolic 2-bridge knot or link (see *e.g.* [16]). Here we assume that p and q are coprime integers such that $2 \le q < p/2$. When p is even, $K_{q/p}$ is link. We use a continued fraction expression of q/p,

$$q/p = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}, \tag{37}$$

where $n \geq 1$, $a_j \in \mathbb{Z}_{>0}$, and $a_n \geq 2$. We set

$$c = \sum_{i=1}^{n} a_i. \tag{38}$$

We then set a sequence of symbols to denote flips as

$$F_1 F_2 \cdots F_{c-3} = \begin{cases} R^{a_1-1} L^{a_2} R^{a_3} \cdots R^{a_{n-1}} L^{a_n-2}, & \text{when } n \text{ is even,} \\ R^{a_1-1} L^{a_2} R^{a_3} \cdots L^{a_{n-1}} R^{a_n-2}, & \text{when } n \text{ is odd.} \end{cases} \tag{39}$$

where F_k denotes a symbol, $F_k = R$ or L.

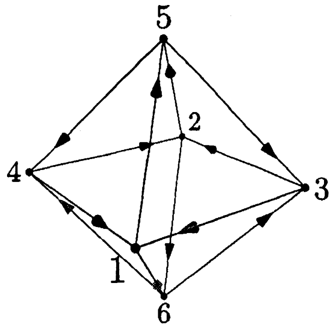

Figure 7: A quiver associated to triangulation of four-punctured sphere $\Sigma_{0,4}$.

A setup for cluster algebra is as follows. We use an exchange matrix

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & -1 & -1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 1 & 1 \\ 1 & 1 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & -1 & -1 \\ -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 1 & 0 & 0 \end{pmatrix}, \tag{40}$$

whose quiver is in Fig. 7. We introduce the flips R and L in (39) as cluster mutations defined by

$$R = s_{5,6}\, s_{1,5}\, s_{2,6}\, \mu_1 \mu_2, \qquad L = s_{5,6}\, s_{3,5}\, s_{4,6}\, \mu_3 \mu_4. \tag{41}$$

Permutations $s_{i,j}$ are used so that the exchange matrix **B** (40) is invariant under these flips. The actions on y-variables are explicitly given by

$$R(\boldsymbol{y}) = \begin{pmatrix} y_5(1+y_1^{-1})^{-1}(1+y_2^{-1})^{-1} \\ y_6(1+y_1^{-1})^{-1}(1+y_2^{-1})^{-1} \\ y_3(1+y_1)(1+y_2) \\ y_4(1+y_1)(1+y_2) \\ y_2^{-1} \\ y_1^{-1} \end{pmatrix}^{\top}, \quad L(\boldsymbol{y}) = \begin{pmatrix} y_1(1+y_3^{-1})^{-1}(1+y_4^{-1})^{-1} \\ y_2(1+y_3^{-1})^{-1}(1+y_4^{-1})^{-1} \\ y_5(1+y_3)(1+y_4) \\ y_6(1+y_3)(1+y_4) \\ y_4^{-1} \\ y_3^{-1} \end{pmatrix}^{\top}. \tag{42}$$

Theorem 4.1 ([9]). *We set* $\boldsymbol{y}[k]$ *recursively by*

$$\boldsymbol{y}[k] \xrightarrow{F_k} \boldsymbol{y}[k+1] \tag{43}$$

where F_k *is* R *or* L *in* (39), *and an initial* y*-variable is given by*

$$\boldsymbol{y}[1] = \left(y, y, -\frac{1}{y}, -\frac{1}{y}, -1, -1\right). \tag{44}$$

Here y *is a geometric solution of*

$$\begin{cases} y[c-2]_3 = y[c-2]_4 = -1, & \text{if } n \text{ is even,} \\ y[c-2]_1 = y[c-2]_2 = -1, & \text{if } n \text{ is odd,} \end{cases} \tag{45}$$

such that each modulus $z_i[k]$ *for* $i = 1, 2$ *and* $k = 1, 2, \ldots, c-3$ *defined by*

$$z_i[k] = \begin{cases} -\dfrac{1}{y[k]_i}, & \text{if } F_k = R, \\ -\dfrac{1}{y[k]_{2+i}}, & \text{if } F_k = L, \end{cases} \tag{46}$$

is in the upper half plane $\mathbb{H}$.

Then $z_i[k]$ *denotes a modulus of tetrahedron* $\triangle_i(F_k)$*, and the hyperbolic volume of the knot complement* $S^3 \setminus K_{q/p}$ *is given by*

$$\mathrm{Vol}(S^3 \setminus K_{q/p}) = \sum_{k=1}^{c-3} \sum_{i=1,2} D(z_i[k]). \tag{47}$$

We can compute the complex volume of 2-bridge knot complement by use of the cluster variables. In this case, we need a specific semi-field to fulfill a "folding condition" at the end, and it is tedious to fix an orientation of tetrahedra. See [9].

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References

[1] J. Cho, H. Kim, and S. Kim, *Optimistic limits of Kashaev invariants and complex volumes of hyperbolic links*, preprint (2013), arXiv:1301.6219 [math.GT].

[2] J. Cho and J. Murakami, *The complex volumes of twist knots via colored Jones polynomials*, J. Knot Theory Ramifications **19**, 1401–1421 (2010).

[3] J. Cho, J. Murakami, and Y. Yokota, *The complex volumes of twist knots*, Proc. Amer. Math. Soc. **137**, 3533–3541 (2009).

[4] I. A. Dynnikov, *On a Yang–Baxter map and the Dehornoy ordering*, Russ. Math. Surveys **57**, 592–594 (2002).

[5] S. Fomin, M. Shapiro, and D. Thurston, *Cluster algebras and triangulated surfaces I. cluster complexes*, Acta Math. **201**, 83–146 (2008), arXiv:math/0608367.

[6] S. Fomin and A. Zelevinsky, *Cluster algebras I. foundations*, J. Amer. Math. Soc. **15**, 497–529 (2002), arXiv:math/0104151.

[7] ———, *Cluster algebras IV: coefficients*, Composito Math. **143**, 112–164 (2007), arXiv:math/0602259.

[8] K. Hikami, *Hyperbolic structure arising from a knot invariant*, Int. J. Mod. Phys. A **16**, 3309–3333 (2001).

[9] K. Hikami and R. Inoue, *Cluster algebra and complex volume of once-punctured torus bundles and two-bridge knots*, preprint (2012), arXiv:1212.6042 [math.GT].

[10] ———, *Braids, complex volume, and cluster algebra*, preprint (2013), arXiv:1304.4776 [math.GT].

[11] K. Hikami and A. N. Kirillov, *Torus knot and minimal model*, Phys. Lett. B **575**, 343–348 (2003), arXiv:hep-th/0308152.

[12] R. M. Kashaev, *A link invariant from quantum dilogarithm*, Mod. Phys. Lett. A **10**, 1409–1418 (1995).

[13] ———, *The hyperbolic volume of knots from quantum dilogarithm*, Lett. Math. Phys. **39**, 269–275 (1997).

[14] R. M. Kashaev and O. Tirkkonen, *Proof of the volume conjecture for torus knots*, J. Math. Sci. **115**, 2033–2036 (2003).

[15] H. Murakami and J. Murakami, *The colored Jones polynomials and the simplicial volume of a knot*, Acta Math. **186**, 85–104 (2001).

[16] K. Murasugi, *Knot Theory and Its Applications*, Birkhäuser, 1996.

[17] K. Nagao, Y. Terashima, and M. Yamazaki, *Hyperbolic geometry and cluster algebra*, preprint (2011), arXiv:1112.3106 [math.GT].

[18] W. D. Neumann, *Combinatorics of triangulations and the Chern–Simons invariant for hyperbolic 3-manifolds*, in B. Apanasov, W. D. Neumann, A. W. Reid, and L. Siebenmann, eds., *Topology '90, Ohio State Univ. Math. Res. Inst. Publ.* vol. **1**, pp. 243–271, de Gruyter, Berlin, 1992.

[19] ———, *Extended Bloch group and the Cheeger–Chern–Simons class*, Geom. Topol. **8**, 413–474 (2004).

[20] W. D. Neumann and D. Zagier, *Volumes of hyperbolic three-manifolds*, Topology **24**, 307–332 (1985).

[21] M. Sakuma and J. Weeks, *Examples of canonical decompositions of hyperbolic link complements*, Japan J. Math. **21**, 393–439 (1995).

[22] D. Thurston, *Hyperbolic volume and the Jones polynomial*, Lecture notes of École d'été de Mathématiques 'Invariants de nœuds et de variétés de dimension 3', Institut Fourier (1999).

[23] W. P. Thurston, *The geometry and topology of three-manifolds*, Lecture Notes in Princeton University, Princeton (1980).

[24] Y. Yokota, *On the complex volume of hyperbolic knots*, J. Knot Theory Ramifications **20**, 955–976 (2011).

[25] D. Zagier, *Vassiliev invariants and a strange identity related to the Dedekind eta-function*, Topology **40**, 945–960 (2001).

[26] ———, *The dilagarithm function*, in P. Cartier, B. Julia, P. Moussa, and P. Vanhove, eds., *Frontiers in Number Theory, Physics, and Geometry II. On Conformal Field Theories, Discrete Groups and Renormalization*, pp. 3–65, Springer, Berlin, 2007.

[27] C. K. Zickert, *The volume and Chern–Simons invariant of a representation*, Duke Math. J. **150**, 489–532 (2009), `arXiv:0710.2049 [math.GT]`.

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