数理解析研究所講究録
第 1868 巻 2013 年 63-73

Notes on liftable vector fields

Yusuke Mizota*

Abstract

We introduce the author's research about an estimate for the highest degrees of liftable vector fields and the module of liftable vector fields for non-singular mono-germs and function mono-germs of one variable.

1 Introduction

In this paper, we introduce the author's research about liftable vector fields. Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. In this paper, suppose that all mappings are smooth (that is, of class C^{∞} if $\mathbb{K}=\mathbb{R}$ or holomorphic if $\mathbb{K}=\mathbb{C}$).

The notion of liftable vector fields was introduced by Arnol'd [1] for studying bifurcations of wave front singularities. As results and applications of liftable vector fields, Bruce and West [2] obtained diffeomorphisms preserving a crosscap to classify functions on it, and Houston and Littlestone [4] obtained generators for the module of vector fields liftable over the generalized cross cap to find $\mathcal{A}_{e}$-codimension 1 maps from $\mathbb{C}^{n}$ to $\mathbb{C}^{n+1}$. Houston and Atique [3] classified ${}_{V}\mathcal{K}$-codimension 2 maps on the generalized crosscap to apply to a classification of $\mathcal{A}_{e}$-codimension 2 maps from $\mathbb{C}^{n}$ to $\mathbb{C}^{n+1}$. Nishimura [8] characterized the minimal number of generators for the module of vector fields liftable over a finitely determined multigerm of corank at most one satisfying a special condition when $n\leq p$.

In previous work [6], the author showed that we can find polynomial vector fields liftable over f if f is a polynomial multigerm and gave an estimate for the highest degrees of liftable vector fields. The highest degree of polynomial vector field ξ means maximum of that of component functions of ξ. Let $[x]$ be the greatest integer not exceeding x. $\mathrm{Lift}(f)$ denotes the module of vector fields liftable over f. We proved the following theorem.

Theorem 1.1 ([6]). *Let $f:(\mathbb{K}^{n},S)\rightarrow(\mathbb{K}^{p},0)$ $(n<p)$ be a polynomial multigerm. Then, there exists a non-zero polynomial vector field in* $\mathrm{Lift}(f)$ *such that*

*Research Fellow DC2 of Japan Society for the Promotion of Science
The author was supported by JSPS and CAPES under the Japan-Brazil Research Cooperative Program.
Graduate School of Mathematics, Kyushu University, 744, Motooka, Nishi-ku, Fukuoka 819-0395, JAPAN.
e-mail: y-mizota@math.kyushu-u.ac.jp

the highest degree is at most

$$N = \left[\sqrt[p-n]{(\alpha+1)(p-1)!} \right] + 1,$$

where

$$\alpha = r \binom{p \cdot 2^n - n}{n!} (D+n-1)^n, \quad r = |S|,$$

$$D = \max\{D_i \mid i \in \{1,2,\ldots,r\}\}, \quad D_i = \max\{\deg(X_j \circ f_i) \mid j \in \{1,2,\ldots,p\}\}.$$

The proof of Theorem 1.1 also gives a method to find a non-zero element of Lift(f). However, we can usually take values of N that are much lower than those calculated in Theorem 1.1. Therefore, we needed to improve this estimate. In [7], a better estimate for the highest degrees of liftable vector fields was discoverd when $n = 1$. It is the following result. Let $\lceil x \rceil$ be the smallest integer greater than or equal to x.

Theorem 1.2 ([7]). *Let $f : (\mathbb{K}, S) \to (\mathbb{K}^p, 0)$ $(p \geq 2)$ be a polynomial multigerm which contains no branch of zero map. Then there exist a non-zero polynomial vector field of* Lift(f) *such that the highest degree is at most*

$$N = \max\left\{ \left\lceil \sqrt[p-1]{(Ap-A)(p-1)!} - \sqrt[p]{p!} \right\rceil, 1 \right\},$$

and the highest degree of a corresponding lowerable vector field for f_i is at most

$$D_i N - D_i + 1,$$

where

$$A = \sum_{i=1}^{r} D_i, \quad D_i = \max\{\deg(X_j \circ f_i) \mid j \in \{1,\ldots,p\}\}.$$

This paper is organized as follows. In Section 2, we explain various definitions, basic facts and examples implying difference of estimates between Theorem 1.1 and Theorem 1.2. In Section 3 the sketch of proof of Theorem 1.2 is described. In Section 4 and 5, topics about the module of liftable vector fields are given. The number of generators for the module of vector fields liftable over a non-singular mono-germ is identified in Section 4. Theorem 1.1 and Theorem 1.2 claims that there exist non-zero polynomial vector fields liftable over f when f is a polynomial. It is natural to ask whether there exist non-zero polynomial liftable vector fields when f is not a polynomial. In Section 5 we investigate the module of vector fields liftable over a function germ of one variable and also consider this problem.

2 Preliminary

Let S be a subset of $\mathbb{K}^n$. A map germ $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ is called a *multigerm*. If S is a singleton, f is called a *mono-germ*. Let C_S (resp., C_0) be the set of function germs $(\mathbb{K}^n, S) \to \mathbb{K}$ (resp., $(\mathbb{K}^p, 0) \to \mathbb{K}$), and let m_S (resp., m_0) be the subset of C_S (resp., C_0) consisting of function germs $(\mathbb{K}^n, S) \to (\mathbb{K}, 0)$ (resp., $(\mathbb{K}^p, 0) \to (\mathbb{K}, 0)$). The sets C_S and C_0 have natural $\mathbb{K}$-algebra structures. A multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ can be defined by $(f_1, f_2 \dots, f_r)$, where $f_i : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$. Each f_i is called a *branch*. In this paper, for a multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ defined by $(f_1, f_2 \dots, f_r)$ with $f_i : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$, we consider S to be a set consisting of r distinct points.

For a map germ $f : (\mathbb{K}^n, S) \to \mathbb{K}^p$, let $\theta_S(f)$ be the set of germs of vector fields along f. The set $\theta_S(f)$ has a natural C_S-module structure and is identified with the direct sum of p copies of C_S. Put $\theta_S(n) = \theta_S(\mathrm{id}_{(\mathbb{K}^n, S)})$ and $\theta_0(p) = \theta_{\{0\}}(\mathrm{id}_{(\mathbb{K}^p, 0)})$, where $\mathrm{id}_{(\mathbb{K}^n, S)}$ (resp., $\mathrm{id}_{(\mathbb{K}^p, 0)}$) is the germ of the identity mapping of $(\mathbb{K}^n, S)$ (resp., $(\mathbb{K}^p, 0)$). For a multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$, following Mather [5], we define tf and ωf as

$$tf : \theta_S(n) \to \theta_S(f), \quad tf(\eta) = df \circ \eta,$$

$$\omega f : \theta_0(p) \to \theta_S(f), \quad \omega f(\xi) = \xi \circ f,$$

where df is the differential of f. Following Wall [9], we put $T\mathcal{R}_e(f) = tf(\theta_S(n))$ and $T\mathcal{L}_e(f) = \omega f(\theta_0(p))$.

For a multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$, a vector field $\xi \in \theta_0(p)$ is said to be *liftable* over f if $\xi \circ f \in T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$. The set of vector fields liftable over f is denoted by $\mathrm{Lift}(f)$. Note that $\mathrm{Lift}(f)$ has a natural C_0-module structure. Let $(x_1, x_2, \dots, x_n)$ (resp., $(X_1, \dots, X_p)$) be the standard local coordinates of $\mathbb{K}^n$ (resp., $\mathbb{K}^p$) at the origin. Sometimes (x_1, x_2) (resp., (X_1, X_2)) is denoted by (x, y) (resp., (X, Y)) and (x_1, x_2, x_3) (resp., (X_1, X_2, X_3)) is denoted by (x, y, z) (resp., (X, Y, Z)). We see easily that

$$\xi = (\psi_1(X_1, X_2, \dots, X_p), \cdots, \psi_p(X_1, X_2, \dots, X_p)) \in \mathrm{Lift}(f),$$

where $\psi_j : (\mathbb{K}^p, 0) \to \mathbb{K}$ $(j = 1, 2, \dots, p)$, if and only if for every $i \in \{1, \dots, r\}$ there exist a vector field

$$\eta_i = (\phi_{i,1}(x_1, x_2, \dots, x_n), \dots, \phi_{i,n}(x_1, x_2, \dots, x_n)),$$

where $\phi_{i,k} : (\mathbb{K}^n, 0) \to \mathbb{K}$ $(k = 1, 2, \dots, n)$, such that $\xi \circ f_i = df_i \circ \eta_i$ i. e.

$$\begin{pmatrix} \psi_1(X_1, X_2, \dots, X_p) \\ \vdots \\ \psi_p(X_1, X_2, \dots, X_p) \end{pmatrix} \circ f_i(x_1, x_2, \dots, x_n)$$

$$= \begin{pmatrix} \frac{\partial(X_1 \circ f_i)}{\partial x_1} & \cdots & \frac{\partial(X_1 \circ f_i)}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial(X_p \circ f_i)}{\partial x_1} & \cdots & \frac{\partial(X_p \circ f_i)}{\partial x_n} \end{pmatrix} \begin{pmatrix} \phi_{i,1}(x_1, x_2, \dots, x_n) \\ \vdots \\ \phi_{i,n}(x_1, x_2, \dots, x_n) \end{pmatrix}.$$

We call this η_i a *lowerable* vector field for f_i corresponding to ξ.

Example 2.1. Let $f : (\mathbb{K},0) \to (\mathbb{K}^2,0)$ be given by $f(x) = (x^2, x^3)$. Then, it can be seen easily that the following vector fields are liftable over f:

$$\begin{pmatrix} 2X \\ 3Y \end{pmatrix}, \begin{pmatrix} 2Y \\ 3X^2 \end{pmatrix}.$$

The forms of vector fields liftable over a polynomial multigerm are complicated generally.

Example 2.2. Let $f : (\mathbb{K}, S) \to (\mathbb{K}^2, 0)$ be given by $f_1(x) = (x^2, x^3)$, $f_2(x) = (x^3, x^2)$. Then, it is known [8] that the following vector fields are liftable over f:

$$(-6X^2Y^2 + 6XY)\frac{\partial}{\partial X} + (-9XY^3 + 5X^3 + 4Y^2)\frac{\partial}{\partial Y},$$

In fact,

$$\begin{pmatrix} -6X^2Y^2 + 6XY \\ -9XY^3 + 5X^3 + 4Y^2 \end{pmatrix} \circ f_1 = \begin{pmatrix} -6x^{10} + 6x^5 \\ -9x^{11} + 9x^6 \end{pmatrix} = \begin{pmatrix} 2x \\ 3x^2 \end{pmatrix} \begin{pmatrix} -3x^9 + 3x^4 \end{pmatrix},$$

$$\begin{pmatrix} -6X^2Y^2 + 6XY \\ -9XY^3 + 5X^3 + 4Y^2 \end{pmatrix} \circ f_2 = \begin{pmatrix} -6x^{10} + 6x^5 \\ -4x^9 + 4x^4 \end{pmatrix} = \begin{pmatrix} 3x^2 \\ 2x \end{pmatrix} \begin{pmatrix} -2x^8 + 2x^3 \end{pmatrix}$$

holds.

The following property is very fundamental and important.

Proposition 2.3. *We assume $g = t \circ f \circ s$, which t and s are diffeomorphisms (that is, f is $\mathcal{A}$-equivalent to g). Then,*

$$\xi \in \mathrm{Lift}(f) \Rightarrow dt \circ \xi \circ t^{-1} \in \mathrm{Lift}(g).$$

This means that only diffeomorphism of the target of f affects liftable vector fields of g. In addition, we will see that only that of the sourse affects lowerable vector fields in the proof.

Proof. There exists $\eta \in \theta_S(n)$ such that

$$\xi \circ f = df \circ \eta.$$

Then, the following holds:

$$\begin{aligned} (dt \circ \xi \circ t^{-1}) \circ g &= dt \circ \xi \circ t^{-1} \circ (t \circ f \circ s) \\ &= dt \circ (\xi \circ f) \circ s \\ &= dt \circ (df \circ \eta) \circ s \\ &= d(t \circ f) \circ \eta \circ s \\ &= d(g \circ s^{-1}) \circ \eta \circ s \\ &= dg \circ (ds^{-1} \circ \eta \circ s) \end{aligned}$$

Thus, $dt \circ \xi \circ t^{-1} \in \mathrm{Lift}(g)$. □

We compare estimates of Theorem 1.1 and Theorem 1.2 using examples.

Example 2.4. When $n = 1$ and $p = 2$, by Theorem 1.1

$$N = 3rD + 2.$$

For $f(x) = (x^2, x^3)$, since $r = 1$ and $D = 3$, we get $N = 11$. On the other hand, when $p = 2$ by Theorem 1.2

$$N = \left\lceil A - \sqrt{2} \right\rceil,$$

Since $A = 3$, we get $N = 2$. In fact, the highest degree of the following liftable vector fields are 1 and 2 respectively;

$$\begin{pmatrix} 2X \\ 3Y \end{pmatrix}, \begin{pmatrix} 2Y \\ 3X^2 \end{pmatrix}.$$

Example 2.5. For $f : (\mathbb{K}, S) \to (\mathbb{K}^2, 0)$ given by $f_1(x) = (x^2, x^3), f_2(x) = (-x^3, x^2), f_3(x) = (x^2 - x^3, x^2 + x^3)$, since $r = 3$ and $D = 3$, by Theorem 1.1 $N = 29$. On the other hand, by Theorem 1.2 since $A = 9$, we get $N = 8$. In fact, the following vector field are liftable over f and the highest degree is 6;

$$\begin{aligned}
&(-15X^6 - 45X^5Y - 45X^4Y^2 + 19X^3Y^3 + 4X^2Y^4 \\
&-4X^5 - 64X^4Y + 45X^3Y^2 + 41X^2Y^3 + 57XY^4 \\
&-7Y^5 + 4X^4 - 12X^3Y - 8X^2Y^2 + 52XY^3 \\
&-14Y^4 + 8X^3 - 16X^2Y)\tfrac{\partial}{\partial X} \\
&+(-10X^5Y - 30X^4Y^2 - 38X^3Y^3 + 18X^2Y^4 + 6XY^5 \\
&+8X^5 - 8X^4Y - 46X^3Y^2 + 34X^2Y^3 + 24XY^4 \\
&+56Y^5 - 4X^4 + 6X^3Y - 26X^2Y^2 - 10XY^3 \\
&+28Y^4 + 12X^2Y - 20XY^2)\tfrac{\partial}{\partial Y}.
\end{aligned}$$

Example 2.6. When $n = 1$ and $p = 3$, by Theorem 1.1

$$N = \left\lceil \sqrt{2(5rD + 1)} \right\rceil.$$

For $f : (\mathbb{K}, S) \to (\mathbb{K}^3, 0)$ given by $f_1(x) = (x^3, x^4, x^5), f_2(x) = (x^4, x^5, x^3), f_3(x) = (x^5, x^3, x^4)$, we get $r = 3$ and $D = 5$. Therefore, $N = 13$. On the other hand, when $p = 3$ by Theorem 1.2

$$N = \left\lceil 2\sqrt{A} - \sqrt[3]{6} \right\rceil.$$

Since $A = 15$, we get $N = 6$. In fact, the following vector field are liftable over f and the highest degree is 5;

$$(12X^3+18X^4-12X^2Y+12X^3Y+18X^4Y+6X^2Y^2-$$
$$6X^3Y^2+18XY^3-18X^2Y^3+6Y^4+18X^3Z-6XYZ-$$
$$12X^2YZ-6X^3YZ-18XY^2Z-18X^2Y^2Z+12Y^3Z+$$
$$6XY^3Z+18Y^4Z-6X^2Z^2-18X^3Z^2-18XYZ^2-$$
$$6X^2YZ^2-18Y^2Z^2+12XZ^3+6X^2Z^3+6YZ^3-$$
$$12Y^2Z^3-6Z^4+12XZ^4)\tfrac{\partial}{\partial X}+$$
$$(-8X^4+16X^2Y+16X^3Y+24X^4Y-8XY^2+8X^2Y^2+$$
$$16X^3Y^2+16XY^3-8Y^4-8X^2Z+8XYZ+16X^2YZ+8X^3YZ-$$
$$8XY^2Z-24X^2Y^2Z-16Y^3Z-16XY^3Z-24X^3Z^2+$$
$$8YZ^2-16X^2YZ^2-8Y^2Z^2-16XZ^3-8X^2Z^3+8YZ^3+$$
$$32XYZ^3-8Z^4)\tfrac{\partial}{\partial Y}+$$
$$(10X^4Y+10XY^2+10X^2Y^2+30X^3Y^2+10XY^3+$$
$$10X^2Y^3+20Y^4+10X^2Z+20X^3Z-20XYZ+10X^2YZ+$$
$$10X^3YZ+10XY^2Z-30XY^3Z-10Y^4Z-10X^3Z^2-$$
$$10YZ^2-20XYZ^2-40X^2YZ^2-20Y^2Z^2-30XY^2Z^2-$$
$$10Y^3Z^2-10X^2Z^3-10YZ^3+30Y^2Z^3+20Z^4+10XZ^4)\tfrac{\partial}{\partial Z}.$$

3 The sketch of proof of Theorem 1.2

We want non-negative integers N and N_i' ($i=1,2,\ldots,r$) such that we can find a coefficient vector

$$(a_1^{(0,0,\ldots,0)},a_1^{(1,0,\ldots,0)},\ldots,a_p^{(0,0,\ldots,N)},a_{1,0},a_{1,1},\ldots,a_{r,N_r'})\neq 0$$

such that for every $i\in\{1,2,\ldots,r\}$, the following polynomial equation with respect to the variable x_1 holds:

$$\begin{pmatrix}\displaystyle\sum_{d=0}^{N}\left(\sum_{i_1+\cdots+i_p=d}a_1^{(i_1,i_2,\ldots,i_p)}\prod_{h=1}^{p}X_h^{i_h}\right)\\ \vdots\\ \displaystyle\sum_{d=0}^{N}\left(\sum_{i_1+\cdots+i_p=d}a_p^{(i_1,i_2,\ldots,i_p)}\prod_{h=1}^{p}X_h^{i_h}\right)\end{pmatrix}\circ f_i = \begin{pmatrix}\dfrac{\partial(X_1\circ f_i)}{\partial x_1}\\ \vdots\\ \dfrac{\partial(X_p\circ f_i)}{\partial x_1}\end{pmatrix}\left(a_{i,0}+a_{i,1}x_1+\cdots+a_{i,N_i'}x_1^{N_i'}\right),$$

where $i_1,i_2,\ldots,i_p$ are non-negative integers. Note that for every $i\in\{1,2,\ldots,r\}$, the highest degree of the left-hand side is at most $N\cdot D_i$ and that of the right-hand side is at most $N_i'+D_i-1$. By comparing the coefficients of the terms on the both sides, a system of linear equations with respect to

$$a_1^{(0,0,\ldots,0)},a_1^{(1,0,\ldots,0)},\ldots,a_p^{(0,0,\ldots,N)},a_{1,0},a_{1,1},\ldots,a_{r,N_r'}$$

is obtained. Let the number of unknowns of the system of linear equations be denoted by U and the number of equations by E. We assume that $N \geq 1$ to put

$$N_i' = ND_i - D_i + 1.$$

The number of combinations of non-negative integers $i_1, \cdots, i_p$ such that $i_1 + \cdots + i_p = d$ is

$$\binom{d+p-1}{d} = \frac{(d+p-1)!}{d!(p-1)!} = \frac{(d+p-1)\cdots(d+1)}{(p-1)!}.$$

Thus, we get

$$U = p\sum_{d=0}^{N} \frac{(d+p-1)\cdots(d+1)}{(p-1)!} + \sum_{i=1}^{r}(ND_i - D_i + 2)$$

and

$$E \leq p\sum_{i=1}^{r}(ND_i + 1).$$

Here, the folloing formula is known.

Proposition 3.1. *For a non-negative integer k,*

$$\sum_{d=1}^{n} d(d+1)\cdots(d+k) = \frac{n(n+1)\cdots(n+k+1)}{k+2}.$$

In addition, the following lemma holds.

Lemma 3.2. *For $p \in \mathbb{N}_{\geq 2}$ and $x \in \mathbb{R}_{>0}$,*

$$\left(x + \sqrt[p]{p!}\right)^p < (x+1)(x+2)\cdots(x+p)$$

Put

$$A = \sum_{i=1}^{r} D_i.$$

By Proposition 3.1 and Lemma 3.2, we get

$$U - E > \frac{1}{(p-1)!}\left(N + \sqrt[p]{p!}\right)\left\{\left(N + \sqrt[p]{p!}\right)^{p-1} - A(p-1)(p-1)!\right\}.$$

Thus, if we put

$$N = \max\left\{\left\lceil \sqrt[p-1]{(Ap-A)(p-1)!} - \sqrt[p]{p!}\right\rceil, 1\right\}$$

and

$$N_i' = ND_i - D_i + 1,$$

then we can obtain $U - E > 0$. This completes the proof.

4 Liftable vector fields for non-singular mono-germs

A mono-germ $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ is *singular* if $\mathrm{rank} Jf(0) < \min\{n, p\}$ holds, where $Jf(0)$ is the Jacobian matrix of f at 0. A mono-germ $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ is *non-singular* if f is not singular.

We identify the number of generators for $\mathrm{Lift}(f)$ for a non-singular mono-germ f.

Proposition 4.1. *Let $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ be non-singular. Then, the number of generators for* $\mathrm{Lift}(f)$ *is*

$$\begin{cases} n + (p-n)(p-n) & (n < p) \\ p & (n \geq p) \end{cases}.$$

Proof. When $n \geq p$ we know that f is $\mathcal{A}$-equivalent to the following form :

$$g(x_1, \ldots, x_n) = (x_1, \ldots, x_p).$$

Then, we can easily check that $\mathrm{Lift}(g) = C_0$. Therefore, the number of generators for $\mathrm{Lift}(f)$ is p.

When $n < p$ we know that f is $\mathcal{A}$-equivalent to the following form :

$$h(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0).$$

Then, we can check that the following vector fields belong to $\mathrm{Lift}(h)$:

$$\begin{array}{cc} \dfrac{\partial}{\partial X_i} & (1 \leq i \leq n) \\ X_{n+i}\dfrac{\partial}{\partial X_{n+j}} & (1 \leq i, j \leq p-n). \end{array}$$

Therefore, we know

$$\left\langle \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}, X_{n+1}\frac{\partial}{\partial X_{n+1}}, X_{n+2}\frac{\partial}{\partial X_{n+1}}, \ldots, X_p\frac{\partial}{\partial X_p} \right\rangle_{C_0}$$

is contained in $\mathrm{Lift}(h)$.

Conversely, for $\xi = (\psi_1(X_1, X_2, \ldots, X_p), \cdots, \psi_p(X_1, X_2, \ldots, X_p)) \in \mathrm{Lift}(h)$, since there exist smooth function germs $\phi_i(x_1, x_2, \ldots, x_n) (i = 1, 2, \ldots, n)$ such that

$$\psi_i(x_1, x_2, \ldots, x_n, 0, 0, \ldots, 0) = \begin{cases} \phi_i(x_1, x_2, \ldots, x_n) & (1 \leq i \leq n) \\ 0 & (n+1 \leq i \leq p) \end{cases},$$

when $n+1 \leq i \leq p$ there exist smooth function germs $\tilde{\psi}_i(X_1, X_2, \ldots, X_p) (i = 1, 2, \ldots, p-n)$ such that

$$\begin{aligned} \psi_i(X_1, X_2, \ldots, X_p) &= \psi_i(X_1, X_2, \ldots, X_n, 0, 0, \ldots, 0) + \tilde{\psi}_1 X_{n+1} + \cdots + \tilde{\psi}_{p-n} X_p \\ &= \tilde{\psi}_1 X_{n+1} + \cdots + \tilde{\psi}_{p-n} X_p. \end{aligned}$$

Therefore, $\xi = (\psi_1(X_1, X_2, \ldots, X_p), \cdots, \psi_p(X_1, X_2, \ldots, X_p)) \in \mathrm{Lift}(h)$ belongs to

$$\left\langle \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}, X_{n+1}\frac{\partial}{\partial X_{n+1}}, X_{n+2}\frac{\partial}{\partial X_{n+1}}, \ldots, X_p\frac{\partial}{\partial X_p} \right\rangle_{C_0}.$$

Thus, the number of generators for $\mathrm{Lift}(f)$ is $n + (p-n)(p-n)$. This completes the proof. □

When f is singular, characterization of the number of generators for $\mathrm{Lift}(f)$ is essentially difficult (see [8]).

5 Liftable vector fields for function mono-germs of one variable

We investigate $\mathrm{Lift}(f)$ for $n = p = 1$.

Proposition 5.1. *Let a smooth function $f : (\mathbb{K}, 0) \to (\mathbb{K}, 0)$ be denoted by $f(x) = \tilde{f}(x)x^n$, where $\tilde{f} : (\mathbb{K}, 0) \to \mathbb{K}$ satisfies $\tilde{f}(0) \neq 0$ and n is an integer greater than 1. Then,* $\mathrm{Lift}(f) = \langle X \rangle_{C_0}$.

Proof. Since $f(x) = \tilde{f}(x)x^n$, we can see easily

$$f'(x) = \tilde{f}'(x)x^n + n\tilde{f}(x)x^{n-1}.$$

At first we show ($\subset$). Since $f(0) = f'(0) = 0$, for $\xi \in \mathrm{Lift}(f)$ $\xi(0) = 0$ holds. Therefore, there exists a function $\psi(X)$ such that $\xi(X) = \psi(X)X$. This means $\mathrm{Lift}(f) \subset \langle X \rangle_{C_0}$.

Next, we show ($\supset$). It is sufficient to show $X \in \mathrm{Lift}(f)$. In fact,

$$\tilde{f}(x)x^n = (\tilde{f}'(x)x^n + n\tilde{f}(x)x^{n-1})\left(\frac{\tilde{f}(x)x}{\tilde{f}'(x)x + n\tilde{f}(x)}\right)$$

holds. Thus, $\mathrm{Lift}(f) \supset \langle X \rangle_{C_0}$. This completes the proof. □

This implies that there exist some cases that we can take non-zero polynomial vector fields liftable over f even though f is not a polynomial. Proposition 5.1 does not hold generally for a flat function f. For example, if

$$f(x) = \begin{cases} x^2 e^{-1/x} & (x > 0) \\ 0 & (x = 0) \\ x^2 e^{1/x} & (x < 0) \end{cases}$$

then X is not liftable. We prove this statement. we show that

$$X \notin \mathrm{Lift}(f).$$

At first, since

$$g(x) = \begin{cases} e^{-1/x} & (x > 0) \\ 0 & (x \leq 0) \end{cases}$$

is a C^{∞} function, so is g(-x). Therefore,

$$h(x)=\begin{cases} e^{-1/x} & (x>0) \\ 0 & (x=0) \\ e^{1/x} & (x<0) \end{cases}$$

is a C^{∞} function. Thus, $f(x)=x^2h(x)$ is a C^{∞} function.

We assume that $X\in \mathrm{Lift}(f)$. Then, a lowerable vector field $\phi(x)$ for a liftable vector field X must be

$$\phi(x)=\begin{cases} \frac{x^2}{2x+1} & (x>0) \\ a & (x=0) \\ \frac{x^2}{2x-1} & (x<0) \end{cases}$$

$(a\in\mathbb{R})$. However, $\phi(x)$ is not class of C^{∞}. Thus, $X\notin \mathrm{Lift}(f)$.

On the other hand, we can show that X^2 is liftable. In fact, we can give a lowerable vector field $\phi(x)$ as follows:

$$\phi(x)=\begin{cases} \frac{x^4e^{-1/x}}{2x+1} & (x>0) \\ 0 & (x=0) \\ \frac{x^4e^{1/x}}{2x-1} & (x<0) \end{cases}.$$

It can be seen easily that $\phi(x)$ is class of C^{∞}. Thus, $X^2\in \mathrm{Lift}(f)$.

The author does not know examples of f such that there exist no polynomial vector fields liftable over f.

References

[1] V. I. Arnol'd, 'Wave front evolution and equivariant Morse lemma', Commun Pure Appl. Math. 29 (1976) 557-582.

[2] J. W. Bruce and J. M. West, 'Function on a crosscap', Math. Proc. Cambridge Philos. Soc. 123 (1998) 19-39.

[3] K. Houston and R. Wik Atique, '$\mathcal{A}$-classification of map-germs via $_V\mathcal{K}$-equivalence', arXiv:1302. 1104v1 [math. AG], 2013.

[4] K. Houston and D. Littlestone, 'Vector fields liftable over corank 1 stable maps', arXiv:0905. 0556v1 [math. DG], 2009.

[5] J. Mather, 'Stability of C^{∞} mappings, III. Finitely determined map-germs', Publ. Math. Inst. Hautes Etudes Sci. 35 (1969) 127-156.

[6] Y. Mizota, 'Construction of generators for the module of liftable vector fields', to appear RIMS Kokyuroku Bessatsu.

[7] Y. Mizota, 'Improving an estimate for the highest degrees of liftable vector fields', preprint, 2013.

[8] T. Nishimura, 'Vector fields liftable over finitely determined multigerms of corank at most one', arXiv:1112. 1214v1 [math. DG], 2011.

[9] C. T. C. Wall, 'Finite determinacy of smooth map-germs', Bull. London Math. Soc. 13 (1981) 481-539.