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The hypersurface in the sphere

Yang Jiang
School of Mathematics and Statistics, Northeast Normal University

Abstract

We consider hypersurfaces in the unit lightlike sphere. The unit sphere can be canonically embedded in the lightcone and de Sitter space in Minkowski space. We investigate these hypersurfaces in the framework of the theory of Legendrian dualities between pseudo-spheres in Minkowski space. This is an anouncement of the results in [15]

1. Introduction

In [2, 3], professor Izumiya has introduced the mandala of Legendrian dualities between pseudo-spheres in Minkowski space. There are three kinds of pseudo-spheres in Minkowski space (i.e., Hyperbolic space, de Sitter space and the lightcone). Especially, if we investigate spacelike submanifolds in the lightcone, those Legendrian dualities are essentially useful (see, also [7]). For de Sitter space and the lightcone in Minkowski $(n+2)$-space, there exist naturally embedded unit n-spheres. Moreover, we have the canonical projection from the lightcone to the unit sphere embedded in the lightcone (cf., §2). In this paper we investigate hypersurfaces in the unit n-sphere in the framework of the theory of Legendrian dualities between pseudo-spheres in Minkowski $(n+2)$-space ([3, 4, 12, 13], etc.). If we have a hypersurface in the unit n-sphere, then we have spacelike hypersurfaces in the embedded unit n-sphere in the lightcone and de Sitter space. Therefore, we naturally have the dual hypersurfaces in the lightcone as an application of the duality theorem in [3]. There are two kinds of lightcone dual hypersurfaces of a hypersurface in the unit n-sphere. One is the dual of the hypersurface of the unit n-sphere embedded in de Sitter space and another is the dual of the hypersurface of the unit n-sphere embedded in the lightcone. By definition, these dual hypersurfaces are different.

On the other hand, we have studied the curves in the unit 2-sphere and the unit 3-sphere from the view point of the Legendrian duality in [5, 6]. In the unit 2-sphere, it is known that the evolute of a curve in the unit 2-sphere is the dual of the tangent indicatrix of the original curve [11]. We have shown that the projection images of the critical value sets of lightcone dual surfaces for a curve in the unit 2-sphere coincide with the evolute of the original curve in [5]. However, this fact doesn't hold for a curve in unit 3-sphere (cf., [6]). For the curve case, these facts has been shown by the direct calculations in [5, 6]. We have not known the geometric reason why the situations are different. In order to clarify these situation, we investigate hypersurfaces in the unit n-sphere from the view point of the theory of Legendrian singularities. The curves in the unit 2-sphere can be considered as a special case of this paper. We can also show that the projection images of the critical value sets of two different lightcone dual hypersurfaces for a hypersurface in the unit n-sphere also coincide with the spherical evolute (cf., [10]) of the original hypersurface. We interpret geometric meanings of the singularities of those two lightcone dual hypersurfaces. Here, we remark that we do not have the notion of tangent indicatrices for higher dimensional submanifolds in the sphere. Therefore, the situation is completely different from the curve case. In [15], we give a classification of the generic singularities of the lightcone duals of the surface in the unit 3-sphere.

All maps and submanifolds considered here are of class C^∞ unless otherwise stated.

2. The basic concepts

Let $\mathbb{R}^{n+2}$ be an $(n+2)$-dimensional vector space. For any two vectors $\boldsymbol{x} = (x_0, x_1, \dots, x_{n+1}), \boldsymbol{y} = (y_0, y_1, \dots, y_{n+1})$ in $\mathbb{R}^{n+2}$, their pseudo scalar product is defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + x_1 y_1 + \dots + x_{n+1} y_{n+1}$. Here, $(\mathbb{R}^{n+2}, \langle , \rangle)$ is called *Lorentz-Minkowski* $(n+2)$*-space* (simply, *Minkowski* $(n+2)$*-space*), which is denoted by $\mathbb{R}^{n+2}_1$. For any $(n+1)$ vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_{n+1} \in \mathbb{R}^{n+2}_1$, their pseudo vector product is defined by

$$\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \dots \wedge \boldsymbol{x}_{n+1} = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \cdots & \boldsymbol{e}_{n+1} \\ x_1^0 & x_1^1 & \cdots & x_1^{n+1} \\ x_2^0 & x_2^1 & \cdots & x_2^{n+1} \\ \vdots & \vdots & \cdots & \vdots \\ x_{n+1}^0 & x_{n+1}^1 & \cdots & x_{n+1}^{n+1} \end{vmatrix},$$

where $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \cdots, \boldsymbol{e}_{n+1}\}$ is the canonical basis of $\mathbb{R}^{n+2}_1$ and $\boldsymbol{x}_i = (x_i^0, x_i^1, \cdots, x_i^{n+1})$. A non-zero vector $\boldsymbol{x} \in \mathbb{R}^{n+2}_1$ is called spacelike, lightlike or timelike if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$ respectively. The norm of $\boldsymbol{x} \in \mathbb{R}^{n+2}_1$ is defined by $\| \boldsymbol{x} \| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. We define the *de Sitter* $(n+1)$*-space* by

$$S^{n+1}_1 = \{\boldsymbol{x} \in \mathbb{R}^{n+2}_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\}.$$

We define the *closed lightcone with the vertex* $\boldsymbol{a}$ by

$$LC_a = \{\boldsymbol{x} \in \mathbb{R}^{n+2}_1 \mid \langle \boldsymbol{x} - \boldsymbol{a}, \boldsymbol{x} - \boldsymbol{a} \rangle = 0\}.$$

We define the *open lightcone at the origin* by

$$LC^* = \{\boldsymbol{x} \in \mathbb{R}^{n+2}_1 \backslash \{\boldsymbol{0}\} \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0\}.$$

We consider a submanifold in the lightcone defined by $S^n_+ = \{\boldsymbol{x} \in LC^* \mid x_1 = 1\}$, which is called the *lightlike unit sphere*. We have a projection $\pi : LC^* \longrightarrow S^n_+$ defined by

$$\pi(\boldsymbol{x}) = \widetilde{\boldsymbol{x}} = \left(1, \frac{x_1}{x_0}, \dots, \frac{x_{n+1}}{x_0}\right),$$

where $\boldsymbol{x} = (x_0, x_1, \dots x_{n+1})$. We also define the *n-dimensional Euclidean unit sphere* in $\mathbb{R}^{n+1}_0$ by $S^n_0 = \{\boldsymbol{x} \in S^{n+1}_1 \mid x_0 = 0\}$, where $\mathbb{R}^{n+1}_0 = \{\boldsymbol{x} \in \mathbb{R}^{n+2}_1 \mid x_0 = 0\}$.

Let $\boldsymbol{x} : U \longrightarrow S^n_+$ be an embedding from an open set $U \subset \mathbb{R}^{n-1}$. We identify $M = \boldsymbol{x}(U)$ with U through the embedding $\boldsymbol{x}$. Obviously, the tangent space T_pM are all spacelike (i.e., consists only spacelike vectors), so M is a spacelike hypersuface in $S^n_+ \subset \mathbb{R}^{n+2}_1$. We have a map $\Phi : S^n_+ \to S^n_0$ defined by $\Phi(\boldsymbol{v}) = \boldsymbol{v} - \boldsymbol{e}_0$, which is an isometry. Then we have a hypersurface $\overline{\boldsymbol{x}} : U \to S^n_0$ defined by $\overline{\boldsymbol{x}}(\boldsymbol{u}) = \Phi(\boldsymbol{x}(\boldsymbol{u})) = \boldsymbol{x}(\boldsymbol{u}) - \boldsymbol{e}_0$, so that $\boldsymbol{x}$ and $\overline{\boldsymbol{x}}$ have the same geometric properties as spherical hypersurfaces. For any $p = \boldsymbol{x}(\boldsymbol{u})$, we can construct a unit normal vector $\boldsymbol{n}(\boldsymbol{u})$ as

$$\boldsymbol{n}(\boldsymbol{u}) = \frac{\overline{\boldsymbol{x}}(\boldsymbol{u}) \wedge \boldsymbol{e}_0 \wedge \boldsymbol{x}_{u_1}(\boldsymbol{u}) \wedge \ldots \wedge \boldsymbol{x}_{u_{n-1}}(\boldsymbol{u})}{\|\overline{\boldsymbol{x}}(\boldsymbol{u}) \wedge \boldsymbol{e}_0 \wedge \boldsymbol{x}_{u_1}(\boldsymbol{u}) \wedge \ldots \wedge \boldsymbol{x}_{u_{n-1}}(\boldsymbol{u})\|}.$$

We have $\langle \boldsymbol{n}(\boldsymbol{u}), \boldsymbol{n}(\boldsymbol{u})\rangle = 1$, $\langle \boldsymbol{e}_0, \boldsymbol{e}_0\rangle = -1$ and $\langle \boldsymbol{e}_0, \boldsymbol{n}\rangle = \langle \boldsymbol{n}, \boldsymbol{x}_{u_i}\rangle = \langle \boldsymbol{n}, \boldsymbol{x}\rangle = 0$. The system $\{\boldsymbol{e}_0, \boldsymbol{n}(\boldsymbol{u}), \overline{\boldsymbol{x}}(\boldsymbol{u}), \boldsymbol{x}_{u_1}(\boldsymbol{u}), \ldots, \boldsymbol{x}_{u_{n-1}}(\boldsymbol{u})\}$ is a basis of $T_p\mathbb{R}^{n+2}_1$. We define a map $G : U \longrightarrow S^n_0$ by $G(\boldsymbol{u}) = \boldsymbol{n}(\boldsymbol{u})$. We call it the *Gauss map* of the hypersurface $M = \boldsymbol{x}(U)$. We have a linear mapping provided by the derivation of the Gauss map at $p \in M$, $dG(\boldsymbol{u}) : T_pM \longrightarrow T_pM$. We call the linear transformation $S_p = -dG(\boldsymbol{u})$ the *shape operator* of M at $p = \boldsymbol{x}(\boldsymbol{u})$. The eigenvalues of S_p denoted by $\{\kappa_i(p)\}_{i=1}^{n-1}$ are called the *principal curvatures* of M at p. The *Gauss-Kronecker curvature* of M at p is defined to be $K(p) = \det S_p$. A point p is called an *umbilic point* if all the principal curvatures coincide at p and thus we have $S_p = \kappa(p)\mathrm{id}_{T_pM}$ for some $\kappa(p) \in \mathbb{R}$. We say that M is *totally umbilic* if all the points on M are umbilic. Since $\boldsymbol{x}$ is a spacelike embedding, we have a *Riemannian metric* (or the *first fundamental form*) on M given by $ds^2 = \sum_{i,j=1}^{n-1} g_{ij}du_idu_j$, where $g_{ij}(\boldsymbol{u}) = \langle \boldsymbol{x}_{u_i}(\boldsymbol{u}), \boldsymbol{x}_{u_j}(\boldsymbol{u})\rangle$ for any $\boldsymbol{u} \in U$. The second fundamental form on M is given by $h_{ij}(\boldsymbol{u}) = -\langle \boldsymbol{n}_{u_i}(\boldsymbol{u}), \boldsymbol{x}_{u_j}(\boldsymbol{u})\rangle$ at any $\boldsymbol{u} \in U$. Under the above notation, we have the following Weingarten formula [9]:

$$G_{u_i} = -\sum_{j=1}^{n-1} h_i^j \boldsymbol{x}_{u_j} (i = 1, \ldots, n-1),$$

where $(h_i^j) = (h_{ik})(g^{kj})$ and $(g^{kj}) = (g_{kj})^{-1}$. This formula induces an explicit expression of the Gauss-Kronecker curvature in terms of the Riemannian metric and the second fundamental invariant given by $K = det(h_{ij}/det(g_{\alpha\beta})$. A point p is a parabolic point if $K(p) = 0$. A point p is a *flat point* if it is an umbilic point and $K(p) = 0$.

In [10] the spherical evolute of a hypersurface has been introduced and investigated the singularities. Each *spherical evolute* of $\overline{M} = \overline{\boldsymbol{x}}(U)$ is defined to be

$$\boldsymbol{\varepsilon}^{\pm}_{\overline{M}} = \bigcup_{i=1}^{n-1} \left\{ \pm \left(\sqrt{\frac{\kappa_i^2(p)}{1+\kappa_i^2(p)}}\overline{\boldsymbol{x}}(\boldsymbol{u}) + \sqrt{\frac{1}{1+\kappa_i^2(p)}}\boldsymbol{n}(\boldsymbol{u}) \right) \Bigm| p = \boldsymbol{x}(\boldsymbol{u}) \in M = \boldsymbol{x}(U) \right\}.$$

3. The lightcone dual surfaces and the lightcone height functions

In [3], professor Izumiya has introduced the Legendrian dualities between pseudo-spheres in Minkowski space which is a basic tool for the study of hypersurfaces in pseudo-spheres in Minkowski space. We define one-forms $\langle d\boldsymbol{v}, \boldsymbol{w}\rangle = -w_0dv_0 + \sum_{i=1}^{n+1} w_idv_i$, $\langle \boldsymbol{v}, d\boldsymbol{w}\rangle = -v_0dw_0 + \sum_{i=1}^{n+1} v_idw_i$ on $\mathbb{R}^{n+2}_1 \times \mathbb{R}^{n+2}_1$ and consider the following two double fibrations:

(1)(a) $LC^* \times S^{n+1}_1 \supset \Delta_3 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = 1\}$,

(b) $\pi_{31} : \Delta_3 \longrightarrow LC^*, \pi_{32} : \Delta_3 \longrightarrow S^{n+1}_1$,

(c) $\theta_{31} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta_3, \theta_{32} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle|\Delta_3$.

(2)(a) $LC^* \times LC^* \supset \Delta_4 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = -2\}$,

(b) $\pi_{41} : \Delta_4 \longrightarrow LC^*, \pi_{42} : \Delta_4 \longrightarrow LC^*$,

(c) $\theta_{41} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta_4, \theta_{42} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle|\Delta_4$.

Here, $\pi_{i1}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{v}$, $\pi_{i2}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{w}$. We remark that $\theta_{i1}^{-1}(0)$ and $\theta_{i2}^{-1}(0)$ define the same tangent hyperplane field over Δ_i which is denoted by $K_i (i = 3, 4)$. It has been shown in [3] that each $(\Delta_i, K_i)(i = 3, 4)$ is a contact manifold and both of $\pi_{ij}(j = 1, 2)$ are Legendrian fibrations. Moreover those contact manifolds are contact diffeomorphic to each other. In [3] we have defined four double fibrations (Δ_i, K_i) $(i = 1, 2, 3, 4)$ such that these are contact diffeomorphic to each other. Here, we only use (Δ_3, K_3) and (Δ_4, K_4). If we have an isotropic mapping $i : L \to \Delta_i$ (i.e., $i^*\theta_{i1} = 0$), we say that $\pi_{i1}(i(L))$ and $\pi_{i2}(i(L))$ are Δ_i-*dual* to each other $(i = 3, 4)$. For detailed properties of Legendrian fibrations, see [1].

We now define hypersurfaces in LC^* associated with the hypersurfaces in S^n_+ or S^n_0. Let $\boldsymbol{x} : U \longrightarrow S^n_+$ be a hypersurface. We define $\overline{LD}^{\pm}_{\overline{M}} : U \times \mathbb{R} \longrightarrow LC^*$ by

$$\overline{LD}^{\pm}_{\overline{M}}(\boldsymbol{u}, \mu) = \overline{\boldsymbol{x}}(\boldsymbol{u}) + \mu\boldsymbol{n}(\boldsymbol{u}) \pm \sqrt{\mu^2 + 1}\boldsymbol{e}_0.$$

We also define $LD_M : U \times \mathbb{R} \longrightarrow LC^*$ by

$$LD_M(\boldsymbol{u}, \mu) = (\mu^2/4 - 1)\overline{\boldsymbol{x}}(\boldsymbol{u}) + \mu\boldsymbol{n}(\boldsymbol{u}) + (\mu^2/4 + 1)\boldsymbol{e}_0.$$

Then we have the following proposition.

Proposition 3.1. *Under the above notation, we have the followings:*
(1) $\overline{\boldsymbol{x}}$ *and* $\overline{LD}^{\pm}_{\overline{M}}$ *are* Δ_3*-dual to each other.*
(2) $\boldsymbol{x}$ *and* LD_M *are* Δ_4*-dual to each other.*

We call each one of $\overline{LD}^{\pm}_{\overline{M}}$ the *lightcone dual hypersurface along* $\overline{M} \subset S^n_0$ and LD_M the *lightcone dual hypersurface along* $M \subset S^n_+$. Then we have two mappings $\pi \circ \overline{LD}^{\pm}_{\overline{M}} : U \times \mathbb{R} \longrightarrow S^n_+$ and $\pi \circ LD_M : U \times \mathbb{R} \longrightarrow S^n_+$ defined by

$$\begin{aligned}
\pi \circ \overline{LD}^{\pm}_{\overline{M}}(\boldsymbol{u}, \mu) &= \pm\left(\frac{1}{\sqrt{\mu^2+1}}\overline{\boldsymbol{x}}(\boldsymbol{u}) + \frac{\mu}{\sqrt{\mu^2+1}}\boldsymbol{n}(\boldsymbol{u})\right) + \boldsymbol{e}_0, \\
\pi \circ LD_M(\boldsymbol{u}, \mu) &= \frac{\mu^2 - 4}{\mu^2 + 4}\overline{\boldsymbol{x}}(\boldsymbol{u}) + \frac{4\mu}{\mu^2+4}\boldsymbol{n}(\boldsymbol{u}) + \boldsymbol{e}_0.
\end{aligned}$$

Let $\boldsymbol{x} : U \longrightarrow S^n_+$ be a hypersurface in the lightlike unit sphere. Then we define two families of functions as follows:

$$\begin{aligned}
&\overline{H} : U \times LC^* \longrightarrow \mathbb{R}; \quad \overline{H}(\boldsymbol{u}, \overline{\boldsymbol{v}}) = \langle \overline{\boldsymbol{x}}(\boldsymbol{u}), \overline{\boldsymbol{v}} \rangle - 1, \\
&H : U \times LC^* \longrightarrow \mathbb{R}; \quad H(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{x}(\boldsymbol{u}), \boldsymbol{v} \rangle + 2.
\end{aligned}$$

We call $\overline{H}$ a *lightcone height function* of the de Sitter spherical hypersurface $\overline{M}$. For any fixed $\overline{\boldsymbol{v}}_0 \in LC^*$, we denote $\overline{h}_{\overline{v}_0}(\boldsymbol{u}) = \overline{H}(\boldsymbol{u}, \overline{\boldsymbol{v}}_0)$. We also call H a *lightcone height function* of the lightlike spherical hypersurface M. For any fixed $\boldsymbol{v}_0 \in LC^*$, we denote $h_{v_0}(\boldsymbol{u}) = H(\boldsymbol{u}, \boldsymbol{v}_0)$.

Proposition 3.2. *Let* $\overline{M}$ *be a hypersurface in* S^n_0 *and* $\overline{H}$ *the lightcone height function on* $\overline{M}$. *For* $p = \boldsymbol{x}(\boldsymbol{u})$ *and* $\overline{p} = \overline{\boldsymbol{x}}(\boldsymbol{u}) \neq \overline{\boldsymbol{v}}^{\pm}$, *we have the followings*:
(1) $\overline{h}_{\overline{v}^{\pm}}(\boldsymbol{u}) = \partial\overline{h}_{\overline{v}^{\pm}}/\partial u_i(\boldsymbol{u}) = 0 (i = 1, \dots, n-1)$ *if and only if*

$$\overline{\boldsymbol{v}}^{\pm} = \overline{LD}^{\pm}_{\overline{M}}(\boldsymbol{u}, \mu) \textit{ for some } \mu \in \mathbb{R}\backslash\{0\}.$$

(2) $\overline{h}_{\overline{v}^{\pm}}(\boldsymbol{u}) = \partial\overline{h}_{\overline{v}^{\pm}}/\partial u_i(\boldsymbol{u}) = 0 (i = 1, \dots, n-1)$ *and* det *Hess* $(\overline{h}_{\overline{v}^{\pm}})(\boldsymbol{u}) = 0$ *if and only if*

$$\overline{\boldsymbol{v}}^{\pm} = \overline{LD}^{\pm}_{\overline{M}}(\boldsymbol{u}, \mu), \ 1/\mu \textit{ is one of the non-zero principle curvatures } \kappa_i(p) \textit{ of } M.$$

Proposition 3.3. *Let M be a hypersurface in S^n_+ and H be the lightcone height function on M. For $p = \boldsymbol{x}(\boldsymbol{u}) \neq \boldsymbol{v}$, we have the followings.*
(1) $h_v(\boldsymbol{u}) = \partial h_v/\partial u_i(\boldsymbol{u}) = 0,\ (i = 1, \dots, n-1)$ *if and only if*

$$\boldsymbol{v} = LD_M(\boldsymbol{u}, \mu) \text{ for some } \mu \in \mathbb{R}\backslash\{0\}.$$

(2) $h_v(\boldsymbol{u}) = \partial h_v/\partial u_i(\boldsymbol{u}) = 0,\ (i = 1, \dots, n-1)$ *and* det *Hess* $(h_v)(\boldsymbol{u}) = 0$ *if and only if*

$$\boldsymbol{v} = LD_M(\boldsymbol{u}, \mu),\ (\mu/4 - 1/\mu) \text{ is one the non-zero principle curvatures } \kappa_i(p) \text{ of } M.$$

Let $(\boldsymbol{u}, \mu)$ be a singular point of each one of $\overline{LD}^{\pm}_{\overline{M}}$. By Proposition 3.2, we have $1/\mu = \kappa_i(p)(1 \leq i \leq n-1)$, where $\kappa_i(p)$ is one of the non-zero principle curvatures of M at $p = \boldsymbol{x}(\boldsymbol{u})$. It follows that $\mu = 1/\kappa_i(p)$. Therefore the critical value sets of $\overline{LD}^{\pm}_{\overline{M}}$ are given by

$$C(\overline{LD}^{\pm}_{\overline{M}}) = \bigcup_{i=1}^{n-1} \left\{ \overline{\boldsymbol{x}}(\boldsymbol{u}) + \frac{1}{\kappa_i(p)}\boldsymbol{n}(\boldsymbol{u}) \pm \sqrt{\frac{1}{\kappa_i^2(p)} + 1}\boldsymbol{e}_0 \middle| \ \boldsymbol{u} \in U \right\}.$$

Let $(\boldsymbol{u}, \mu)$ be a singular point of $LD_M(\boldsymbol{u}, \mu)$. By Proposition 3.3, we have $\mu/4 - 1/\mu = \kappa_i(p)(1 \leq i \leq n-1)$. It follows that we have $\mu = 2(\kappa_i(p) \pm \sqrt{1 + \kappa_i^2(p)})$. For simplification, we write that $\sigma^{\pm}(\kappa_i(p)) = \kappa_i(p) \pm \sqrt{1 + \kappa_i^2(p)}$. Then the critical value sets of LD_M are given by

$$C(LD_M)^{\pm} = \bigcup_{i=1}^{n-1} \{((\sigma^{\pm}(\kappa_i(p)))^2 - 1)\overline{\boldsymbol{x}}(\boldsymbol{u}) + 2\sigma^{\pm}(\kappa_i(p))\boldsymbol{n}(\boldsymbol{u}) + ((\sigma^{\pm}(\kappa_i(p)))^2 + 1)\boldsymbol{e}_0 \mid \boldsymbol{u} \in U \}.$$

We respectively denote that

$$LF^{\pm}_{\overline{M}} = \bigcup_{i=1}^{n-1} \left\{ \overline{\boldsymbol{x}}(\boldsymbol{u}) + \frac{1}{\kappa_i(p)}\boldsymbol{n}(\boldsymbol{u}) \pm \sqrt{\frac{1}{\kappa_i^2(p)} + 1}\boldsymbol{e}_0 \middle| \ \boldsymbol{u} \in U \right\},$$

$$LF^{\pm}_{M} = \bigcup_{i=1}^{n-1} \{((\sigma^{\pm}(\kappa_i(p)))^2 - 1)\overline{\boldsymbol{x}}(\boldsymbol{u}) + 2\sigma^{\pm}(\kappa_i(p))\boldsymbol{n}(\boldsymbol{u}) + ((\sigma^{\pm}(\kappa_i(p)))^2 + 1)\boldsymbol{e}_0 \mid \boldsymbol{u} \in U \}.$$

We respectively call each one of $LF^{\pm}_{\overline{M}}$ the *lightcone focal surface* of the de Sitter spherical hypersurface $\overline{\boldsymbol{x}}$ and each one of $LF^{\pm}_{M}$ the *ligtcone focal surface* of the lightcone spherical hypersurface $\boldsymbol{x}$. Then the projections of these surfaces to S^n_+ are given as follows:

$$\pi(C(\overline{LD}^{\pm}_{\overline{M}})) = \bigcup_{i=1}^{n-1} \left\{ \pm \left(\sqrt{\frac{\kappa_i^2(p)}{1 + \kappa_i^2(p)}}\overline{\boldsymbol{x}}(\boldsymbol{u}) + \sqrt{\frac{1}{1 + \kappa_i^2(p)}}\boldsymbol{n}(\boldsymbol{u}) \right) + \boldsymbol{e}_0 \middle| \ \boldsymbol{u} \in U \right\},$$

$$\pi(C(LD_M)^{\pm}) = \bigcup_{i=1}^{n-1} \left\{ \frac{(\sigma^{\pm}(\kappa_i(p)))^2 - 1}{(\sigma^{\pm}(\kappa_i(p)))^2 + 1}\overline{\boldsymbol{x}}(\boldsymbol{u}) + \frac{2\sigma^{\pm}(\kappa_i(p))}{(\sigma^{\pm}(\kappa_i(p)))^2 + 1}\boldsymbol{n}(\boldsymbol{u}) + \boldsymbol{e}_0 \mid \boldsymbol{u} \in U \right\}.$$

By definition, we have $\boldsymbol{\varepsilon}^{\pm}_{\overline{M}} = \Phi \circ \pi(C(\overline{LD}^{\pm}_{\overline{M}}))$, where each one of $\boldsymbol{\varepsilon}^{\pm}_{\overline{M}}$ is the spherical evolute of $\overline{M} = \overline{\boldsymbol{x}}(U)$. This means that the spherical evolutes are obtained from the critical value sets of the lightcone dual hypersurfaces of $\overline{M} = \overline{\boldsymbol{x}}(U)$. Since $\sigma^{\pm}(\kappa_i(p)) = \kappa_i(p) \pm \sqrt{1+\kappa_i^2(p)}$, we have$(\sigma^{\pm}(\kappa_i(p)))^2 = 2\kappa_i(p)\sigma^{\pm}(\kappa_i(p)) + 1$. By straightforward calculations, we have

$$\left(\frac{(\sigma^{\pm}(\kappa_i(p)))^2 - 1}{(\sigma^{\pm}(\kappa_i(p)))^2 + 1}\right)^2 = \frac{\kappa_i^2(p)(\sigma^{\pm}(\kappa_i(p)))^2}{\kappa_i^2(p)(\sigma^{\pm}(\kappa_i(p)))^2 + (\sigma^{\pm}(\kappa_i(p)))^2} = \frac{\kappa_i^2(p)}{1+\kappa_i^2(p)}$$

and

$$\left(\frac{2\sigma^{\pm}(\kappa_i(p))}{(\sigma^{\pm}(\kappa_i(p)))^2 + 1}\right)^2 = \frac{(\sigma^{\pm}(\kappa_i(p)))^2}{\kappa_i^2(p)(\sigma^{\pm}(\kappa_i(p)))^2 + (\sigma^{\pm}(\kappa_i(p)))^2} = \frac{1}{1+\kappa_i^2(p)}.$$

Thus we have the following proposition.

Proposition 3.4. *Let $\boldsymbol{x} : U \longrightarrow S^n_+$ be a hypersurface in S^n_+. Then*

$$\frac{(\sigma^{\pm}(\kappa_i(p)))^2 - 1}{(\sigma^{\pm}(\kappa_i(p)))^2 + 1}\overline{\boldsymbol{x}}(\boldsymbol{u}) + \frac{2\sigma^{\pm}(\kappa_i(p))}{(\sigma^{\pm}(\kappa_i(p)))^2 + 1}\boldsymbol{n}(\boldsymbol{u}) = \pm\left(\sqrt{\frac{\kappa_i^2(p)}{1+\kappa_i^2(p)}}\overline{\boldsymbol{x}}(\boldsymbol{u}) + \sqrt{\frac{1}{1+\kappa_i^2(p)}}\boldsymbol{n}(\boldsymbol{u})\right).$$

We define $\tilde{\pi} = \Phi \circ \pi : LC^* \longrightarrow S^n_0$. Then we have the following theorem as a corollary of Proposition 3.4.

Theorem 3.5. *Both of the projections of the critical value sets $C(\overline{LD^{\pm}_{M}})$ and $C(LD_M)^{\pm}$ in the unit sphere S^n_0 are the images of the spherical evolutes of $\overline{M}$.*

$$\tilde{\pi}(C(\overline{LD^{\pm}_{M}})) = \tilde{\pi}(C(LD_M)^{\pm}) = \boldsymbol{\varepsilon}^{\pm}_{\overline{M}}.$$

4. The lightcone dual hypersurfaces as wave fronts

We now naturally interpret the lightcone dual hypersurfaces of the submanifolds in S^n_+ as wave front sets in the theory of Legendrian singularities. Let $\overline{\pi} : PT^*(LC^*) \longrightarrow LC^*$ be the projective cotangent bundles with canonical contact structures. Consider the tangent bundle $\tau : TPT^*(LC^*) \longrightarrow PT^*(LC^*)$ and the differential map $d\overline{\pi} : TPT^*(LC^*) \longrightarrow T(LC^*)$ of $\overline{\pi}$. For any $X \in TPT^*(LC^*)$, there exists an element $\alpha \in T^*(LC^*)$ such that $\tau(X) = [\alpha]$. For an element $V \in T_{\boldsymbol{v}}(LC^*)$, the property $\alpha(V) = 0$ dose not depend on the choice of representative of the class $[\alpha]$. Thus we have the canonical contact structure on $PT^*(LC^*)$ by

$$K = \{X \in TPT^*(LC^*) \mid \tau(X)(d\overline{\pi}(X))\} = 0.$$

On the other hand, we consider a point $\boldsymbol{v} = (v_0, v_1, \ldots, v_{n+1}) \in LC^*$, then we have $v_0 = \pm\sqrt{v_1^2 + \ldots + v_{n+1}^2}$. So we adopt the coordinate system $(v_1, \ldots, v_{n+1})$ of LC^*. For the local coordinate neighborhood $(U, (\pm\sqrt{v_1^2 + \ldots + v_{n+1}^2}, v_1, \ldots, v_{n+1}))$ in LC^*, we have a trivialization $PT^*(LC^*) \equiv LC^* \times P(\mathbb{R}^n)^*$ and we call $((\pm\sqrt{v_1^2 + \ldots + v_{n+1}^2}, v_1, \ldots, v_{n+1}), [\xi_1 : \cdots : \xi_{n+1}])$ homogeneous coordinates of $PT^*(LC^*)$, where $[\xi_1 : \cdots : \xi_{n+1}]$ are the homogeneous coordinates of the dual projective space $P(\mathbb{R}^n)^*$. It is easy to show that $X \in K_{(\boldsymbol{v},[\xi])}$ if and only if $\sum_1^{n+1} \mu_i \xi_i = 0$, where $d\overline{\pi}(X) = \sum_1^{n+1} \mu_i \partial/\partial v_i \in T_{\boldsymbol{v}}LC^*$. An immersion $i : L \longrightarrow PT^*(LC^*)$ is said to be a Legendrian immersion if dim $L = n$ and $di_q(T_qL) \subset K_{i(q)}$ for any $q \in L$. The map $\overline{\pi} \circ i$ is also called the Legendrian map and we call the set $W(i)$=image$\overline{\pi} \circ i$ the wave front of i. Moreover, i(or the image of i) is called the Legendrian lift of $W(i)$. Let $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$ be a function germ. We say that F is a Morse family of hypersurfaces if the map germ $\Delta^* F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^{k+1}, 0)$ defined by $\Delta^* F = (F, \partial F/\partial u_1, \cdots, \partial F/\partial u_k)$. is nonsingular. In this case, we have the following smooth $(n-1)$-dimensional smooth submanifold.

$$\Sigma_*(F) = \left\{ (\boldsymbol{u}, \boldsymbol{v}) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \mid F(\boldsymbol{u}, \boldsymbol{v}) = \frac{\partial F}{\partial u_1}(\boldsymbol{u}, \boldsymbol{v}) = \cdots = \frac{\partial F}{\partial u_k}(\boldsymbol{u}, \boldsymbol{v}) = 0 \right\} = (\Delta^* F)^{-1}(\mathbf{0}).$$

The map germ $\mathcal{L}_F : (\Sigma_*(F), \mathbf{0}) \longrightarrow PT^*\mathbb{R}^n$ defined by

$$\mathcal{L}_F(\boldsymbol{u}, \boldsymbol{v}) = \left(\boldsymbol{v}, \left[\frac{\partial F}{\partial v_1}(\boldsymbol{u}, \boldsymbol{v}) : \ldots : \frac{\partial F}{\partial v_n}(\boldsymbol{u}, \boldsymbol{v}) \right] \right).$$

is a Legendrian immersion germ. Then we have the following fundamental theorem of Arnol'd and Zakalyukin [1, 14].

Proposition 4.1. *All Legendrian submanifold germs in $PT^*\mathbb{R}^n$ are constructed by the above method.*

We call F a generating family of $\mathcal{L}_F(\Sigma_*(F))$. Therefore the wave front of $\mathcal{L}_F$ is

$$W(\mathcal{L}_F) = \left\{ \boldsymbol{v} \in \mathbb{R}^n \mid \exists \boldsymbol{u} \in \mathbb{R}^k \text{ such that } F(\boldsymbol{u}, \boldsymbol{v}) = \frac{\partial F}{\partial u_1}(\boldsymbol{u}, \boldsymbol{v}) = \ldots = \frac{\partial F}{\partial u_k}(\boldsymbol{u}, \boldsymbol{v}) = 0 \right\}.$$

We claim here that we have a trivialization as follows:

$$\Phi : PT^*(LC^*) \equiv LC^* \times P(\mathbb{R}^n)^*; \Phi([\sum_{i=1}^{n+1} \xi_i dv_i]) = (v_0, v_1, \cdots, v_{n+1}), [\xi_1 : \cdots \xi_{n+1}])$$

by using the above coordinate system.

Proposition 4.2. *The lightcone height function $H : U \times LC^* \longrightarrow \mathbb{R}$ is a Morse family of the hypersurface around $(\boldsymbol{u}, \boldsymbol{v}) \in \Sigma_*(H)$.*

We also have the following proposition.

Proposition 4.3. *The lightcone height function $\overline{H} : U \times LC^* \longrightarrow \mathbb{R}$ is a Morse family of the hypersurface around $(\boldsymbol{u}, \boldsymbol{v}) \in \Sigma_*(\overline{H})$.*

Here, we consider the Legendrian immersion

$$\mathcal{L}_4 : (\boldsymbol{u}, \mu) \longrightarrow \Delta_4;\ \mathcal{L}_4(\boldsymbol{u}, \mu) = (LD_M(\boldsymbol{u}, \mu), \boldsymbol{x}(\boldsymbol{u})).$$

We define the following mapping:

$$\Psi : \Delta_4 \longrightarrow LC^* \times P(\mathbb{R}^n)^*; \Psi(\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{v}, [v_0 w_1 - v_1 w_0 : \cdots : v_0 w_{n+1} - v_{n+1} w_0]).$$

For the canonical contact form $\theta = \sum_{i=1}^{n+1} \xi_i dv_i$ on $PT^*(LC^*)$, we have $\Psi^*\theta = (v_0 w_1 - v_1 w_0) dv_1 + \cdots + (v_0 w_{n+1} - v_{n+1} w_0) dv_{n+1}|_{\Delta_4} = v_0(-w_0 dv_0 + w_1 dv_1 + \cdots + w_{n+1} dv_{n+1}) - w_0(-v_0 dv_0 + v_1 dv_1 + \cdots + v_{n+1} dv_{n+1})|_{\Delta_4} = v_0 \langle \boldsymbol{w}, d\boldsymbol{v} \rangle|_{\Delta_4} = v_0 \theta_{42}|_{\Delta_4}$. Thus Ψ is a contact morphism.

Theorem 4.4. *For any hypersurface $\boldsymbol{x} : U \longrightarrow S^n_+$, the lightcone height function $H : U \times LC^* \longrightarrow \mathbb{R}$ is a generating family of the Legendrian immersion $\mathcal{L}_4$.*

Similarly, we consider the Legendrian immersions $\mathcal{L}_3^{\pm} : (\boldsymbol{u}, \mu) \longrightarrow \Delta_3$ defined by $\mathcal{L}_3^{\pm}(\boldsymbol{u}, \mu) = (\overline{LD}^{\pm}_M(\boldsymbol{u}, \mu), \overline{\boldsymbol{x}}(\boldsymbol{u}))$. Then we have the following theorem.

Theorem 4.5. *For any hypersurface $\overline{\boldsymbol{x}} : U \longrightarrow S^n_0$, the lightcone height function $\overline{H} : U \times LC^* \longrightarrow \mathbb{R}$ is a generating family of the Legendrian immersions $\mathcal{L}_3^{\pm}$.*

5. Contact with parabolic $(n-1)$-spheres and parabolic n-hyperquadrics

Before we start to consider the contact between hypersurfaces in the sphere with parabolic $(n-1)$-sphere and parabolic n-hyperquadrics, we briefly review the theory of contact due to Montaldi[8]. Let $X_i, Y_i (i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with dim X_1=dim X_2 and dimY_1=dim Y_2. We say that the contact of X_1 and Y_1 at y_1 is *the same type as the contact of X_2 and Y_2 at y_2* if there is a diffeomorphism $\Phi : (\mathbb{R}^n, y_1) \longrightarrow (\mathbb{R}^n, y_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1) = Y_2$. In this case, we write $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$. Of course, in the definition, $\mathbb{R}^n$ can be replaced by any manifold. Two function germs $f_i : (\mathbb{R}^n, a_i) \longrightarrow \mathbb{R} (i = 1, 2)$ are called $\mathcal{K}$-equivalent if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, a_1) \longrightarrow (\mathbb{R}^n, a_2)$, and a function germ $\lambda : (\mathbb{R}^n, a_1) \longrightarrow \mathbb{R}$ with $\lambda(a_1) \neq 0$ such that $f_1 = \lambda \cdot (f_2 \circ \Phi)$.

Theorem 5.1 (Montaldi [8]). *Let X_i, Y_i (for i=1,2) be submanifolds of $\mathbb{R}^n$ with dimX_1=dimX_2 and dimY_1=dimY_2. Let $g_i : (X_i, x_i) \longrightarrow (\mathbb{R}^n, y_i)$ be immersion germs and $f_i : (\mathbb{R}^n, y_i) \longrightarrow (\mathbb{R}^p, 0)$ be submersion germs with $(Y_i, y_i) = (f_i^{-1}(0), y_i)$. Then $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$ if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathcal{K}$-equivalent.*

Returning to the lightcone dual hypersurface LD_M, we now consider the function $\mathfrak{h} : S^n_+ \times LC^* \longrightarrow \mathbb{R}$ defined by $\mathfrak{h}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{u}, \boldsymbol{v} \rangle + 2$ and the function $\mathfrak{g} : LC^* \times LC^* \longrightarrow \mathbb{R}$ defined by $\mathfrak{g}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{u}, \boldsymbol{v} \rangle + 2$. For a given $\boldsymbol{v}_0 \in LC^*$, we denote $\mathfrak{h}_{v_0}(\boldsymbol{u}) = \mathfrak{h}(\boldsymbol{u}, \boldsymbol{v}_0)$ and $\mathfrak{g}_{v_0}(\boldsymbol{u}) = \mathfrak{g}(\boldsymbol{u}, \boldsymbol{v}_0)$, then we have $\mathfrak{h}_{v_0}^{-1}(0) = S^n_+ \cap HP(\boldsymbol{v}_0, -2)$ and $\mathfrak{g}_{v_0}^{-1}(0) = LC^* \cap HP(\boldsymbol{v}_0, -2)$. For any $\boldsymbol{u}_0 \in U$, $\mu_0 \in \mathbb{R}$, we take the point $\boldsymbol{v}_0 = LD_M(\boldsymbol{u}_0, \mu_0)$. Then we have

$$\mathfrak{g}_{v_0} \circ \boldsymbol{x}(\boldsymbol{u}_0) = \mathfrak{g} \circ (\boldsymbol{x} \times id_{LC^*})(\boldsymbol{u}_0, \boldsymbol{v}_0) = \mathfrak{h}_{v_0} \circ \boldsymbol{x}(\boldsymbol{u}_0) = \mathfrak{h} \circ (\boldsymbol{x} \times id_{LC^*})(\boldsymbol{u}_0, \boldsymbol{v}_0) = H(\boldsymbol{u}_0, \boldsymbol{v}_0) = 0.$$

We also have

$$\frac{\partial(\mathfrak{g}_{v_0} \circ \boldsymbol{x})}{\partial u_i}(\boldsymbol{u}_0) = \frac{\partial(\mathfrak{h}_{v_0} \circ \boldsymbol{x})}{\partial u_i}(\boldsymbol{u}_0) = \frac{\partial H}{\partial u_i}(\boldsymbol{u}_0, \boldsymbol{v}_0) = 0$$

for $i = 1, \cdots, n-1$. This means that the $(n-1)$-sphere $\mathfrak{h}_{v_0}^{-1}(0) = S^n_+ \cap HP(\boldsymbol{v}_0, -2)$ is tangent to $M = \boldsymbol{x}(U)$ at $p_0 = \boldsymbol{x}(\boldsymbol{u}_0)$. In this case, we call it the *lightcone tangent parabolic $(n-1)$-sphere* of M at p_0, which is denoted by $TPS^{n-1}_+(\boldsymbol{x}, \boldsymbol{u}_0)$. The n-hyperquadric $\mathfrak{g}_{v_0}^{-1}(0) = LC^* \cap HP(\boldsymbol{v}_0, -2)$ is also tangent to M at p_0. In this case, we call it the *lightcone tangent parabolic n-hyperquadric* of M at p_0, which is denoted by $TPH^n(\boldsymbol{x}, \boldsymbol{u}_0)$. For the lightcone dual surfaces $\overline{LD}^{\pm}_{\overline{M}}$, we consider a function $\overline{\mathfrak{h}} : S^n_0 \times LC^* \longrightarrow \mathbb{R}$ defined by $\overline{\mathfrak{h}}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{u}, \boldsymbol{v} \rangle - 1$ and a function $\overline{\mathfrak{g}} : S^{n+1}_1 \times LC^* \longrightarrow \mathbb{R}$ defined by $\overline{\mathfrak{g}}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{u}, \boldsymbol{v} \rangle - 1$. For a given $\boldsymbol{v}_0 \in LC^*$, we denote that $\overline{\mathfrak{h}}_{v_0}(\boldsymbol{u}) = \overline{\mathfrak{h}}(\boldsymbol{u}, \boldsymbol{v}_0)$ and $\overline{\mathfrak{g}}_{v_0}(\boldsymbol{u}) = \overline{\mathfrak{g}}(\boldsymbol{u}, \boldsymbol{v}_0)$. Then we have $\overline{\mathfrak{h}}_{v_0}^{-1}(0) = S^n_0 \cap HP(\boldsymbol{v}_0, 1)$ and $\overline{\mathfrak{g}}_{v_0}^{-1}(0) = S^{n+1}_1 \cap HP(\boldsymbol{v}_0, 1)$. For any $\boldsymbol{u}_0 \in U$ and the points $\overline{\boldsymbol{v}}^{\pm}_0 = \overline{LD}^{\pm}_{\overline{M}}(\boldsymbol{u}_0, \mu_0)$, we have

$$\overline{\mathfrak{g}}_{\overline{v}^{\pm}_0} \circ \overline{\boldsymbol{x}}(\boldsymbol{u}_0) = \overline{\mathfrak{g}} \circ (\overline{\boldsymbol{x}} \times id_{LC^*})(\boldsymbol{u}_0, \overline{\boldsymbol{v}}^{\pm}_0) = \overline{\mathfrak{h}}_{\overline{v}^{\pm}_0} \circ \overline{\boldsymbol{x}}(\boldsymbol{u}_0) = \overline{\mathfrak{h}} \circ (\overline{\boldsymbol{x}} \times id_{LC^*})(\boldsymbol{u}_0, \overline{\boldsymbol{v}}^{\pm}_0) = \overline{H}(\boldsymbol{u}_0, \overline{\boldsymbol{v}}^{\pm}_0) = 0.$$

We also have

$$\frac{\partial(\overline{\mathfrak{g}}_{\overline{v}^{\pm}_0} \circ \overline{\boldsymbol{x}})}{\partial u_i}(\boldsymbol{u}_0) = \frac{\partial(\overline{\mathfrak{h}}_{\overline{v}^{\pm}_0} \circ \overline{\boldsymbol{x}})}{\partial u_i}(\boldsymbol{u}_0) = \frac{\partial \overline{H}}{\partial u_i}(\boldsymbol{u}_0, \overline{\boldsymbol{v}}^{\pm}_0) = 0$$

for $i = 1, \cdots, n-1$. It follows that each one of the $(n-1)$-sphere $\overline{\mathfrak{h}}_{\overline{v}^{\pm}_0}^{-1}(0) = S^n_0 \cap HP(\overline{\boldsymbol{v}}^{\pm}_0, 1)$ is tangent to $\overline{M}$ at $\overline{p}_0 = \overline{\boldsymbol{x}}(\boldsymbol{u}_0)$. In this case, we call each one the tangent parabolic $(n-1)$-sphere of $\overline{M}$ at $\overline{p}_0$, which are denoted by $TPS^{n-1\pm}_0(\boldsymbol{x}, \boldsymbol{u}_0)$. Also we have each of the n-hyperquadric $\overline{\mathfrak{g}}^{-1}_{\overline{v}_0^\pm}(0) = S^{n+1}_1 \cap HP(\overline{v}_0^\pm, 1)$ is tangent to $\overline{M}$ at $\overline{p}_0$. In this case, we call each one the de-Sitter tangent parabolic n-hyperquadric of $\overline{M}$ at $\overline{p}_0$, which are denoted by $TPS^{n\pm}_1(\overline{\boldsymbol{x}}, \boldsymbol{u}_0)$.

Let $\boldsymbol{x}_i : (U, u_i) \longrightarrow (S^n_+, p_i)(i = 1, 2)$ be hypersurface germs. For $\boldsymbol{v}_i = LD_{M_i}(\boldsymbol{u}_i, \mu_i)$, we denote $h_{i,v_i} : (U, \boldsymbol{u}_i) \longrightarrow (\mathbb{R}, 0)$ by $h_{i,v_i}(\boldsymbol{u}_i) = H(\boldsymbol{u}_i, \boldsymbol{v}_i)$. Then we have $h_{i,v_i}(\boldsymbol{u}) = (\mathfrak{h}_{i,v_i} \circ \boldsymbol{x}_i)(\boldsymbol{u}) = (\mathfrak{g}_{i,v_i} \circ \boldsymbol{x}_i)(\boldsymbol{u})$. For $\overline{\boldsymbol{v}}_i^\pm = \overline{LD}^\pm_{\overline{M}_i}(\boldsymbol{u}_i, \mu_i)$, We denote $\overline{h}_{i,\overline{v}_i^\pm} : (U, \boldsymbol{u}_i) \longrightarrow (\mathbb{R}, 0)$ by $\overline{h}_{i,\overline{v}_i^\pm}(\boldsymbol{u}_i) = \overline{H}(\boldsymbol{u}_i, \overline{\boldsymbol{v}}_i^\pm)$. Then we have $\overline{h}_{i,\overline{v}_i^\pm}(\boldsymbol{u}) = (\overline{\mathfrak{h}}_{i,\overline{v}_i^\pm} \circ \overline{\boldsymbol{x}}_i)(\boldsymbol{u}) = (\overline{\mathfrak{g}}_{i,\overline{v}_i^\pm} \circ \overline{\boldsymbol{x}}_i)(\boldsymbol{u})$. By Theorem 5.1, we have the following proposition.

Proposition 5.2. *Let $\boldsymbol{x}_i : (U, u_i) \longrightarrow (S^n_+, p_i)(i = 1, 2)$ be hypersurface germs. For $\boldsymbol{v}_i = LD_{M_i}(\boldsymbol{u}_i, \mu_i)$, the following conditions are equivalent:*
(1) $K(\boldsymbol{x}_1(U), TPS^{n-1}_+(\boldsymbol{x}_1, \boldsymbol{u}_1), \boldsymbol{v}_1) = K(\boldsymbol{x}_2(U), TPS^{n-1}_+(\boldsymbol{x}_2, \boldsymbol{u}_2), \boldsymbol{v}_2)$.
(2) $K(\boldsymbol{x}_1(U), TPH^n(\boldsymbol{x}_1, \boldsymbol{u}_1), \boldsymbol{v}_1) = K(\boldsymbol{x}_2(U), TPH^n(\boldsymbol{x}_2, \boldsymbol{u}_2), \boldsymbol{v}_2)$.
(3) h_{1,v_1} *and* h_{2,v_2} *are $\mathcal{K}$-equivalent.*
Moreover, for $\overline{\boldsymbol{v}}_i^\pm = \overline{LD}^\pm_{\overline{M}_i}(\boldsymbol{u}_i, \mu_i)$, the following conditions are equivalent:
(4) $K(\boldsymbol{x}_1(U), TPS^{n-1\pm}_0(\boldsymbol{x}_1, \boldsymbol{u}_1), \overline{\boldsymbol{v}}_1^\pm) = K(\boldsymbol{x}_2(U), TPS^{n-1\pm}_0(\boldsymbol{x}_2, \boldsymbol{u}_2), \overline{\boldsymbol{v}}_2^\pm)$.
(5) $K(\boldsymbol{x}_1(U), TPS^{n\pm}_1(\boldsymbol{x}_1, \boldsymbol{u}_1), \overline{\boldsymbol{v}}_1^\pm) = K(\boldsymbol{x}_2(U), TPS^{n\pm}_1(\boldsymbol{x}_2, \boldsymbol{u}_2), \overline{\boldsymbol{v}}_2^\pm)$.
(6) $\overline{h}_{1,\overline{v}_1^\pm}$ *and* $\overline{h}_{2,\overline{v}_2^\pm}$ *are $\mathcal{K}$-equivalent.*

On the other hand, we return to the review on the theory of Legendrian singularities. We introduce a natural equivalence relation among Legendrian submanifold germs. Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$ be Morse families of hypersurfaces. Then we say that $\mathcal{L}_F(\Sigma_*(F))$ and $\mathcal{L}_G(\Sigma_*(G))$ are *Legendrian equivalent* if there exists a contact diffeomorphism germ $H : (PT^*\mathbb{R}^n, z) \longrightarrow (PT^*\mathbb{R}^n, z')$ such that H preserves fibers of π and that $H(\mathcal{L}_F(\Sigma_*(F))) = \mathcal{L}_G(\Sigma_*(G))$, where $z = \mathcal{L}_F(0), z' = \mathcal{L}_G(0)$. By using the Legendrian equivalence, we can define the notion of Legendrian stability for Legendrian submanifold germs by the ordinary way (see, [1][Part III]). We can interpret the Legendrian equivalence by using the notion of generating families. We denote by $\mathcal{E}_n$ the local ring of function germs $(\mathbb{R}^n, 0) \longrightarrow \mathbb{R}$ with the unique maximal ideal $\mathfrak{M}_n = \{h \in \mathcal{E}_n \mid h(0) = 0\}$. Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$ be function germs.

Let $Q_{n+1}(\boldsymbol{x}, u_0)$ be the local ring of the function germ $h_{v_0} : (U, u_0) \longrightarrow \mathbb{R}$ defined by

$$Q_{n+1}(\boldsymbol{x}, \boldsymbol{u}_0) = C^\infty_{u_0}(U)/(\langle h_{v_0}\rangle_{C^\infty_{u_0}(U)} + \mathfrak{M}^{n+2}_{n-1}),$$

and $Q^\pm_{n+1}(\overline{\boldsymbol{x}}, u_0)$ be the local rings of the function germs $\overline{h}_{\overline{v}_0^\pm} : (U, u_0) \longrightarrow \mathbb{R}$ defined by

$$Q^\pm_{n+1}(\overline{\boldsymbol{x}}, \boldsymbol{u}_0) = C^\infty_{u_0}(U)/(\langle \overline{h}_{\overline{v}_0^\pm}\rangle_{C^\infty_{u_0}(U)} + \mathfrak{M}^{n+2}_{n-1}),$$

where $\boldsymbol{v}_0 = LD_M(\boldsymbol{u}_0, \mu_0)$, $\overline{\boldsymbol{v}}_0^\pm = \overline{LD}^\pm_{\overline{M}}(\boldsymbol{u}_0, \mu_0)$ and $C^\infty_{u_0}(U)$ is the local ring of function germs at $\boldsymbol{u}_0$ with the unique maximal ideal $\mathfrak{M}_{n-1}$.

Theorem 5.3. *Let $\boldsymbol{x}_i : (U, u_i) \longrightarrow (S^n_+, p_i)(i = 1, 2)$ be hypersurface germs such that the corresponding Legendrian immersion germs are Legendrian stable. Then the following conditions are equivalent.*
(1) *The lightcone hypersurface germs $LD_{M_1}(U \times \mathbb{R})$ and $LD_{M_2}(U \times \mathbb{R})$ are diffeomorphic.*
(2) *Legendrian immersion germs $\mathcal{L}^1_4$ and $\mathcal{L}^2_4$ are Legendrian equivalent.*
(3) *The lightcone height functions germs H_1 and H_2 are $\mathcal{P}$-$\mathcal{K}$-equivalent.*
(4) h_{1,v_1} *and* h_{2,v_2} *are $\mathcal{K}$-equivalent.*
(5) $K(\boldsymbol{x}_1(U), TPS^{n-1}_+(\boldsymbol{x}_1, \boldsymbol{u}_1), \boldsymbol{v}_1) = K(\boldsymbol{x}_2(U), TPS^{n-1}_+(\boldsymbol{x}_2, \boldsymbol{u}_2), \boldsymbol{v}_2)$.

(6) $K(\boldsymbol{x}_1(U), TPH^n(\boldsymbol{x}_1, \boldsymbol{u}_1), \boldsymbol{v}_1) = K(\boldsymbol{x}_2(U), TPH^n(\boldsymbol{x}_2, \boldsymbol{u}_2), \boldsymbol{v}_2)$.
(7) *Local rings* $Q_{n+1}(\boldsymbol{x}_1, \boldsymbol{u}_1)$ *and* $Q_{n+1}(\boldsymbol{x}_2, \boldsymbol{u}_2)$ *are isomorphic as* $\mathbb{R}$*-algebras.*

Theorem 5.4. *Let* $\overline{\boldsymbol{x}}_i : (U, \boldsymbol{u}_i) \longrightarrow (S_0^n, p_i)(i = 1, 2)$ *be hypersurface germs such that the corresponding Legendrian immersion germs are Legendrian stable. Then the following conditions are equivalent.*

(1) *The lightcone hypersurface germs* $\overline{LD}^{\pm}_{\overline{M}_1}(U \times \mathbb{R})$ *and* $\overline{LD}^{\pm}_{\overline{M}_2}(U \times \mathbb{R})$ *are diffeomorphic.*
(2) *Legendrian immersion germs* $\mathcal{L}_3^{1\pm}$ *and* $\mathcal{L}_3^{2\pm}$ *are Legendrian equivalent.*
(3) *The lightcone height functions germs* $\overline{H}_1$ *and* $\overline{H}_2$ *are* $\mathcal{P}$-$\mathcal{K}$*-equivalent.*
(4) $\overline{h}_{1,\overline{\boldsymbol{v}}_1^{\pm}}$ *and* $\overline{h}_{2,\overline{\boldsymbol{v}}_2^{\pm}}$ *are* $\mathcal{K}$*-equivalent.*
(5) $K(\overline{\boldsymbol{x}}_1(U), TPS_0^{n-1\pm}(\overline{\boldsymbol{x}}_1, \boldsymbol{u}_1), \overline{\boldsymbol{v}}_1^{\pm}) = K(\overline{\boldsymbol{x}}_2(U), TPS_0^{n-1\pm}(\overline{\boldsymbol{x}}_2, \boldsymbol{u}_2), \overline{\boldsymbol{v}}_2^{\pm})$.
(6) $K(\overline{\boldsymbol{x}}_1(U), TPS_1^{n\pm}(\overline{\boldsymbol{x}}_1, \boldsymbol{u}_1), \overline{\boldsymbol{v}}_1^{\pm}) = K(\overline{\boldsymbol{x}}_2(U), TPS_1^{n\pm}(\overline{\boldsymbol{x}}_2, \boldsymbol{u}_2), \overline{\boldsymbol{v}}_2^{\pm})$.
(7) *Local rings* $Q_{n+1}^{\pm}(\overline{\boldsymbol{x}}_1, \boldsymbol{u}_1)$ *and* $Q_{n+1}^{\pm}(\overline{\boldsymbol{x}}_2, \boldsymbol{u}_2)$ *are isomorphic as* $\mathbb{R}$*-algebras.*

Lemma 5.5. *Let* $\boldsymbol{x} : U \longrightarrow S_+^n$ *be a hypersurface germ such that the corresponding Legendrian immersion germs* $\mathcal{L}_4$ *and* $\mathcal{L}_3^{\pm}$ *are Legendrian stable. Then at the singular point* $\boldsymbol{v}_0 = LD_M(\boldsymbol{u}_0, 2\sigma^{\pm}(\kappa_i(p_0)))(1 \leq i \leq n-1)$ *of* LD_M *and the singular points* $\overline{\boldsymbol{v}}_0^{\pm} = \overline{LD}^{\pm}_{\overline{M}}(\boldsymbol{u}_0, 1/\kappa_i(p_0))$ *of* $\overline{LD}^{\pm}_{\overline{M}}$*, we have the following equivalent assertions:*

(1) *The lightcone hypersurface germs* $LD_M(U \times \mathbb{R})$ *and* $\overline{LD}^{\pm}_{\overline{M}}(U \times \mathbb{R})$ *are diffeomorphic.*
(2) *Legendrian immersion germs* $\mathcal{L}_3^{\pm}$ *and* $\mathcal{L}_4$ *are Legendrian equivalent.*
(3) *The lightcone height functions germs* H *and* $\overline{H}$ *are* $\mathcal{P}$-$\mathcal{K}$*-equivalent.*
(4) $h_{\boldsymbol{v}_0}$ *and* $\overline{h}_{\overline{\boldsymbol{v}}_0^{\pm}}$ *are* $\mathcal{K}$*-equivalent.*
(5) $K(\boldsymbol{x}(U), TPS_+^{n-1}(\boldsymbol{x}, \boldsymbol{u}_0), \boldsymbol{v}_0) = K(\overline{\boldsymbol{x}}(U), TPS_0^{n-1\pm}(\overline{\boldsymbol{x}}, \boldsymbol{u}_0), \overline{\boldsymbol{v}}_0^{\pm})$.
(6) $K(\boldsymbol{x}(U), TPH^n(\boldsymbol{x}, \boldsymbol{u}_0), \boldsymbol{v}_0) = K(\overline{\boldsymbol{x}}(U), TPS_1^{n\pm}(\overline{\boldsymbol{x}}, \boldsymbol{u}_0), \overline{\boldsymbol{v}}_0^{\pm})$.
(7) *Local rings* $Q_{n+1}^{\pm}(\overline{\boldsymbol{x}}, \boldsymbol{u}_0)$ *and* $Q_{n+1}(\boldsymbol{x}, \boldsymbol{u}_0)$ *are isomorphic as* $\mathbb{R}$*-algebras.*

By Lemma 5.5, we have our main result as the following theorem.

Theorem 5.6. *Let* $\boldsymbol{x}_i : (U, \boldsymbol{u}_i) \longrightarrow (S_+^n, p_i)(i = 1, 2)$ *be hypersurface germs such that the corresponding Legendrian immersion germs are Legendrian stable. At the singular points* $\overline{\boldsymbol{v}}_i^{\pm} = \overline{LD}^{\pm}_{\overline{M}}(\boldsymbol{u}_0, 1/\kappa_j(p))(1 \leq j \leq n-1)$ *of* $\overline{LD}^{\pm}_{\overline{M}}$*, and the singular points* $\boldsymbol{v}_i = LD_M(\boldsymbol{u}_0, 2\sigma^{\pm}(\kappa_j(p)))$ *of* LD_M*, the conditions* (1) $\sim$ (7) *in Theorem 5.3 and the conditions* (1) $\sim$ (7) *in Theorem 5.4 are all equivalent.*

References

[1] V. I. Arnol'd, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of Differentiable Maps vol. I.* Birkhäuser, 1986.

[2] L. Chen and S. Izumiya, *A mandala of Legendrian dualities for pseudo-spheres in semi-Euclidean space*, Proceedings of the Japan Academy, 85 Ser. A, (2009), 49–54.

[3] S. Izumiya, *Lengendrian dualities and spacelike hypersurfaces in the lightcone*, Mosc. Math. J. 9 (2009) 325–357.

[4] S. Izumiya, *Differential Geometry from the viewpoint of Lagrangian or Legendrian singularity theory*, in Singularity Theory (ed., D. Chéniot et al), World Scientific (2007), 241–275.

[5] S. Izumiya, Y. Jiang and D.-H. Pei, *Lightcone dualities for curves in the sphere*, The Quart. J. Math. (2012) DOI:10.1093/qmath/has002

[6] S. Izumiya, Y. Jiang and T. Sato, *Lightcone dualities for curves in the 3-sphere*, preprint (2012).

[7] H. L. Liu, S. D. Jung, *Hypersurfaces in lightlike cone*, J. Geom. Phys. 58 (2008) 913–922.

[8] J.A.Montaldi,*On ctact between submanifolds*, Michigan Math. J., 33(1986), 81-85.

[9] T. Nagai, *The Gauss map of a hypersurface in Euclidean sphere and the spherical Legendrian duality*, Topology and its Applications, 159 (2011) 545–554.

[10] T. Nagai, *The spherical evolute of a hypersurface in Euclidean sphere*, in preparation.

[11] I. R. Porteous, *Some remrks on duality in* S^3, Geometry and topology of caustics, Banach Center Publ. 50, Polish Acad. Sci., Warsaw, (2004) 217–226.

[12] M. C. Romero Fuster, *Sphere stratifications and the Gauss map*, Proceedings of the Royal Soc. Edinburgh, **95A** (1983), 115–136.

[13] O. P. Shcherbak, *Projectively dual space curves and Legendre singularities*, Sel. Math. Sov 5 (1986), 391–421.

[14] V. M. Zakalyukin, *Lagrangian and Legendrian singularities*, Funct. Anal. Appl. 10(1976), No. 1, 23-31.

[15] S. Izumiya, Y. Jiang and D.-H. Pei, Lightcone dualities for hypersurface in the sphere, preprint.

YANG JIANG
A. SCHOOL OF MATHEMATICS AND STATISTICS
NORTHEAST NORMAL UNIVERSITY
CHANGCHUN 130024
P.R.CHINA
B. DEPARTMENT OF MATHEMATICS
HOKKAIDO UNIVERSITY
SAPPORO 060-0810
JAPAN