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SPECIAL LAGRANGIAN SUBMANIFOLDS INVARIANT UNDER THE ISOTROPY ACTION OF SYMMETRIC SPACES OF RANK TWO

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1. Introduction

Special Lagrangian submanifold in Calabi-Yau manifolds play an important role in the explanation of Mirror symmetry. They are examples of calibrated submanifolds, appearing in Harvey and Lawson ([1]), which generalizes the concept of volume minimizing property of complex submanifolds of Käher manifolds. Let M be a Calabi-Yau manifold with a complex volume form Ω. Then naturally $\mathrm{Re}\,\Omega$ is a calibration on M, and a calibrated submanifold is called a *special Lagrangian* submanifold. For examples, Joyce constructed many interesting examples of special Lagrangian submanifolds in $\mathbb{C}^n$, using various methods. In particular, cohomogeneity one special Lagrangian submanifolds are constructed using moment map techniques.

The cohomogeneity one actions on spheres have been classified by Hsiang and Lawson ([2]). Every cohomogeneity one action on S^n is orbit equivalent to the isotropy representation of a Riemannian symmetric space of rank 2.

A compact hypersurface N in the unit standard sphere S^n is *homogeneous* if it is obtained as an orbit of a compact connected subgroup of $SO(n+1)$. It is well known that any homogeneous hypersurface in S^n can be obtained as a principal orbit of the isotropy representation of a Riemannian symmetric space of rank 2 ([2]). A homogeneous hypersurface N in S^n is a hypersurface with constant principal curvatures, which is called *isoparametric* ([13]). Then the number g of distinct principal curvatures must be 1, 2, 3, 4 or 6 ([13], [8] and see [9], [10] for general isoparametric hypersurfaces). Denote by (m_1, m_2)

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the multiplicities of its principal curvatures. The isotropy representation of a Riemannian symmetric space G/K of rank 2 induces a group action of K on S^n and thus T^*S^n in a natural way. In the cases of $g = 1, 2$ such group actions of $SO(p) \times SO(n+1-p)$ $(1 \leq p \leq n)$ are induced on T^*S^n. We classified cohomogeneity one special Lagrangian submanifolds in T^*S^n under the group actions ([5]).

In this paper we shall discuss the construction of cohomogeneity one special Lagrangian submanifolds in the case when $g = 1, 2, 3, 4$ and G/K is of classical type. We refer Halgason's textbook ([7])

TABLE 1. Homogeneous hypersurfaces in spheres

g	(G, K)	m_1, m_2	$\dim N$
1	$(S^1 \times SO(n+1), SO(n))$	$n-1$	$n-1$
2	$(SO(p+1) \times SO(n+1-p),$ $SO(p) \times SO(n-p))$	$p-1, n-p$	$n-1$
3	$(SU(3), SO(3))$	$1, 1$	3
3	$(SU(3) \times SU(3), SU(3))$	$2, 2$	6
3	$(SU(6), Sp(3))$	$4, 4$	12
3	(E_6, F_4)	$8, 8$	24
4	$(SO(5) \times SO(5), SO(5))$	$2, 2$	8
4	$(SO(2+m), SO(2) \times SO(m))$	$1, m-2$	$2m-2$
4	$(SU(2+m), S(U(2) \times U(m)))$	$2, 2m-3$	$4m-2$
4	$(Sp(2+m), Sp(2) \times Sp(m))$	$4, 4m-5$	$8m-2$
4	$(SO(10), U(5))$	$4, 5$	18
4	$(E_6, U(1) \times Spin(10))$	$6, 9$	30
6	$(G_2, SO(4))$	$1, 1$	6
6	$(G_2 \times G_2, G_2)$	$2, 2$	12

for the general theory of Riemannian symmetric space. Let M be a simply connected semisimple Riemannian symmetric space. If G is the identity component of the full group of isometrics of M, then G acts transitively on M and we can write $M = G/K$, where K is the isotropy subgroup of G at a point $p \in M$. Since S is simply connected and G is connected, K is also connected. If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the canonical decomposition of $\mathfrak{g}$ associated to the symmetric pair (G, K), then the isotropy representation of K on T_pM is equivalent to the adjoint representation of K on $\mathfrak{p}$. The isotropy representation of G/K

at p is a Lie group homomorphism $\mathrm{Ad}_{\mathfrak{p}} : K \to SO(\mathfrak{p})$. So a K-orbit through $X \in \mathfrak{p}$ is denoted by $\mathrm{Ad}_{\mathfrak{p}}(K)X$.

Let G/K be an $(n+1)$-dimensional rank 2 symmetric space of compact type. Define an $\mathrm{Ad}_{\mathfrak{p}}(K)$-invariant inner product of $\mathfrak{p}$ from the Killing from of $\mathfrak{g}$. Then the vector space $\mathfrak{p}$ can be identified with $\mathbb{R}^{n+1}$ with respect to the inner product. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Since for each $X \in \mathfrak{p}$ there is an element $k \in K$ such that $\mathrm{Ad}_{\mathfrak{p}}(K)X \in \mathfrak{a}$, every orbit in $\mathfrak{p}$ under K meets $\mathfrak{p}$. The unit hypersphere in $\mathfrak{p}$ is denoted by S^n. Since the action of K on $\mathfrak{p}$ is an orthogonal representation, an orbit $\mathrm{Ad}_{\mathfrak{p}}(K)X$ is a submanifold of the hypersphere S^n in $\mathfrak{p}$. For a *regular* element $H \in \mathfrak{a} \cap S^n$, we obtain a homogeneous hypersurface $N = \mathrm{Ad}_{\mathfrak{p}}(K)H \subset S^n \subset \mathfrak{p} \cong \mathbb{R}^{n+1}$. Conversely, every homogeneous hypersurfaces in a sphere is obtained in this way ([2]).

2. Calabi-Yau manifolds and special Lagrangian submanifolds

We shall review some definitions and basic notions of Calabi-Yau manifolds and special Lagrangian submanifolds. See [4] for details.

There are several different definitions of Calabi-Yau manifolds. In this paper, we use the following definition.

Definition 2.1. Let $n \geq 2$. An *almost Calabi-Yau* n-fold is a quadruple (M, J, ω, Ω) such that (M, J, ω) is a Kähler manifold of complex dimension n with a complex structure J and a Kähler form ω, and Ω is a nonvanishing holomorphic $(n, 0)$-form on M. In addition, if ω and Ω satisfy

$$\frac{\omega^n}{n!} = (-1)^{n(n-1)/2} \left(\frac{\sqrt{-1}}{2}\right)^n \Omega \wedge \bar{\Omega}, \tag{1}$$

then we call (M, J, ω, Ω) a *Calabi-Yau* n-fold.

If ω and Ω satisfy (1), then the Kähler metric g of (M, J, ω) is Ricci-flat. Its holonomy group $\mathrm{Hol}(g)$ is a subgroup of $SU(n)$, and this is another definition of a Calabi-Yau manifold.

A closed p-form φ on a Riemannian manifold (M, g) is called a *calibration* if $\varphi|_V \leq \mathrm{vol}_V$ for any oriented p-plane $V \subset T_xM$ for all $x \in M$. A p-dimensional submanifold N of M is said to be *calibrated* by a calibration φ if $\varphi|_{T_xN} = \mathrm{vol}_{T_xN}$ for all $x \in N$.

Remark 2.2. The constant factor in (1) is chosen so that $\mathrm{Re}(e^{\sqrt{-1}\theta}\Omega)$ is a calibration for any $\theta \in \mathbb{R}$.

Definition 2.3. Let (M, J, ω, Ω) be a Calabi-Yau n-fold and L be a real n-dimensional submanifold of M. Then, for $\theta \in \mathbb{R}$, L is called a *special Lagrangian submanifold* of phase θ if it is calibrated by the calibration $\mathrm{Re}(e^{\sqrt{-1}\theta}\Omega)$.

Harvey and Lawson gave the following alternative characterization of special Lagrangina submanifolds.

Proposition 2.4 ([1]). *Let (M, J, ω, Ω) be a Calabi-Yau n-fold and L be a real n-dimensional submanifold of M. Then L is a special Lagrangian submanifold of phase θ if and only if $\omega|_L \equiv 0$ and $\mathrm{Im}(e^{\sqrt{-1}\theta}\Omega)|_L \equiv 0$.*

3. Stenzel metric and Momenta maps

We briefly recall the Stenzel metric on T^*S^n. We denote the cotangent bundle of the n-sphere $S^n \cong SO(n+1)/SO(n)$ by $T^*S^n = \{(x, \xi) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \|x\| = 1, \langle x, \xi \rangle = 0\}$. We identify the tangent bundle and the cotangent bundle of S^n by the Riemannian metric on S^n. Since any unit cotangent vector of S^n can be translated to another one, the Lie group $SO(n+1)$ acts on T^*S^n with cohomogeneity one by $g \cdot (x, \xi) = (gx, g\xi)$ for $g \in SO(n+1)$. Let Q^n be a complex quadric in $\mathbb{C}^{n+1}$ defined by

$$Q^n = \left\{ z = (z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \;\middle|\; \sum_{i=1}^{n+1} z_i^2 = 1 \right\}.$$

The Lie group $SO(n+1, \mathbb{C})$ acts on Q^n transitively, hence $Q^n \cong SO(n+1, \mathbb{C})/SO(n, \mathbb{C})$. According to Szöke ([12]), we can identify T^*S^n with Q^n through the following diffeomorphism:

$$\Phi : T^*S^n \ni (x, \xi) \longmapsto x\cosh(\|\xi\|) + \sqrt{-1}\frac{\xi}{\|\xi\|}\sinh(\|\xi\|) \in Q^n.$$

The diffeomorphism Φ is equivariant under the action of $SO(n+1)$. Thus we frequently identify T^*S^n with Q^n. Then consider a holomorphic n-form Ω_{Stz} given by

$$\frac{1}{2}d(z_1^2 + z_2^2 + \cdots + z_{n+1}^2 - 1) \wedge \Omega_{Stz} = dz_1 \wedge \ldots \wedge dz_{n+1}.$$

The *Stenzel metric* is a complete Ricci-flat Kähler metric on Q^n defined by $\omega_{Stz} := \sqrt{-1}\partial\bar{\partial}\, u(r^2)$, where $r^2 = \|z\|^2 = \sum_{i=0}^{n+1} z_i\bar{z}_i$ and u is a smooth real-valued function satisfying the differential equation

$$\frac{d}{dt}(U'(t))^n = cn\,(\sinh t)^{n-1} \quad (c > 0)$$

where $U(t) = u(\cosh t)$. The Kähler form ω_{Stz} is exact, that is, $\omega_{Stz} = d\alpha_{Stz}$ where $\alpha_{Stz} := -\mathrm{Im}(\bar{\partial}u(r^2))$.

Let K be a compact connected Lie subgroup of $SO(n+1)$ with Lie algebra $\mathfrak{k}$. Then the group action of K on Q^n is Hamiltonian with respect to ω_{Stz} and its moment map $\mu : Q^n \to \mathfrak{k}^*$ is given by

$$(2) \quad \langle \mu(z), X\rangle = \alpha_{Stz}(Xz) = u'(\|z\|^2)\langle Jz, Xz\rangle \qquad (z \in Q^n, X \in \mathfrak{k}).$$

Choose a subset Σ of T^*S^n such that every K-orbit in T^*S^n meets Σ. In general assume that K has the Hamiltonian group action on a symplectic manifold M. We define the center of $\mathfrak{k}^*$ to be $Z(\mathfrak{k}^*) = \{X \in \mathfrak{k}^* \mid \mathrm{Ad}^*(k)X = X \ (\forall k \in K)\}$. It is easy to see that the inverse image $\mu^{-1}(c)$ of $c \in \mathfrak{k}^*$ is invariant under the group action of K if and only if $c \in Z(\mathfrak{k}^*)$.

Proposition 3.1. *Let L be a connected isotropic submanifold, i.e., $\omega|_L \equiv 0$, of M invariant under the action of K. Then $L \subset \mu^{-1}(c)$ for some $c \in Z(\mathfrak{k}^*)$.*

Proposition 3.2. *Let L be a connected submanifold of M invariant under the action of K. Suppose that the action of K on L is of cohomogeneity one (possibly transitive). Then L is an isotropic submanifold, i.e., $\omega|_L \equiv 0$, if and only if $L \subset \mu^{-1}(c)$ for some $c \in Z(\mathfrak{k}^*)$.*

For the group action of K induced by the isotropy representation of G/K, the moment map formula (2) becomes

$$(3) \quad \mu(Z) = -u'(\|Z\|^2)\sqrt{-1}[Z, \overline{Z}] = -2u'(\|Z\|^2)\,[X, Y] \in \mathfrak{k} \cong \mathfrak{k}^*$$

for each $Z = X + \sqrt{-1}Y \in Q^n \subset \mathfrak{p}^{\mathbb{C}} \cong \mathbb{C}^{n+1}$ with $X, Y \in \mathfrak{p} \cong \mathbb{R}^{n+1}$.

Now we consider only the case where the inverse image $\mu^{-1}(0)$ of $0 \in \mathfrak{k}^*$. In the same way as [2], the orbit space of K-action on $\mu^{-1}(0)$ can be explicitly parametrized by a complex coordinate $\tau = t + \sqrt{-1}\xi_1 \in \mathbb{C}$.

4. Main results

In [5] we studied in detail and classified cohomogeneity one special Lagrangian submanifolds in T^*S^n under the group action of $SO(p) \times$

$SO(n+1-p)$ $(1 \leq p \leq n)$. In these cases, we give explicit descriptions of the special Lagrangian submanifolds in terms of ordinary differential equations and determine the diffeomorphism type of the principal orbits. These special Lagrangian submanifolds are generically smooth, but in some degenerate cases they are singular and we explicitly describe the form of the singularities. We observe the asymptotic behavior of the ends and singularities of special Lagrangian submanifolds in T^*S^n.

In this section, by generalizing the arguments of [5], we provide a construction of cohomogeneity one special Lagrangian submanifolds in T^*S^n under the group action induced by the isotropy representation of a Riemannian symmetric space G/K of rank 2.

4.1. **Case** $g = 1$.

4.1.1. $(S^1 \times SO(n+1), SO(n))$.

Theorem 4.1. *Let τ be a regular curve in the complex plane $\mathbb{C}$. Define a curve σ in $\mu^{-1}(0) \cap \Phi(\Sigma)$ by*

$$\sigma(s) = (\cos\tau(s), \sin\tau(s), 0, \ldots, 0).$$

Then the K-orbit $L = K \cdot \sigma$ through σ is a Lagrangian submanifold in Q^n. Moreover, the smooth part of L is a special Lagrangian submanifold of phase θ if and only if τ satisfies

$$\mathrm{Im}\left(e^{\sqrt{-1}\theta}\tau'(s)(\sin\tau(s))^{n-1}\right) = 0. \tag{4}$$

4.2. **Case** $g = 2$.

4.2.1. $(SO(p+1) \times SO(n+1-p), SO(p) \times SO(n+1-p))$.

Theorem 4.2. *Let τ be a regular curve in the complex plane $\mathbb{C}$. Define a curve σ in $\mu^{-1}(0) \cap \Phi(\Sigma)$ by*

$$\sigma(s) = (\cos\tau(s), 0, \ldots, 0, \sin\tau(s), 0, \ldots, 0).$$

Then the K-orbit $L = K \cdot \sigma$ through σ is a Lagrangian submanifold in Q^n. Moreover, the smooth part of L is a special Lagrangian submanifold of phase θ if and only if τ satisfies

$$\mathrm{Im}\left(e^{\sqrt{-1}\theta}\tau'(s)(\cos\tau(s))^{p-1}(\sin\tau(s))^{q-1}\right) = 0. \tag{5}$$

4.3. **Case** $g = 3$.

4.3.1. $(G,K) = (SU(3), SO(3))$. We consider the case of $(G,K) = (SU(3), SO(3))$. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K respectively. The canonical decomposition of $\mathfrak{g}$ is given by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where

$$\mathfrak{k} = \mathfrak{so}(3) \quad \text{and} \quad \mathfrak{p} = \left\{ \sqrt{-1}X \in M_3(\mathbb{R}) \mid {}^tX = X, \mathrm{Tr}X = 0 \right\}.$$

Then the isotropy representation of K is defined by $\mathrm{Ad}_{\mathfrak{p}}(k)X = kX{}^tk$ for $k \in SO(3)$ and $X \in \mathfrak{p}$. We define an inner product on $\mathfrak{p}$ by $\langle X, Y \rangle = -\mathrm{Tr}(XY)$ for $X, Y \in \mathfrak{p}$. Let

$$\mathfrak{a} = \left\{ \sqrt{-1} \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} \;\middle|\; \begin{array}{l} a_1 + a_2 + a_3 = 0, \\ a_1, a_2, a_3 \in \mathbb{R} \end{array} \right\}.$$

Then $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$. The group action of $K = SO(3)$ is naturally induced on the complex quadric Q^4 in $\mathfrak{p}^{\mathbb{C}} = \{Z \in M_3(\mathbb{C}) \mid {}^tZ = Z, \mathrm{Tr}Z = 0\}$.

Theorem 4.3. *Let τ be a regular curve in the complex plane $\mathbb{C}$. Define a curve σ in $\mu^{-1}(0) \cap \Phi(\Sigma)$ by*

$$\sigma(s) = \frac{1}{\sqrt{6}} \begin{pmatrix} 2\cos\tau(s) & & \\ & -\cos\tau(s) + \sqrt{3}\sin\tau(s) & \\ & & -\cos\tau(s) - \sqrt{3}\sin\tau(s) \end{pmatrix} \in \mathfrak{a}^{\mathbb{C}}.$$

Then the K-orbit $L = K \cdot \sigma$ through a curve σ is a cohomogeneity one Lagrangian submanifold under the group action of K in Q^4. Conversely, such a cohomogeneity one Lagrangian submanifold in Q^4 is obtained in this way. Moreover, L is a special Lagrangian submanifold of phase θ if and only if τ satisfies

$$\mathrm{Im}\left(e^{\sqrt{-1}\theta} \tau'(s) (3\cos^2\tau(s) - \sin^2\tau(s)) \sin\tau(s) \right) = 0. \tag{6}$$

4.3.2. $(G,K) = (SU(3) \times SU(3), SU(3))$. We consider the case of $(G,K) = (SU(3) \times SU(3), SU(3))$. The canonical decomposition of $\mathfrak{g}$ is given by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where

$$\mathfrak{k} = \{(X, X) \mid X \in \mathfrak{su}(3)\}$$

and

$$\mathfrak{p} = \{(X, -X) \mid X \in \mathfrak{su}(3)\} \cong \mathfrak{su}(3).$$

Since $\mathfrak{p}$ is linearly isomorphic to $\mathfrak{su}(3)$, we identify them. Then the linearly isotropy representation of K is defined by $\mathrm{Ad}_{\mathfrak{p}}(k)X = kX^tk$ for $k \in SU(3)$ and $X \in \mathfrak{p}$. We define an inner product on $\mathfrak{p}$ by $\langle X, Y\rangle = -\mathrm{Tr}(XY)$ for $X, Y \in \mathfrak{p} = \mathfrak{su}(3)$. Let

$$\mathfrak{a} = \left\{ \sqrt{-1} \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} \;\middle|\; \begin{array}{l} a_1 + a_2 + a_3 = 0, \\ a_1, a_2, a_3 \in \mathbb{R} \end{array} \right\}.$$

Then $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$. The group action of $K = SU(3)$ is naturally induced on Q^7 in $\mathfrak{p}^{\mathbb{C}} = \mathfrak{sl}(3, \mathbb{C})$.

Theorem 4.4. *Let τ be a regular curve in the complex plane $\mathbb{C}$. Define a curve σ in $\mu^{-1}(0) \cap \Phi(\Sigma)$ by*

$$\sigma(s) = \frac{1}{\sqrt{6}} \begin{pmatrix} 2\cos\tau(s) & & \\ & -\cos\tau(s) + \sqrt{3}\sin\tau(s) & \\ & & -\cos\tau(s) - \sqrt{3}\sin\tau(s) \end{pmatrix} \in \mathfrak{a}^{\mathbb{C}}.$$

Then the K-orbit $L = K \cdot \sigma$ through a curve σ is a cohomogeneity one Lagrangian submanifold under the group action of K in Q^7. Conversely, such a cohomogeneity one Lagrangian submanifold in Q^7 is obtained in this way. Moreover, L is a special Lagrangian submanifold of phase θ if and only if τ satisfies

$$\mathrm{Im}\left(e^{\sqrt{-1}\theta}\tau'(s)(3\cos^2\tau(s) - \sin^2\tau(s))^2(\sin\tau(s))^2\right) = 0. \tag{7}$$

4.3.3. $(G, K) = (SU(6), Sp(3))$. We consider the case of $(G, K) = (SU(6), Sp(3))$. The canonical decomposition of $\mathfrak{g}$ is given by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where

$$\mathfrak{k} = \mathfrak{su}(6)$$

and

$$\mathfrak{p} = \left\{ \begin{pmatrix} X & Y \\ \overline{Y} & -\overline{X} \end{pmatrix} \mid X \in \mathfrak{su}(n), Y \in \mathfrak{o}(n, \mathbb{C}) \right\}.$$

Then

$$\mathfrak{p}^{\mathbb{C}} = \left\{ \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \;\middle|\; V_{11}, V_{22} \in \mathfrak{sl}(n, \mathbb{C}), V_{12}, V_{21} \in \mathfrak{o}(n, \mathbb{C}) \right\}.$$

We define an inner product on $\mathfrak{p}$ by $\langle X, Y\rangle = -\mathrm{Tr}(XY)$ for $X, Y \in \mathfrak{p}$. Then the isotropy representation of K is defined by $\mathrm{Ad}_{\mathfrak{p}}(k)X = kX^tk$

for $k \in Sp(3)$ and $X \in \mathfrak{p}$. Let

$$\mathfrak{a} = \left\{ \sqrt{-1}\begin{pmatrix} H & O \\ O & H \end{pmatrix} \;\middle|\; H = \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix}, \begin{array}{l} a_1 + a_2 + a_3 = 0, \\ a_1, a_2, a_3 \in \mathbb{R} \end{array} \right\}.$$

Then $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$. The group action of $K = Sp(3)$ is naturally induced on Q^{14} in $\mathfrak{p}^{\mathbb{C}}$.

Theorem 4.5. *Let τ be a regular curve in the complex plane $\mathbb{C}$. Define a curve σ in $\mu^{-1}(0) \cap \Phi(\Sigma)$ by $\sigma(s) = \sqrt{-1}\begin{pmatrix} H & O \\ O & H \end{pmatrix} \in \mathfrak{a}^{\mathbb{C}}$, where*

$$H = \frac{1}{\sqrt{6}} \begin{pmatrix} 2\cos\tau(s) & & \\ & -\cos\tau(s) + \sqrt{3}\sin\tau(s) & \\ & & -\cos\tau(s) - \sqrt{3}\sin\tau(s) \end{pmatrix}.$$

Then the K-orbit $L = K \cdot \sigma$ through a curve σ is a cohomogeneity one Lagrangian submanifold under the group action of K in Q^{14}. Conversely, such a cohomogeneity one Lagrangian submanifold in Q^{14} is obtained in this way. Moreover, L is a special Lagrangian submanifold of phase θ if and only if τ satisfies

$$\text{(8)} \qquad \operatorname{Im}\left(e^{\sqrt{-1}\theta}\tau'(s)(3\cos^2\tau(s) - \sin^2\tau(s))^4(\sin\tau(s))^4\right) = 0.$$

4.4. **Case $g = 4$.**

4.4.1. $(G, K) = (SO(m+2), SO(2) \times SO(m))$. We consider the case of $(G, K) = (SO(m+2), SO(2) \times SO(m))$. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K respectively. The canonical decomposition of $\mathfrak{g}$ is given by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & O \\ O & B \end{pmatrix} \;\middle|\; A \in \mathfrak{o}(2), B \in \mathfrak{o}(m) \right\}$$

and

$$\mathfrak{p} = \left\{ \begin{pmatrix} O & X \\ -{}^tX & O \end{pmatrix} \;\middle|\; X \in M_{2,m}(\mathbb{R}) \right\}.$$

Since $\mathfrak{p}$ is linearly isomorphic to $M_{2,m}(\mathbb{R})$, we identify them. We define an inner product by $\langle X, Y \rangle = \operatorname{Tr}(X^t Y)$ for $X, Y \in M_{2,m}(\mathbb{R})$. Then the isotropy representation of K is defined by $\mathrm{Ad}_{\mathfrak{p}}(k)X = k_1 X k_2^{-1}$ for $k = \begin{pmatrix} k_1 & O \\ O & k_2 \end{pmatrix} \in SO(2) \times SO(m+1)$ and $X \in \mathfrak{p}$. We take a maximal

abelian subspace of $\mathfrak{p}$ as

$$\mathfrak{a} = \left\{ \begin{pmatrix} O & H \\ -{}^tH & O \end{pmatrix} \;\middle|\; H = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \end{pmatrix} \in M_{2,m}(\mathbb{R}) \right\}.$$

The group action of $K = SO(2) \times SO(m)$ is naturally induced on Q^{2m-1} in $\mathfrak{p}^{\mathbb{C}} \cong M_{2,m}(\mathbb{C})$.

Theorem 4.6. *Let τ be a regular curve in the complex plane $\mathbb{C}$. Define a curve σ in $\mu^{-1}(0) \cap \Phi(\Sigma)$ by $\sigma(s) = \begin{pmatrix} O & H \\ -{}^tH & O \end{pmatrix} \in \mathfrak{a}^{\mathbb{C}}$, where*

$$H = \begin{pmatrix} \cos\tau(s) & 0 & 0 & \cdots & 0 \\ 0 & \sin\tau(s) & 0 & \cdots & 0 \end{pmatrix}.$$

Then the K-orbit $L = K \cdot \sigma$ through σ is a cohomogeneity one Lagrangian submanifold under the group action of K in Q^{2m-1}. Conversely, such a cohomogeneity one Lagrangian submanifold in Q^{2m-1} is obtained in this way. Moreover, L is a special Lagrangian submanifold of phase θ if and only if τ satisfies

$$\mathrm{Im}\left(e^{\sqrt{-1}\theta}\tau'(s)\cos 2\tau(s)(\sin 2\tau(s))^{m-2}\right) = 0. \tag{9}$$

4.4.2. $(G, K) = (SU(m+2), S(U(2)\times U(m)))$. We consider the case of $(G, K) = (SU(m+2), S(U(2) \times U(m)))$. The canonical decomposition of $\mathfrak{g}$ is given by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & O \\ O & B \end{pmatrix} \;\middle|\; \begin{array}{l} A \in \mathfrak{u}(2), \\ B \in \mathfrak{u}(m) \end{array} \right\}$$

and

$$\mathfrak{p} = \left\{ \begin{pmatrix} O & X \\ -{}^t\overline{X} & O \end{pmatrix} \;\middle|\; X \in M_{2,m}(\mathbb{C}) \right\}.$$

Then

$$\mathfrak{p}^{\mathbb{C}} = \left\{ \begin{pmatrix} O & V \\ {}^tW & O \end{pmatrix} \;\middle|\; V, W \in M_{2,m}(\mathbb{C}) \right\}.$$

We define an inner product by $\langle X, Y \rangle = -\mathrm{Tr}(XY)$ for $X, Y \in \mathfrak{p}$. Then the isotropy representation of K is defined by $\mathrm{Ad}_{\mathfrak{p}}(k)X = k_1 X {}^t\overline{k_2}$ for $k = \begin{pmatrix} k_1 & O \\ O & k_2 \end{pmatrix} \in S(U(2) \times U(m))$ and $X \in \mathfrak{p}$. We take a maximal abelian subspace of $\mathfrak{p}$ as

$$\mathfrak{a} = \left\{ \begin{pmatrix} O & H \\ -{}^tH & O \end{pmatrix} \;\middle|\; H = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \end{pmatrix} \in M_{2,m}(\mathbb{R}) \right\}.$$

The group action of $K = S(U(2) \times U(m))$ is naturally induced on Q^{4m-1} in $\mathfrak{p}^{\mathbb{C}}$.

Theorem 4.7. *Let τ be a regular curve in the complex plane $\mathbb{C}$. Define a curve σ in $\mu^{-1}(0) \cap \Phi(\Sigma)$ by $\sigma(s) = \begin{pmatrix} O & H \\ -{}^tH & O \end{pmatrix} \in \mathfrak{a}^{\mathbb{C}}$, where*

$$H = \begin{pmatrix} \cos\tau(s) & 0 & 0 & \cdots & 0 \\ 0 & \sin\tau(s) & 0 & \cdots & 0 \end{pmatrix}.$$

Then the K-orbit $L = K \cdot \sigma$ through a curve σ is a cohomogeneity one Lagrangian submanifold under the group action of K in Q^{4m-1}. Moreover, L is a special Lagrangian submanifold of phase θ if and only if τ satisfies

$$\text{(10)} \qquad \operatorname{Im}\left(e^{\sqrt{-1}\theta}\tau'(s)(\cos 2\tau(s))^2(\sin 2\tau(s))^{2m-3}\right) = 0.$$

4.4.3. $(G, K) = (Sp(m+2), Sp(2) \times Sp(m))$. We consider the case of $(G, K) = (Sp(m+2), Sp(2) \times Sp(m))$. The canonical decomposition of $\mathfrak{g}$ is given by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where

$$\mathfrak{k} = \left\{ \begin{pmatrix} A_{11} & O & B_{11} & O \\ O & A_{22} & O & B_{22} \\ -\overline{B}_{11} & O & \overline{A}_{11} & O \\ O & -\overline{B}_{22} & O & \overline{A}_{22} \end{pmatrix} \;\middle|\; \begin{array}{l} A_{11} \in \mathfrak{u}(2), A_{22} \in \mathfrak{u}(m), \\ B_{11} \in M_2(\mathbb{C}), {}^tB_{11} = B_{11}, \\ B_{22} = M_m(\mathbb{C}), {}^tB_{22} = B_{22} \end{array} \right\}$$

and

$$\mathfrak{p} = \left\{ \begin{pmatrix} O & X_{12} & O & Y_{12} \\ -{}^t\overline{X}_{12} & O & -{}^tY_{12} & O \\ O & -\overline{Y}_{12} & O & \overline{X}_{12} \\ -{}^t\overline{Y}_{12} & O & -{}^tX_{12} & O \end{pmatrix} \;\middle|\; X_{12}, Y_{12} \in M_{2,m}(\mathbb{C}) \right\}.$$

Then

$$\mathfrak{p}^{\mathbb{C}} = \left\{ \begin{pmatrix} O & V_{12} & O & V_{14} \\ -{}^tW_{12} & O & {}^tV_{14} & O \\ O & -W_{14} & O & W_{12} \\ -{}^tW_{14} & O & -{}^tV_{12} & O \end{pmatrix} \;\middle|\; V_{12}, V_{14}, W_{12}, W_{14} \in M_{2,m}(\mathbb{C}) \right\}.$$

We define an inner product by $\langle X, Y \rangle = -\operatorname{Tr}(XY)$ for $X, Y \in \mathfrak{p}$. Then the isotropy representation of K is defined by $\mathrm{Ad}_{\mathfrak{p}}(k)X = k_1 X {}^t\overline{k_2}$ for

$k = \begin{pmatrix} k_1 & O \\ O & k_2 \end{pmatrix} \in Sp(2) \times Sp(m)$ and $X \in \mathfrak{p}$. Then

$$\mathfrak{a} = \left\{ \begin{pmatrix} O & H & O & O \\ -{}^tH & O & O & O \\ O & O & O & H \\ O & O & -{}^tH & O \end{pmatrix} \;\middle|\; H = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \end{pmatrix} \right\}$$
$$a_1, a_2 \in \mathbb{R}$$

is a maximal abelian subspace of $\mathfrak{p}$. The group action of $K = Sp(2) \times Sp(m)$ is naturally induced on Q^{8m-1} in $\mathfrak{p}^{\mathbb{C}}$.

Theorem 4.8. *Let τ be a regular curve in the complex plane $\mathbb{C}$. Define a curve σ in $\mu^{-1}(0) \cap \Phi(\Sigma)$ by*

$$\sigma(s) = \begin{pmatrix} O & H & O & O \\ -{}^tH & O & O & O \\ O & O & O & H \\ O & O & -{}^tH & O \end{pmatrix} \in \mathfrak{a}^{\mathbb{C}},$$

where

$$H = \begin{pmatrix} \cos\tau(s) & 0 & 0 & \cdots & 0 \\ 0 & \sin\tau(s) & 0 & \cdots & 0 \end{pmatrix}.$$

Then the K-orbit $L = K \cdot \sigma$ through a curve σ is a cohomogeneity one Lagrangian submanifold under the group action of K in Q^{8m-1}. Conversely, such a cohomogeneity one Lagrangian submanifold in Q^{8m-1} is obtained in this way. Moreover, L is a special Lagrangian submanifold of phase θ if and only if τ satisfies

$$\text{(11)} \qquad \operatorname{Im}\left(e^{\sqrt{-1}\theta}\tau'(s)(\cos 2\tau(s))^4(\sin 2\tau(s))^{4m-5}\right) = 0.$$

4.4.4. $(G, K) = (SO(5) \times SO(5),\ SO(5))$. We consider the case of $(G, K) = (SO(5) \times SO(5),\ SO(5))$. The canonical decomposition of $\mathfrak{g}$ is given by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where

$$\mathfrak{k} = \{(X, X) \mid X \in \mathfrak{o}(5)\} \cong \mathfrak{o}(5)$$

and

$$\mathfrak{p} = \{(X, -X) \mid X \in \mathfrak{o}(5)\}.$$

Then $\mathfrak{p}^{\mathbb{C}} = \mathfrak{o}(5, \mathbb{C})$. We use the inner product by $\langle X, Y \rangle = -\frac{1}{2}\mathrm{Tr}(XY)$ for $X,\ Y \in \mathfrak{p}$. Then the isotropy representation of K is defined by

$\mathrm{Ad}_{\mathfrak{p}}(k)X = kXk^{-1}$ for $k \in SO(5)$ and $X \in \mathfrak{p}$. Let

$$\mathfrak{a} = \left\{ (H, -H) \;\middle|\; H = \begin{pmatrix} 0 & a_1 & & & \\ -a_1 & 0 & & & \\ & & 0 & a_2 & \\ & & -a_2 & 0 & \\ & & & & 0 \end{pmatrix}, a_1, a_2 \in \mathbb{R} \right\}.$$

Then $\mathfrak{a}$ is a maximal abelian subalgebra in $\mathfrak{p}$. The group action of $K = SO(5)$ is naturally induced on Q^9 in $\mathfrak{p}^{\mathbb{C}}$.

Theorem 4.9. *Let τ be a regular curve in the complex plane $\mathbb{C}$. Define a curve σ in $\mu^{-1}(0) \cap \Phi(\Sigma)$ by $\sigma(s) = (H, -H) \in \mathfrak{a}^{\mathbb{C}}$, where*

$$H = \begin{pmatrix} 0 & \cos\tau(s) & & & \\ -\cos\tau(s) & 0 & & & \\ & & 0 & \sin\tau(s) & \\ & & -\sin\tau(s) & 0 & \\ & & & & 0 \end{pmatrix}.$$

Then the K-orbit $L = K \cdot \sigma$ through a curve σ is a cohomogeneity one Lagrangian submanifold under the group action of K in Q^9. Conversely, such a cohomogeneity one Lagrangian submanifold in Q^9 is obtained in the way. Moreover, L is a special Lagrangian submanifold of phase θ if and only if there exists a constant $c \in \mathbb{R}$ so that τ satisfies

$$\mathrm{Im}\left(e^{\sqrt{-1}\theta}\tau'(s)(\cos 2\tau(s))^2(\sin 2\tau(s))^2\right) = 0. \tag{12}$$

4.4.5. $(G, K) = (SO(10), U(5))$. We consider the case of $(G, K) = (SO(10), U(5))$. The canonical decomposition of $\mathfrak{g}$ is given by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \;\middle|\; {}^tA = -A, {}^tB = B \right\} \cong \mathfrak{u}(5)$$

and

$$\mathfrak{p} = \left\{ \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \;\middle|\; X, Y \in \mathfrak{so}(5) \right\}.$$

Then

$$\mathfrak{p}^{\mathbb{C}} = \left\{ \begin{pmatrix} V & W \\ W & -V \end{pmatrix} \in \mathfrak{o}(10, \mathbb{C}) \;\middle|\; V, W \in \mathfrak{o}(5, \mathbb{C}) \right\}.$$

We define an inner product by $\langle X, Y \rangle = -\frac{1}{2}\mathrm{Tr}(XY)$ for $X, Y \in \mathfrak{p}$. Then the isotropy representation of K is defined by $\mathrm{Ad}_{\mathfrak{p}}(k)X = kX{}^t\overline{k}$

for $k \in U(5)$ and $X \in \mathfrak{p}$. Then

$$\mathfrak{a} = \left\{ \begin{pmatrix} H & O \\ O & -H \end{pmatrix} \;\middle|\; H = \begin{pmatrix} 0 & a_1 & & & \\ -a_1 & 0 & & & \\ & & 0 & a_2 & \\ & & -a_2 & 0 & \\ & & & & 0 \end{pmatrix}, a_1, a_2 \in \mathbb{R} \right\}$$

is a maximal abelian subspace of $\mathfrak{p}$. The group action of $K = U(5)$ is naturally induced on Q^{19} in $\mathfrak{p}^{\mathbb{C}}$.

Theorem 4.10. *Let τ be a regular curve in the complex plane $\mathbb{C}$. Define a curve σ in $\mu^{-1}(0) \cap \Phi(\Sigma)$ by $\sigma(s) = \begin{pmatrix} H & O \\ O & -H \end{pmatrix} \in \mathfrak{a}^{\mathbb{C}}$, where*

$$H = \begin{pmatrix} 0 & \cos\tau(s) & & & \\ -\cos\tau(s) & 0 & & & \\ & & 0 & \sin\tau(s) & \\ & & -\sin\tau(s) & 0 & \\ & & & & 0 \end{pmatrix}.$$

Then the K-orbit $L = K \cdot \sigma$ through a curve σ is a cohomogeneity one Lagrangian submanifold under the group action of K in Q^{19}. Conversely, such a cohomogeneity one Lagrangian submanifold in Q^{19} is obtained in this way. Moreover, L is a special Lagrangian submanifold of phase θ if and only if τ satisfies

$$\text{Im}\left(e^{\sqrt{-1}\theta}\tau'(s)(\cos 2\tau(s))^4(\sin 2\tau(s))^5\right) = 0. \tag{13}$$

In the forthcoming paper we will study the remaining cases when G/K are of exceptional type.

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