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Singularities of the discriminant of pairs of regular foliations in $\mathbb{R}^3$

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1 Introduction

An important feature of pairs of codimension one regular foliations in $\mathbb{R}^3$ is its discriminant which is the locus of points where the foliations are tangent. Assuming that the foliations are the leaves of germs of differential 1-forms ω and η, then the discriminant $D(\omega,\eta)$ of the pair (ω,η) is the zero locus of $\omega\wedge\eta$, that is, the locus where the 1-form ω is a multiple of η. This is generically a germ of a space curve. In local coordinates, the discriminant is given by the fibre of a map-germ (the discriminant map-germ) $F:(\mathbb{R}^3,0)\rightarrow(\mathbb{R}^2,0)$. In [2] the authors show that the discriminant $D(\omega,\eta)$ determines the local topological type of the pair (ω,η) and obtain a complete list of discrete topological models.

More precisely, assume that the discriminant $D(\omega,\eta)$ is transverse away from 0 to the pair of foliations. When the number of branches of the discriminant in each half-region delimited by the leaf of ω (or η) through 0 is at most 2, then it is showed in [2] that the local topological type is determined by their configuration. The authors also remark that if the number of the branches of the discriminant in a half-region delimited by the leaf of ω (or η) through 0 is more than 2, then there is no discrete topological model.

Since the discriminant plays a key role in the topological classification of the pair (ω,η), it is of interest to analyse its singularities. This is carried out in [2, 3]. We give in this paper a résumé on the various classifications that are given in those articles.

2 Classifications

In the coordinates system (x,y,z), we can set $\omega=df$ and $\eta=dz$, where $f:\mathbb{R}^3,0\rightarrow\mathbb{R},0$ is a germ of a smooth function. Then $D(\omega,\eta)$ is the zero fibre of the map-germ $F_{\omega,\eta}$: $\mathbb{R}^3,0\rightarrow\mathbb{R}^2$ given by $F_{\omega,\eta}(x,y,z)=(f_x(x,y,z),f_y(x,y,z))$.

Let $\mathcal{E}_n$ be the local ring of germs of functions $\mathbb{R}^n,0\rightarrow\mathbb{R}$ and m_n its maximal ideal (which is the subset of germs that vanish at the origin). Denote by $\mathcal{E}(n,p)$ the p-tuples of elements in $\mathcal{E}_n$. Let $\mathcal{A} = \mathcal{R} \times \mathcal{L} = Diff(\mathbb{R}^n, 0) \times Diff(\mathbb{R}^p, 0)$ denotes the group of right-left equivalence which acts smoothly on $m_n.\mathcal{E}(n,p)$ by $(h,k).G = k \circ G \circ h^{-1}$. We have another group of interest, namely the contact group $\mathcal{K}$. The group $\mathcal{K}$ is the set of germs of diffeomorphisms $H = (h, H_1) \in Diff(\mathbb{R}^{n+p}, 0)$, with $h \in Diff(\mathbb{R}^n, 0)$. Then H acts on $m_n.\mathcal{E}(n,p)$ as follows: $G = H.F$ if and only if $H(x, F(x)) = (h(x), G(h(x)))$.

2.1 $\mathcal{K}$-singularities of the discriminant

The action of the group $\mathcal{K}$ is a natural one to use when one seeks to understand the singularities of the zero fibres of germs in $m_n\mathcal{E}(n,p)$.

When at least one foliation is regular, the description of the singularities of the discriminant of the pair (ω, η) via an action of the contact group $\mathcal{K}$ on $m_3.\mathcal{E}(3,2)$ reduces to an action on families of matrices. The corank 2 simple germs are given in Table 1 ([1]). See [2], Sect. 5.1 for details.

Table 1: The $\mathcal{K}$-simples singularities of corank 2 germs $\mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ ([1]).

Name	Normal form	$\mathcal{K}_e$ − *codimension*
$P_{2,k}$	$(xy, x^k + y^2 + z^2)), k \geq 2$	$k+1$
$P_{3,3}$	$(xy, x^3 + y^3 + z^2)$	5
$P_{3,4}$	$(xy, x^4 + y^3 + z^2)$	6
$P_{3,5}$	$(xy, x^5 + y^3 + z^2)$	7
J_7	$(y^2 + xz, x^3 + yz)$	5
J_8	$(y^2 + xz, x^2y + yz)$	6
J_9	$(y^2 + xz, x^4 + yz)$	7
K_8	$(y^2 + xz, x^3 + z^2)$	6
K_9	$(y^2 + xz, x^2y + z^2)$	7
G_{10}	$(y^2 + x^3, x^3 + z^2)$	7
G_{11}	$(y^2 + x^3, x^2y + z^2)$	8

2.2 $\mathcal{K}_V$-singularities of the discriminant

The action of the subgroup $\mathcal{K}_V$ of $\mathcal{K}$ on $m_3.\mathcal{E}(3,2)$, where the changes of coordinates in the source preserve the variety V given by $z = 0$, preserves the smooth type of the zero fibre of $F : \mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ as well as the leaf $z = 0$ of dz. Since the key ingredient in the topological classification of the pairs (df, dz) is the number of branches of the discriminant in each semi-space delimited by the plane $z = 0$, it is natural to seek a classification of F under such action of $\mathcal{K}_V$.

Theorem 2.1 (Theorem 5.1, [2]) *The $\mathcal{K}_V$-simple map-germ $\mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ are given in Table 2.*

It follows that the $\mathcal{K}_V$-simple singularities of the discriminant $D(df, dz)$ of pairs (df, dz) are given in Table 2. The action of the subgroup $\mathcal{K}_V$ of $\mathcal{K}$ preserves the number of branches, in each semi-space $z > 0$ and $z < 0$, of the zero fibre of any map-germ $\mathbb{R}^3, 0 \to \mathbb{R}^2, 0$.

Table 2: Normal forms of $\mathcal{K}_V$-simple map-germs $\mathbb{R}^3, 0 \to \mathbb{R}^2, 0$.

Normal form F	$d_e(F, \mathcal{K}_V)$
(x, y)	0
$(x, z + y^k)$, $k \geq 2$	$k - 2$
$(x, y^2 + z^k)$, $k \geq 2$	$k - 1$
$(x, y^2 - z^{2k})$, $k \geq 1$	$2k - 1$
$(x, yz + y^k)$, $k \geq 3$	$k - 1$
$(x, z^2 + y^3)$	3

2.3 $\mathcal{K}^*$-singularities of the discriminant

It is of interest to classify the singularities of the discriminant up to an equivalence that preserves the leaves of the foliation ω or η. See [2], Sect. 5.3. Let $\mathcal{K}^*$ be the subgroup of $\mathcal{K}$ (and indeed of $\mathcal{K}_V$) where the changes of coordinates in the source preserve the foliation defined by dz, i.e., the horizontal planes. Recall that the discriminant is the zero fibre of map germ $F_{\omega,\eta} : \mathbb{R}^3, 0 \to \mathbb{R}^2, 0$, so we are seeking a classification of the singularities of map-germs $\mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ under the action of the subgroup $\mathcal{K}^*$, which is a Damon geometric subgroup. The classification of the $\mathcal{K}^*$-simple map-germs $\mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ coincide with the $\mathcal{K}_V$-simple singularities.

Theorem 2.2 (Theorem 5.4, [2]) *A map-germ $\mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ is $\mathcal{K}^*$-simple if and only if it is $\mathcal{K}_V$-simple. So the $\mathcal{K}^*$-simple germs are also those given in Table 2.*

The action $\mathcal{K}^*$ does not only preserves the number of branches in each semi-space $z > 0$ and $z < 0$ of the zero fibre of any map-germ $\mathbb{R}^3, 0 \to \mathbb{R}^2, 0$, but also the contact of the zero fibre with the horizontal planes.

2.4 $\mathcal{B}$-singularities of the discriminant

We also can study the singularities of the discriminant up to an action that preserves the leaf of $\eta = dz$ through the origin considering the subgroup $\mathcal{B}$ of $\mathcal{A}$ which consists of pairs of diffeomorphisms (h, k), where h preserves the plane $z = 0$ in the source and k is any diffeomorphism in the target. The group $\mathcal{B}$ acts on $m(x, y, z).\mathcal{E}(3, 2)$ and is a Damon geometric subgroup. Theorem 2.3 models codimension less than or equal to four simple singularities of the discriminant map-germ up to subgroup $\mathcal{B}$.

Theorem 2.3 (Theorem 1.1, [3]) *The $\mathcal{B}$-simple map-germs $(\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ of corank at most 1 and $\mathcal{B}_e$-codimension ≤ 4 are given in Table 3.*

Table 3: Normal forms of $\mathcal{B}_e$-codimension ≤ 4 ($\epsilon, \epsilon_1 = \pm 1$).

Normal form	$\mathcal{B}_e$-codimension
(x, y)	0
$(x, z + \epsilon y^2)$	0
$(x, z + xy + y^3)$	0
$(x, z + y^3 + \epsilon^{k-1} x^k y)$, $k \geq 2$	$k - 1$
$(x, z + xy + \epsilon y^4)$	1
$(x, z + xy + y^5 + \epsilon y^7)$	2
$(x, z + xy + y^5)$	3
$(x, z + xy + \epsilon y^6 + y^9)$	4
$(x, z + xy^2 + \epsilon y^4 + y^{2k+1})$, $k \geq 2$	k
$(x, z + xy^2 + y^5 + \epsilon y^6)$	3
$(x, z + xy^2 + y^5 + \epsilon y^9)$	4
$(x, z + x^2 y + \epsilon y^4 + \epsilon_1 y^5)$	3
$(x, z + x^2 y + \epsilon y^4)$	4
$(x, y^2 + \epsilon z^2 + \epsilon_1{}^k x^{k-1} z)$, $k \geq 2$	$k - 2$
$(x, y^2 + xz + \epsilon z^3)$	1
$(x, y^2 + xz + \epsilon z^4 + \epsilon_1 z^6)$,	2
$(x, y^2 + xz + \epsilon z^4)$	3
$(x, yz + xy + \epsilon y^3)$	1
$(x, yz + xy + y^4 + \epsilon y^6)$	2
$(x, yz + xy + y^4)$	3
$(x, xy + z^2 + y^3 + \epsilon y^k z)$, $k \geq 2$	k

We observe that the classification under the action of the group $\mathcal{B}$ also models the singularities of projections of hypersurfaces with boundary in $\mathbb{R}^4$ to planes and those of invariant map-germs $\mathbb{R}^3 \to \mathbb{R}^2$. See [3] for details.

References

[1] A. A. du Plessis and C. T. C. Wall, *The geometry of topological stability.* London Mathematical Society Monographs. New Series, 9. Oxford Science Publications. Oxford University Press, 1995.

[2] L. F. Martins, R. D. S. Oliveira, and F. Tari, "On pairs of regular foliations in $\mathbb{R}^3$ and singularities of map-germs", *Geometriae Dedicata,* **135** (2008), 103-118.

[3] L. F. Martins and A. C. Nabarro, "Projections of hypersurfaces in $\mathbb{R}^4$ with boundary to planes", *to appear in Glasgow Mathematical Journal,* 2013.