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Problems on Low-dimensional Topology, 2013

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This is a list of open problems on low-dimensional topology with expositions of their history, background, significance, or importance. This list was made by editing manuscripts written by contributors of open problems to the problem session of the conference "Intelligence of Low-dimensional Topology" held at Research Institute for Mathematical Sciences, Kyoto University in May 22–24, 2013.

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1 Super-A-polynomials of knots

(Hiroyuki Fuji)

In the generalized volume conjecture, the A-polynomial appears in the asymptotic expansion of the colored Jones polynomial $J_n(K;q)$:

$$J_n(K;q=\mathrm{e}^{\hbar}) \sim \exp\Big(\frac{1}{\hbar}S_0(x)+\sum_{k=1}^{\infty}\hbar^{k-1}S_k(x)\Big) \quad (x:=q^n),$$

$$\frac{\partial S_0(x)}{\partial x}=\log y(x), \quad A_K\big(x,y(x)\big)=0.$$

The higher order terms $S_k(x)$ in the above asymptotic expansion is found by solving the q-difference equation iteratively [11]:

$$\begin{aligned} \hat{A}_K(\hat{x},\hat{y};q)\,J_n(K;q) &= 0 \\ \hat{x}J_n(K;q) &= q^nJ_n(K;q), \quad \hat{x}J_n(K;q) \;=\; J_{n+1}(K;q). \end{aligned}$$

Such q-difference operator $\hat{A}_K(\hat{x},\hat{y};q)$ reduces to the A-polynomial in $q\to 1$ limit, and its existence is conjectured in the quantum volume conjecture/AJ conjecture [21, 18].

In [2, 1], the quantum volume conjecture is generalized to the colored HOMFLY polynomial $H_R(K;a,q)$ for the completely symmetric representation $R=S^r$. From q-difference equation of the colored HOMFLY polynomial, one finds a 1-parameter generalization of the A-polynomial $A_K^{\mathrm{Q-def}}(x,y;a)$ that is named as the *Q-deformed A-polynomial.* In [2], this polynomial is proposed to be equivalent to the augmentation polynomial of the knot contact homology [36]. In [16] we proposed the further generalization of the quantum volume conjecture for the colored superpolynomial $\mathcal{P}^R(K;a,q,t)$ with $R=S^r$, and the 2-parameter generalization of the A-polynomial is found. This (a,t)-deformed polynomial is named as *super-A-polynomial.*

For $\mathbf{3_1}$ and $\mathbf{4_1}$ knots, the super-A-polynomial are obtained as follows:

$$\begin{aligned} A^{\mathrm{super}}_{\mathbf{3_1}}(x,y;a,t) &= a^2t^4(x-1)x^3+(1+at^3x)y^2 \\ &\quad -a\big(1-t^2x+2t^2(1+at)x^2+at^5x^3+a^2t^6x^4\big)y, \\ A^{\mathrm{super}}_{\mathbf{4_1}}(x,y;a,t) &= a^2t^5(x-1)^2x^2+at^2x^2(1+at^3x)^2y^3 \\ &\quad +at(x-1)(1+t(1-t)x+2at^3(t+1)x^2-2at^4(t+1)x^3+a^2t^6(1-t)x^4-a^2t^8x^5)y \\ &\quad -(1+at^3x)(1+at(1-t)x+2at^2(t+1)x^2+2a^2t^4(t+1)x^3+a^2t^5(t-1)x^4+a^3t^7x^5)y^2. \end{aligned}$$

Question 1.1 (H. Fuji). *Is it possible to express the augmentation polynomial and the super-A-polynomial via some (a,t)-deformed holonomy representation? Can we extract the non-trivial geometric information of $S^3\backslash K$ for $A_K^{\mathrm{super}}(x,y;a,t)$?*

In the context of the matrix models and topological strings, we find a similar integrable structure as the quantum volume conjecture from the Eynard-Orantin's topological recursion on the spectral curve $\mathcal{C}$ [12]:

$$\mathcal{C} \;=\; \Big\{(x,y)\in\mathbb{C}\times\mathbb{C}^2 \;\Big|\; y^2=M(x)^2S(x),\; S(x)=\prod_{i=1}^{2n}(x-x_i)\Big\}.$$

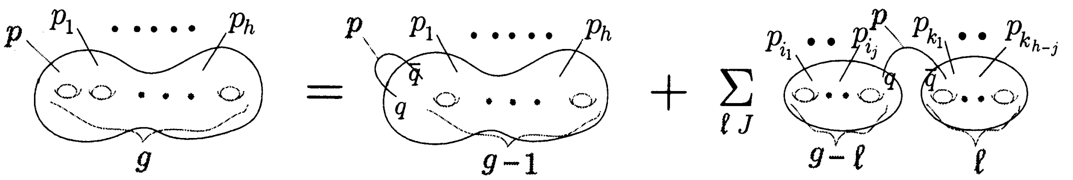


Figure 1: Diagrammatic representation of the Eynard-Orantin's topological recursion relation.

The topological recursion solves the meromorphic $(1,\cdots,1)$-forms $W^{(h,n)}(p_1,\cdots,p_n)$ on $(p_1,\cdots,p_n)\in\mathcal{C}^{\otimes n}$ in an iterative manner:

$$W^{(0,1)}(p)=y(p)dx(p),\quad W^{(0,2)}(p_1,p_2)=B(p_1,p_2),\quad K(p_0,p)=\frac{-\frac{1}{2}\int_{\bar{p}}^{p}B(\cdot,p_0)}{(y(p)-y(\bar{p}))dx(p)},$$

$$W^{(h,n)}(p_1,\cdots,p_n)=\sum_{a:\text{Branch pts.}}\operatorname*{Res}_{q\to a}K(p_0,q)\Big[W^{(h-1,n+1)}(q,\bar{q},p_I)$$
$$+\sum_{h',J}W^{(h',|J|+1)}(q,p_J)W^{(h-h',n-|J|)}(\bar{q},p_{I\setminus J})\Big],$$

where $B(x,y)$ is the Bergman kernel on $\mathcal{C}$ which is $(1,1)$-form on $\mathcal{C}\times\mathcal{C}$.

Using the theta function of the spectral curve $\mathcal{C}$ and meromorphic forms $W^{(h,n)}$, the tau function $\mathcal{T}_{g_s}[ydx]$ is studied systematically in [4]. $\mathcal{T}_{g_s}[ydx]$ is known as the *non-perturbative partition function*, and the $(n|n)$*-Baker-Akhierzer kernel* which is denoted as $\psi_{g_s}^{[n|n]}(p_1,o_1;\cdots;p_n,o_n)$ is defined as the Schlesinger transformation of the non-perturbative partition function:

$$\psi_{g_s}^{[n|n]}(p_1,o_1;\cdots;p_n,o_n)\;=\;\frac{\mathcal{T}_{g_s}[ydx+\sum_{k=1}^{n}dS_{o_k,p_k}]}{\mathcal{T}_{g_s}[ydx]}$$

where the meromorphic 1-form dS_{o_k,p_k} is given by $dS_{o_k,p_k}=\int_{o_k}^{p_k}B(\cdot,p)$.

In [10, 4], the topological recursion on the character variety

$$\mathcal{C}_K\;=\;\big\{(x,y)\in\mathbb{C}^*\times\mathbb{C}^*\;\big|\;A_K(x,y)=0\big\}$$

is studied for the figure eight knot and the SnapPea census manifold $m009$. More precisely speaking, the formal power series for the colored Jones polynomial around the exponential growth point is conjectured to be realized as the $(2|2)$ *Baker-Akhierzer kernel* $\mathcal{T}_{g_s}$ [4]:

Conjecture 1.2 ([4]). *As formal power series, the colored Jones polynomial expanded around the exponential growth point coincides with the* $(2|2)$*-Baker-Akhierzer kernel on the character variety* $\mathcal{C}_K=\{(x,y)\in\mathbb{C}^*\times\mathbb{C}^*\,|\,A_K(x,y)=0\}$,

$$J_n(K;q=e^{\hbar/2})\;\simeq\;\psi_{\hbar}^{[2|2]}(p,o;\iota(p),\iota(o))^{1/2},$$

where $x(\iota(p))=x$ *and the involution* ι *acts as* $(x(\iota(p)),y(\iota(p)))=(x(\iota(p))^{-1},y(\iota(p))^{-1})$.

Here we consider the possibility for the generalization of this conjecture to the colored HOMFLY polynomial and superpolynomial.

Question 1.3 (H. Fuji). *Is the Baker-Akhierzer* (2|2)*-kernel on the the super-character variety whose defining equation is given by* $A_K^{\rm super}(x,t;a,t)$ *related with the asymptotic expansion of the colored superpolynomial* $\mathcal{P}^{S^r}(K;a,q,t)$*?*

In the case of $K = \mathbf{3_1}$, the super-character variety is expressed in $\mathbb{C}^2$ variables as:

$$\begin{aligned}
&\mathcal{C} = \{(x,y) \in \mathbb{C}^2 \mid y^2 = M(x)^2 S(x)\},\\
&S(x) = x^4 + 2a^{-1}t^{-1}u^3 + (a^{-2}t^2 + 2a^{-1}t^{-3} + 4a^{-2}t^{-4}) - 2a^{-2}t{-}4 + a^{-2}t^{-6},\\
&A(x) = \frac{1 - t^2x + 2t^2x^2 + 2at^3x^2 + at^5x^3 + a^2t^6x^4}{at^3(1+at^3x^2)},\\
&M(x) = \frac{1}{2x\sqrt{S(x)}} \log \frac{A(x)+\sqrt{S(x)}}{A(x)-\sqrt{S(x)}}.
\end{aligned}$$

and $A_{\mathbf{3_1}}^{\rm super}(x,y;a,t)$ has deformed reciprocal symmetry $\iota : x \mapsto -1/(at^3x)$. The S_0 and S_1 terms are given by definition of the free energy as the Abel-Jacobi map and discriminant on $\mathcal{C}_0$ respectively [10]:

$$\begin{aligned}
S_0(x) &= \int_o^x \frac{dx}{x} \log y(x),\\
S_1(x) &= \frac{1}{2}\log\frac{\gamma}{\sqrt{S(x)}},
\end{aligned}$$

where γ is a constant. The next order S_2 is computed via the topological recursion [4], and the result is

$$\begin{aligned}
S_2(x) = \Big(&1 + 2at + 2t^2 + 4at^3 + 2a^2t^4 + \big(3t^2 - 2t^4 - 4at^5 - 2a^2t^6\big)x\\
&+ \big(-16t^2 - 11at^3 - 21t^4 + 2a^2t^4 - 28at^5 - 2t^6 - 8a^2t^6 - 4at^7 + 2a^3t^7 - 2a^2t^8\big)x^2\\
&+ \big(-30at^5 - 7t^6 - 24a^2t^6 - 48at^7 + 2t^8 - 68a^2t^8 + 4at^9 - 28a^3t^9 + 2a^2t^{10}\big)x^3\\
&+ \big(16at^5 + 11a^2t^6 + 21at^7 - 2a^3t^7 + 28a^2t^8 + 2at^9 + 8a^3t^9\\
&\qquad + 4a^2t^{10} - 2a^4t^{10} + 2a^3t^{11}\big)x^4\\
&+ \big(3a^2t^8 - 2a^2t^{10} - 4a^3t^{11} - 2a^4t^{12}\big)x^5\\
&+ \big(-a^3t^9 - 2a^4t^{10} - 2a^3t^{11} - 4a^4t^{12} - 2a^5t^{13}\big)x^6\Big)\\
&\Big/\Big(48(1+at)(1+t^2+at^3)(1 - 2t^2x + 4t^2x^2 + 2at^3x^2 + t^4x^2 + 2at^5x^3 + a^2t^6x^4)^{3/2}\Big).
\end{aligned}$$

The details of the computations are given in [15]. The above result and the asymptotic expansion of $\mathcal{P}^{S^r}(K;a,q,t)$ behave differently. This discrepancy would be related with the q-dependence of a and x. This point may be cured by taking into account of the mirror map of the "D-mirror symmetry" proposed in [2] carefully.

2 Quasi-trivial quandles and link homotopy invariants

(Ayumu Inoue)

A quandle X is said to be *quasi-trivial* if it satisfies the condition $x * \varphi(x) = x$ for each $x \in X$ and $\varphi \in \mathrm{Inn}(X)$ [25]. Here, $\mathrm{Inn}(X)$ denotes the inner automorphism group of X. Modifying usual quandle homology slightly, we have homology of a quasi-trivial quandle [25]. We let $H_n^{Q,qt}(X)$ and $H^n_{Q,qt}(X)$ denote the n-th homology and cohomology groups of a quasi-trivial quandle X respectively.

For each oriented and ordered link L in S^3, we have the *reduced knot quandle* $RQ(L)$ which is invariant under link-homotopy [24]. Here, given two links are said to be *link-homotopic* if they are related to each other by a finite sequence of ambient isotopies and self-crossing changes [32]. The reduced knot quandle $RQ(L)$ is quasi-trivial. Associated with each component K_i of L, we have the *fundamental class* $[K_i]^{qt} \in H_2^{Q,qt}(L)$ which is invariant under link-homotopy [25, 26].

Conjecture 2.1 (A. Inoue). *Two n-component links $L = K_1 \cup \cdots \cup K_n$ and $L' = K_1' \cup \cdots \cup K_n'$ are link-homotopic if and only if there is an isomorphism $\varphi : RQ(L) \to RQ(L')$ such that $\varphi_\sharp([K_i]^{qt}) = [K_i']^{qt}$ for each i $(1 \leq i \leq n)$. Here, $\varphi_\sharp : H_2^{Q,qt}(RQ(L)) \to H_2^{Q,qt}(RQ(L'))$ denotes the isomorphism induced from φ.*

This conjecture is true in the case $n \leq 3$. Prove the conjecture, or give a counter example.

Suppose that X is a quasi-trivial quandle. By definition, for each $x \in X$ and $\varphi \in \mathrm{Inn}(X)$, the pair $(x, \varphi(x))$ is a 2-cycle.

Question 2.2 (A. Inoue). *Let $L = K_1 \cup \cdots \cup K_n$ be an n-component link. For all $x \in RQ(L)$ and $\varphi \in \mathrm{Inn}(X)$, is not the class $[x, \varphi(x)]$ in $\langle [K_1]^{qt} \rangle \oplus \cdots \oplus \langle [K_n]^{qt} \rangle \triangleleft H_2^{Q,qt}(RQ(L))$?*

We have another homology group $\widetilde{H}_n^{Q,qt}(RQ(L))$ which is a certain quotient of $H_n^{Q,qt}(RQ(L))$. By definition, $[x, \varphi(x)] = 0$ in $\widetilde{H}_n^{Q,qt}(RQ(L))$. We can show that $[K_i]^{qt} \in \widetilde{H}_n^{Q,qt}(RQ(L))$ for each i and $\widetilde{H}_n^{Q,qt}(RQ(L)) = \langle [K_1]^{qt} \rangle \oplus \cdots \oplus \langle [K_n]^{qt} \rangle$. If Question 2.2 is true, considering $\widetilde{H}_2^{Q,qt}(RQ(L))$ instead of $H_2^{Q,qt}(RQ(L))$ is sufficient to classifying links up to link-homotopy.

Choose and fix a quasi-trivial quandle X and its 2-cocycle θ with coefficients in an abelian group A. For each n-component link $L = K_1 \cup \cdots \cup K_n$, consider all homomorphisms $\varphi : RQ(L) \to X$. Then the multiset consisting of all elements $\langle \theta, \varphi_\sharp([K_i]^{qt}) \rangle \in A$ obviously gives rise to a link-homotopy invariant. We call this numerical invariant a *quandle cocycle invariant.*

Problem 2.3 (A. Inoue). *Classify links up to link-homotopy using quandle cocycle invariants.*

For example, we can distinguish the trivial 3-component link and the Borromean rings using a certain quandle cocycle invariant [25]. Can we distinguish the n-component trivial link and Brunnian links using quandle cocycle invariants?

We have other numerical link-homotopy invariants called Milnor's $\overline{\mu}$-invariants [33].

Question 2.4 (A. Inoue). *How are quandle cocycle invariants related with Milnor's $\overline{\mu}$-invariants?*

We hope that (quasi-trivial) quandles and their homology would be useful to investigate other areas in knot theory, *e.g.*, tunnel number, sliceness, etc.

3 Ordering of knot and 3-manifold groups

(Tetsuya Ito)

A total ordering $<_G$ of a group G is called a *left-ordering* (resp. *right-ordering*) if $a <_G b$ implies $ga <_G gb$ (resp. $ag <_G bg$) for all $a, b, g \in G$. $<_G$ is called a *bi-ordering* if it is both left- and right-ordering. A group G is *left-orderable* (resp. *bi-orderable*) if G has at least one left-ordering (resp. bi-ordering).

The problem that which knot group or 3-manifold group admits a left- or bi-ordering recently gathered much attention, but many basic questions remain unsolved. Moreover, we have few (non-trivial) examples of orderings. For backgrounds on orderability and 3-dimensional topology, see [5, 6]. Here we list several problems concerning orderings and orderability which may not be too hard.

For bi-orderings of knot groups, our knowledge is limited to fibered knot groups [8, 27, 41, 42]. Thus we pose the following questions.

Question 3.1 (T. Ito (Bi-ordering of knot groups)).

(1) *Find an example of bi-ordering of non-fibered knot groups.*

(2) *Is the knot group of the 5_2 knot bi-orderable ?*

(3) *Is the knot group of the 6_2 knot bi-orderable ?*

(4) *Find an example of not bi-orderable, but virtually bi-orderable knot groups. (More generally, is a knot group virtually bi-orderable ?)*

As for (2), the 5_2 knot is the simplest non-fibered knot. As for (3), the 6_2 knot is the first example of fibered knot whose bi-orderability is unknown.

Contrary, it is known that *all* knot groups are left-orderable, although still we have few concrete examples. We have several interesting and highly non-trivial left-orderings (such as, isolated orderings) of torus knot groups [28, 35]. An element a of a left-orderable group G is called *universally cofinal* if for every left-ordering $<_G$ of G and $g \in G$, there exists $N \in \mathbb{Z}$ such that $a^{-N} <_G g <_G a^N$ holds. A left ordering $<_G$ of G is called *discrete* if $<_G$ admits a $<_G$-positive minimum element.

Question 3.2 (T. Ito (Left-ordering of knot groups)).

(1) *Give an* "explicit" *example of left-ordering $<$ of knot groups.*

(2) *Can we generalize the above mentioned orderings of torus knot groups to other class of knot groups, such as, 2-bridge knot groups ?*

(3) *Which knot group has (does not have) universally cofinal element ?*

(4) *Do every knot group $<_G$ admit a discrete ordering $<_G$ such that its meridian is the $<_G$-minimum positive element ?*

As for (1), *"explicit"* means that we have a reasonable characterization of $<$-positive elements, or, one can check given element x of the knot group is $<$-positive or not in an reasonable way. As for (3), it is easy to see that a torus knot group admits universally cofinal element. As for (4), of course, constructing discrete ordering for knot group is already interesting.

It is conjectured [5] that 3-manifold M is non-left-orderable (that is, M has non-left-orderable fundamental group) if and only if M is an L-space. Several supporting evidences of this conjecture have been established by many researchers, but general case is widely open.

Question 3.3 (T. Ito ((Non-)left-orderable 3-manifold)).
(1) *Is the double branched covering of a quasi-alternating link non-left-orderable ?*
(2) *Characterize non-left-orderable 3-manifold of Heegaard genus two.*
(3) *Let M be non-left-orderable (hyperbolic) 3-manifold. Is there a finite covering $\widetilde{M}$ of M with property (a) and (b) below ?*
  (a) *$\widetilde{M}$ is left-orderable.*
  (b) *$\widetilde{M}$ is a rational homology 3-sphere.*

As for (1), see [29] for related results. As for (2), see [31] for the corresponding results for 3-manifolds admitting genus one open book decomposition. As for (3), by virtual fibered conjecture, if we drop the condition (b), we can always find such $\widetilde{M}$.

4 Left-orderable surgeries and L–space surgeries

(Kimihiko Motegi)[2]

L–spaces

Heegaard Floer homology is a package of 3–manifold invariants introduced by Ozsváth and Szabó [37, 38]. In the following we consider the simplest one called the "hat" version, denoted by $\widehat{HF}(Y)$. The Heegaard Floer homology $\widehat{HF}(Y)$ has a relative Z_2–grading $\widehat{HF}(Y) = \widehat{HF}_0(M) \oplus \widehat{HF}_1(Y)$. For a rational homology 3–sphere M, we have $\chi(\widehat{HF}(Y)) = |H_1(M;\mathbb{Z})|$. In particular, rank$\widehat{HF}(M) \geq |H_1(M;\mathbb{Z})|$.

There are several kinds of definitions for L–spaces. In particular, we often find the following definitions in literatures.

Definition (L–space)
(1) A rational homology 3–sphere M is called an *L–space* if the Heegaard Floer homology with $\mathbb{Z}$–coefficients $\widehat{HF}(Y;\mathbb{Z})$ is a free abelian group of rank $|H_1(Y;\mathbb{Z})|$.
(2) A rational homology 3–sphere M is called an *L–space* if the Heegaard Floer

[2] I would like to thank Cameron Gordon, Hiroshi Matsuda, Yi Ni, Motoo Tange for private communications concerning Question 4.1. I would also like to thank Masakazu Teragaito for useful discussions.

homology with $\mathbb{Z}$–coefficients $\widehat{HF}(Y;\mathbb{Z})$ has the rank $|H_1(Y;\mathbb{Z})|$.
(3) A rational homology 3–sphere M is called an *L–space* if the Heegaard Floer homology with $\mathbb{Z}_2$–coefficients $\widehat{HF}(Y;\mathbb{Z}_2)$ has the rank $|H_1(Y;\mathbb{Z})|$.

(1) is the original definition of Ozsváth and Szabó [39], which does *not* allow "torsion", but (2) allows "torsion". The difference between (2) and (3) is the choice of "coefficients" for $\widehat{HF}(Y)$ which we use.

Question 4.1 (K. Motegi). *Are these three conditions equivalent? In particular, for any rational homology sphere Y, does $\widehat{HF}(Y;\mathbb{Z})$ have no torsion?*

For rational homology 3–spheres, no examples are known where $\widehat{HF}(Y;\mathbb{Z})$ has torsion. So possibly definitions (1), (2) and (3) are equivalent. However, at the moment, it seems convenient to adopt the definition (3) for homogeneity; see [5, 1.1].

The set of left-orderable surgeries

We say that a nontrivial group G is *left-orderable* if there exists a strict total ordering $<$ on its elements such that $g < h$ implies $fg < fh$ for all elements $f, g, h \in G$. The left-orderability of fundamental groups of 3–manifolds has been studied by Boyer, Rolfsen and Wiest [6]. In particular, it is known that P^2–irreducible 3–manifold M with first Betti number $b_1 \geq 1$ has the left-orderable fundamental group. Hence for any knot K in S^3, its exterior $E(K)$ has the left-orderable fundamental group. However if $r \neq 0$, the result $K(r)$ of r–Dehn surgery on K, which is a rational homology 3–sphere, may not have such a fundamental group. A Dehn surgery is said to be *left-orderable* if the resulting manifold of the surgery has the left-orderable fundamental group, and the set of left-orderable surgery slopes for K is defined as

$$\mathcal{S}_{LO}(K) \ = \ \{r \in \mathbb{Q} \mid \pi_1(K(r)) \text{ is left-orderable}\}.$$

For any nontrivial knot K, $K(0)$ is irreducible [17, Corollary 8.3] and $H_1(K(0)) \cong \mathbb{Z}$, and hence $\{0\} \subset \mathcal{S}_{LO}(K)$ [6, Theorem 1.1]. If K is a trivial knot then $\mathcal{S}_{LO}(K) = \{0\}$, which has the smallest size. In the opposite direction, in [34], we demonstrate that there are infinitely many hyperbolic knots K with $\mathcal{S}_{LO}(K) = \mathbb{Q}$. It seems interesting to ask:

Question 4.2 (K. Motegi). *If K is a nontrivial knot in S^3, then does $\mathcal{S}_{LO}(K)$ contain $(-1,1) \cap \mathbb{Q}$?*

Recently Li and Roberts [30, Corollary 1.2] prove that for any hyperbolic knot K, there exists a constant N_K such that $\{\frac{1}{n} \mid |n| > N_K\} \subset \mathcal{S}_{LO}(K)$. More strongly we would like to ask:

Question 4.3 (K. Motegi). *If K is a nontrivial knot in S^3, then does $\mathcal{S}_{LO}(K)$ contain $(-\infty,1) \cap \mathbb{Q}$ or $(-1,\infty) \cap \mathbb{Q}$?*

For the simplest nontrivial knot $T_{3,2}$ (resp. $T_{-3,2}$), the argument in the proof of [9, Theorem 1.4] shows that $\mathcal{S}_{LO}(T_{3,2}) = (-\infty,1) \cap \mathbb{Q}$ (resp. $\mathcal{S}_{LO}(T_{-3,2}) = (-1,\infty) \cap \mathbb{Q}$).

Question 4.4 (K. Motegi). *If $\mathcal{S}_{LO}(K) = (-\infty, 1) \cap \mathbb{Q}$ or $\mathcal{S}_{LO}(K) = (-1, \infty) \cap \mathbb{Q}$, then is K a trefoil knot $T_{3,2}$ or $T_{-3,2}$, respectively?*

Question 4.5 (K. Motegi). *Let K be a nontrivial knot in S^3. Then does $\mathcal{S}_{LO}(K)$ have no maximum and minimum?*

Boyer-Gordon-Watson conjecture

Results in [5] and previously known examples suggest that there exists a strange correspondence between 3–manifolds whose fundamental groups are left-orderable and L–spaces. For instance, lens spaces, spherical 3–manifolds are L–spaces, on the other hand their fundamental groups are *not* left-orderable (because they have torsions). The following conjecture is formulated by Boyer, Gordon and Watson [5].

Conjecture 4.6 ([5]). *An irreducible rational homology 3-sphere is an L–space if and only if its fundamental group is not left-orderable.*

A Dehn surgery is called an *L–space surgery* if the resulting manifold of the surgery is an L–space. Define the set of L–space surgery slopes for K as

$$\mathcal{S}_L(K) = \{r \in \mathbb{Q} \mid K(r) \text{ is an } L\text{–space}\}.$$

Now let us look at the shape of $\mathcal{S}_L(K)$, which is well-understood by Proposition 9.6 in [40] ([22, Lemma 2.13]). It is known [40, 22] that, if K is a nontrivial knot and $\mathcal{S}_L(K) \neq \emptyset$, then $\mathcal{S}_L(K) = [2g(K)-1, \infty) \cap \mathbb{Q}$ or $\mathcal{S}_L(K) = (-\infty, -2g(K)+1] \cap \mathbb{Q}$. Since Conjecture 4.6 says that $\mathcal{S}_L(K)$ and $\mathcal{S}_{LO}(K)$ are complementary to each other in $\mathbb{Q}$ if K has no reducing surgery, we expect the following explicit form of $\mathcal{S}_{LO}(K)$.

Conjecture 4.7. *Let K be a nontrivial knot in S^3 which has no reducing surgery. Then $\mathcal{S}_{LO}(K)$ coincides with one of $\mathbb{Q}$, $(-\infty, 2g(K)-1) \cap \mathbb{Q}$ and $(-2g(K)+1, \infty) \cap \mathbb{Q}$.*

Finally we give a comment on Question 4.4 in case of $\mathcal{S}_{LO}(K) = (-\infty, 1) \cap \mathbb{Q}$; the other case follows by taking the mirror image. By the assumption $1 \notin \mathcal{S}_{LO}(K)$, if Conjecture 4.6 is true, then $1 \in \mathcal{S}_L(K)$ or $K(1)$ is reducible. The latter possibility is eliminated by [20, Corollary 3.1], and hence $K(1)$ is an L–space. Further, it is known [23] that, if K is a nontrivial knot and $K(\frac{1}{n})$ is an L–space, then $n = 1$ (resp. -1) and K is a trefoil knot $T_{3,2}$ (resp. $T_{-3,2}$). Therefore, K is a trefoil knot $T_{3,2}$.

5 Groups with addition

(Rinat M. Kashaev)

A *group with addition* is a group with an additional commutative and associative binary operation $(x, y) \mapsto x + y$ with respect to which the group product is distributive from both sides.

Examples (groups with addition)

(1) $\mathbb{Q}_{>0}$ or $\mathbb{R}_{>0}$ with the usual addition.

(2) $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$ with tropical addition $\max(x,y)$.
(3) The group of upper triangular n-by-n matrices over $\mathbb{Q}$ or $\mathbb{R}$ with positive diagonal elements and with the addition of matrices.

One can easily show that no finite group can be a group with addition.

Question 5.1.
(1) *Are there other examples of groups with addition?*
(2) *What are the conditions under which a given infinite group admits an addition?*
(3) *Is there a systematic procedure for constructing groups with addition?*

Motivation for groups with addition

Groups with addition can be used for construction of representations of mapping class groups of punctured surfaces as follows.

Groupoid of ideal triangulations. Recall that for any set S freely acted upon by a group G, one can associate a connected groupoid $\mathcal{G}_{S,G}$ with the set of G-orbits in S as the set of objects $\mathrm{Ob}\,\mathcal{G}_{S,G}$, the G-orbits in $S\times S$ (with respect to the diagonal action) as the set of morphisms $\mathrm{Mor}\,\mathcal{G}_{S,G}$, the domain and the codomain maps

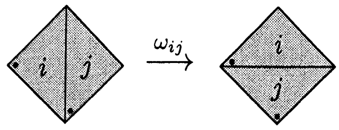
$$\mathrm{dom}, \mathrm{cod}\colon \mathrm{Mor}\,\mathcal{G}_{S,G} \to \mathrm{Ob}\,\mathcal{G}_{S,G}, \quad \mathrm{dom}[x,y]=[x],\ \mathrm{cod}[x,y]=[y],$$

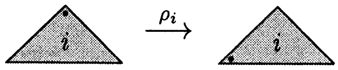
so that two morphisms $[x,y]$ and $[u,v]$ are composable if and only if $\mathrm{cod}[x,y] = \mathrm{dom}[u,v]$, i.e. $[y]=[u]$ with the composition (adopting the convention used for the composition in fundamental groupoids of topological spaces) $[x,y]\circ[u,v]=[x,gv]$, where g is the unique group element such that $y=gu$. The groupoid $\mathcal{G}_{S,G}$ is a connected groupoid with any vertex group being isomorphic to G, so that any representation of $\mathcal{G}_{S,G}$ restricts to a representation of G.

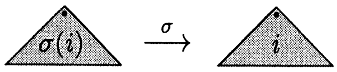
Let $\Sigma=\Sigma_{g,s}$ be a closed oriented surface of genus g with s punctures such that $(2g-2+s)s>0$. An *ideal triangulation* of Σ is a CW-decomposition of the closed surface $\overline{\Sigma}$ where the set of vertices coincides with the set of punctures, and all cells are simplices. A *decorated ideal triangulation* of Σ is an ideal triangulation where each triangle is provided with a distinguished corner and the set of all triangles is linearly ordered. Let Δ_Σ be the set of decorated ideal triangulations of Σ. The mapping class group of Σ, Γ_Σ, freely acts on Δ_Σ. The associated groupoid $\mathcal{G}_{\Delta_\Sigma,\Gamma_\Sigma}$ is called the groupoid of (decorated) ideal triangulations of Σ.

A presentation of the groupoid of ideal triangulations. One can show that

$\mathcal{G}_{\Delta_\Sigma,\Gamma_\Sigma}$ is generated by the following three types of morphisms:

Diagonal flips: 

Corner changes: 

Permutations: 

which satisfy the following relations:

Pentagons:	$\omega_{ij} \circ \omega_{ik} \circ \omega_{jk} = \omega_{jk} \circ \omega_{ij}$
Triple corner changes:	$\rho_i \circ \rho_i = \rho_i^{-1}$
Double flips:	$\omega_{ij} \circ \rho_i \circ (ij) \circ \omega_{ij} = \rho_i \circ \rho_j$
Permutation relations:	$(ij) \circ (jk) \circ (ij) = (jk) \circ (ij) \circ (jk)$ etc.

Semisymmetric T-matrices. A *T-matrix* in a symmetric monodical category $\mathcal{C} = (\mathcal{C}, \otimes, s)$ is an automorphism $T \in \mathrm{Aut}(V^{\otimes 2})$, with $V \in \mathrm{Ob}\,\mathcal{C}$, which satisfies the equation $T_{12}T_{13}T_{23} = T_{23}T_{12}$ in $\mathrm{Aut}(V^{\otimes 3})$. A T-matrix $T \in \mathrm{Aut}(V^{\otimes 2})$ is *semisymmetric* if there exists a *symmetry* $A \in \mathrm{Aut}(V)$ such that $A^3 = \mathrm{id}_V$ and $T(A \otimes \mathrm{id}_V)s_{V,V}T = A \otimes A$.

Theorem. *Let Σ be an oriented surface of genus g with s punctures such that $(2g - 2 + s)s > 0$. Then, for any semisymmetric T-matrix $T \in \mathrm{Aut}(V^{\otimes 2})$ there exists a unique representation $\pi_T\colon \mathcal{G}_{\Delta_\Sigma,\Gamma_\Sigma} \to Aut(V^{\otimes n_{g,s}})$, $n_{g,s} := 4g - 4 + 2s$, such that $\pi_T(\omega_{ij}) = T_{ij}$, $\pi_T(\rho_i) = A_i$, $\pi_T((1,2)) = s_{V,V} \otimes \mathrm{id}_{V^{\otimes(n_{g,s}-2)}}$.*

Theorem. *Let G be a group with addition, $c \in G$ a central element, and $X = G \times G$. Then, there exists a semisymmetric set-theoretical T-matrix $t\colon X^2 \ni (x,y) \mapsto (x \cdot y, x * y) \in X^2$ with symmetry $a\colon X \ni (x_1, x_2) \mapsto (cx_1^{-1}x_2, x_1^{-1}) \in X$, $x \cdot y = (x_1y_1, x_1y_2 + x_2)$, and $x * y = a^{-1}(a(y) \cdot a^{-1}(x))$.*

6 Periodic orbits of pseudo-Anosov flows

(Sérgio R. Fenley)

A pseudo-Anosov flow Φ has a countable number of periodic orbits. Each orbit defines a free homotopy class, or a conjugacy class in the fundamental group of M. The orbit is traversed in the positive flow direction.

Question 6.1 (S. R. Fenley). *Do the conjugacy classes of the periodic orbits of the pseudo-Anosov flow Φ generate $\pi_1(M)$?*

This is not always true. For example if Φ is a suspension pseudo-Anosov flow then this does not happen. Is that the only situation where this does not happen?

Question 6.2 (S. R. Fenley). *Suppose that Φ is a pseudo-Anosov flow transverse to an $\mathbb{R}$-covered foliation and that Φ is regulating for $\mathcal{F}$. Does this imply that the periodic orbits do not generate $\pi_1(M)$?*

Here $\mathbb{R}$-covered means that in the universal cover, the leaf space of the lifted foliation is homeomorphic to the set of real numbers $\mathbb{R}$. Regulating means that in the universal cover, every orbit of the lifted flow intersects every leaf of the lifted foliation and vice versa. This is very common: whenever the foliation $\mathcal{F}$ is $\mathbb{R}$-covered, transversely orientable, and M is aspherical and atoroidal, there is such a flow. This was proved in [7] and [14].

An Anosov flow Φ is said to be $\mathbb{R}$-covered if its stable (or unstable) foliation is $\mathbb{R}$-covered. They are very common and there are infinitely many examples where M is hyperbolic as proved in [13]. Suppose that M is atoroidal. It was proved in [13] that every closed orbit of Φ is freely homotopic to infinitely many other closed orbits.

Question 6.3 (S. R. Fenley). *Are there other examples where some closed orbits are freely homotopic to infinitely many other closed orbits of the flow? Does it imply that M has to be atoroidal?*

In [3] it was proved that if Φ is an $\mathbb{R}$-covered Anosov flow so that its stable foliation is transversely orientable then the following happens: given γ a periodic orbit, it is freely homotopic to infinitely many other closed orbits. In [3] it is proved that γ is in fact isotopic to each of these other closed orbits. That is, they represent the same knot in M.

Question 6.4 (S. R. Fenley). *Suppose that α and β are closed orbits of a pseudo-Anosov flow Φ which are freely homotopic. Does it follow that α and β are isotopic, as is the case when Φ is an $\mathbb{R}$-covered Anosov flow? When are they isotopic?*

The proof in [3] uses the universal circle of Thurston [43] in an essential way. In the case that the stable foliation of Φ is not an $\mathbb{R}$-covered foliation, the universal circle is much more complicated and the analysis of Question 6.4 is bound to be more complicated too. There may be some simple examples or counterexamples which should be analysed first.

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