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NEGATIVELY CURVED MÖBIUS STRIPS ON A GIVEN KNOT

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1. INTRODUCTION

In Euclidean 3-space $\mathbb{R}^3$, a space curve is called *regular* if its velocity vector never vanishes. A closed regular space curve γ in $\mathbb{R}^3$ is called a *knot* (or *simple closed curve*) if it has no self-intersections. We consider the following problem:

Problem. Give a necessary and sufficient condition for the existence of closed strips along a given knot γ which have prescribed negative Gaussian curvature at each point on γ.

The positively curved case has been studied by Gluck-Pan [3]. It has been proved that any positively curved strip along γ admits only one isotopy type. On the other hand, the flat (zero Gaussian curvature) case has been studied in Chicone-Kalton [1], Røgen [8], and in the author's previous work [5]. The purpose of this paper is to report on the outline of the answer [6] for the above problem in the case that γ is real-analytic.

2. PRELIMINARIES

We first define some terminologies. Let $\gamma=\gamma(s):\mathbb{S}^1\longrightarrow\mathbb{R}^3$ be a C^∞-knot with arc-length parameter s, where $\mathbb{S}^1:=\mathbb{R}/l\mathbb{Z}$ and l is the total arc-length of γ. We assume that the curvature function $\kappa(s):=|\gamma''(s)|$ of γ is positive everywhere. The torsion function of γ is defined by

$$\tau:=\frac{\det(\gamma',\gamma'',\gamma''')}{\kappa^2}.$$

Under this definition, for example, the torsion function of the clockwise helix $t\mapsto(\cos t,\ \sin t,\ t)$ is positive. We fix a sufficiently small $\epsilon>0$. The integer

$$\mathrm{SL}(\gamma):=\mathrm{Link}(\gamma(s),\gamma(s)+\epsilon\boldsymbol{n}(s))\in\mathbb{Z} \tag{2.1}$$

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is called the *self-linking number* of γ, where $\mathrm{Link}(\alpha,\beta)$ is the linking number between two knots α and β (cf. [7], [8]).

Let $\sim_+$ (resp. $\sim_-$) be the equivalence relation in $\mathbb{R}\times(-\epsilon,\epsilon)$ generated by

$$(s,u)\sim_\pm(s+l,\pm u)\qquad(s\in\mathbb{R},\ |u|<\epsilon),$$

and set $M_\pm:=\mathbb{R}\times(-\epsilon,\epsilon)/\sim_\pm$. Then, M_+ (resp. M_-) is homeomorphic to a cylinder (resp. a Möbius strip). Suppose that M is either M_+ or M_-. For a sufficiently small $\epsilon>0$, a C^∞-embedding $F: M\longrightarrow\mathbb{R}^3$ satisfying $F(s,0)=\gamma(s)$ for each $s\in\mathbb{S}^1$ is called a *closed strip* along γ. Let B be the boundary of the image of a closed strip F along γ. The number of connected components of B is one or two, and the former case occurs if and only if M is non-orientable. We assign the orientation induced by γ to all (one or two) connected components of B. Then,

$$\mathrm{Mtn}(F):=\frac{1}{2}\mathrm{Link}(\gamma,B)\in(1/2)\mathbb{Z}\tag{2.2}$$

is called the *topological twisting number*, or the *twisting number* for short, of F, where $\mathrm{Link}(\gamma,B)$ is the sum of linking numbers of all connected components of B. The isotopy type of F is determined by the isotopy type of γ and its twisting number. The twisting number $\mathrm{Mtn}(F)$ becomes a half-integer if and only if M is non-orientable. If the twisting direction of F is clockwise, then the twisting number $\mathrm{Mtn}(F)$ is positive.

3. Main theorem

Let $\gamma=\gamma(s):\mathbb{S}^1\longrightarrow\mathbb{R}^3$ be a C^ω-knot (i.e. real-analytic knot) whose curvature function is positive everywhere. In this case, we can give a necessary and sufficient condition for the isotopy types of negatively curved strips on γ, using a prescribed negative Gaussian curvature function $K=K(s)$ along γ. The following assertion is an answer to the problem in the introduction:

Main Theorem([6]). Let $K=K(s):\mathbb{S}^1\longrightarrow(-\infty,0)$ be a negatively valued C^ω-function. Define two C^ω-functions

$$b_K^-(s):=-\tau(s)-\sqrt{|K(s)|},\quad b_K^+(s):=-\tau(s)+\sqrt{|K(s)|}\qquad(s\in\mathbb{S}^1).$$

Denote by B_K^- (resp. B_K^+) the minimum of $b_K^-(s)$ (resp. maximum of $b_K^+(s)$), and define the open interval $I_K:=(B_K^-,B_K^+)$.

(1) Suppose $I_K\subset(0,\infty)$. Then,

  (i) for each $n\geq\mathrm{SL}(\gamma)$ ($n\in(1/2)\mathbb{Z}$), there exists a real-analytic closed strip along γ such that its Gaussian curvature at each point on $\gamma(s)$ is $K(s)$ and its topological twisting number is n. On the other hand,

(ii) for each $n < \mathrm{SL}(\gamma)$ ($n \in (1/2)\mathbb{Z}$), there does not exist such a strip.

(2) Suppose $0 \in I_K$. Then, for each $n \in (1/2)\mathbb{Z}$, there exists a real-analytic closed strip along γ such that its Gaussian curvature at each point on $\gamma(s)$ is $K(s)$ and its topological twisting number is n.

(3) Suppose $I_K \subset (-\infty, 0)$. Then,

(i) for each $n \leq \mathrm{SL}(\gamma)$ ($n \in (1/2)\mathbb{Z}$), there exists a real-analytic closed strip along γ such that its Gaussian curvature at each point on $\gamma(s)$ is $K(s)$ and its topological twisting number is n. On the other hand,

(ii) for each $n > \mathrm{SL}(\gamma)$ ($n \in (1/2)\mathbb{Z}$), there does not exist such a strip.

Moreover, in (1)–(i), (2) and (3)–(i), the closed strip can be constructed as a ruled surface.

Any pair of C^ω-knot γ and negatively valued C^ω-function f satisfies one of the three cases of $I_K \subset (0, \infty)$, $0 \in I_K$ or $I_K \subset (-\infty, 0)$. On the other hand, if γ and K are of class C^∞, instead of C^ω, then it seems difficult to describe such a necessary and sufficient condition. However, if $B_K^- \neq 0$ and $B_K^+ \neq 0$, then the same assertion as the Main Theorem holds, as follows:

Corollary 3.1. *Let $\gamma(s)$ be a C^∞-knot with positive curvature, and $K(s)$ a negatively valued C^∞-function. If $I_K \subset (0, \infty)$ and $B_K^- \neq 0$, then the item* (1) *holds as in the Main Theorem. Similarly, if $I_K \subset (-\infty, 0)$ and $B_K^+ \neq 0$, then the item* (3) *holds. On the other hand, if $0 \in I_K$, then the item* (2) *holds without any modifications.*

Example 3.2. We consider twisting numbers of closed strips along the unit circle

$$\gamma_0(s) := (\cos s, \sin s, 0) \quad (s \in \mathbb{R})$$

which have Gaussian curvature $K(s) = -1$ at each point on γ_0. Since the torsion function of γ_0 vanishes identically, we can calculate $b_K^-(s) = -1$ and $b_K^+(s) = 1$ for $s \in \mathbb{R}$. Therefore, we have $I_K = (-1, 1)$. By (2) of the Main Theorem, for any twisting number $n \in (1/2)\mathbb{Z}$, we can construct a strip containing γ_0 and having Gaussian curvature -1 along γ_0. For example, the four pictures in Figure 1 are all such examples which have twisting number $n = -1/2,\ 0,\ 1/2,\ 1$, respectively. The first figure is the reflection of the Möbius strip of the third example (which will be defined soon) with respect to the xy-plane. The second figure is the image of

$$F_0(s,u) := \gamma_0(s) + u(\sin s, -\cos s, 1) \quad (s \in \mathbb{R}, |u| < 1/5),$$

which is a part of a one-sheeted hyperboloid. The fourth figure is the image of

$$F_1(s,u) = \gamma_0(s) + u(-\cos^2 s, -\cos s \sin s, \sin s) \quad (s \in \mathbb{R}, |u| < 1/5).$$

This example is a ruled closed strip generated by a ruling vector which twists just once clockwise at constant speed when γ_0 is traversed once. The third figure is a ruled Möbius strip defined by

$$F_{1/2}(s,u) = \gamma_0(s) + u\xi(s) \quad (s \in \mathbb{R}, |u| < 1/5),$$

where

$$\xi(s) := p(s)\boldsymbol{e}(s) + \cos\theta(s)\boldsymbol{n}(s) + \sin\theta(s)\boldsymbol{b}(s),$$
$$p(s) := \frac{-1+\cos s}{2\sin\theta(s)}, \quad \theta(s) := \frac{s+\sin s}{2},$$

and $\{\boldsymbol{e}, \boldsymbol{n}, \boldsymbol{b}\}$ is the Frenet frame of γ. Since the pair of the two functions $p(s)$ and $\theta(s)$ satisfies the three conditions (4.9), (4.11), (4.12), this strip is a Möbius strip with Gaussian curvature -1 at each point on γ_0. We remark that $p(s)$ can be also real-analytic at $s = 0 \pmod{2\pi}$, so $p(s)$ is well-defined on $\mathbb{R}$. To find such functions $p(s)$ and $\theta(s)$, we must control the jets of these functions at all points where $\sin\theta(s) = 0$.

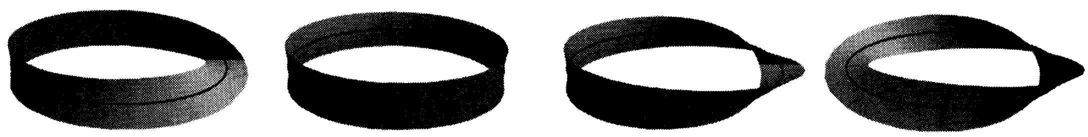

FIGURE 1. Closed ruled strips containing γ_0 which have Gaussian curvature -1 along γ_0. Their twisting numbers are $-1/2$, 0, $1/2$, 1, respectively.

Example 3.3. The torus knot defined by

$$\gamma(t) := \left(\cos t - \frac{1}{10}\cos t\cos 5t, \sin t - \frac{1}{10}\cos 5t \sin t, -\frac{1}{10}\sin 5t\right)$$

coils 5 times anti-clockwise around a solid torus as in Figure 2. The curvature function $\kappa(s)$ of γ does not vanish, and the torsion function τ of γ is less than -1. Since the knot $t \mapsto \gamma(t) + \epsilon\boldsymbol{n}(s)$ is isotopic to the unit circle centered the origin in the xy-plane, the self-linking number $\mathrm{SL}(\gamma)$ is equal to -5, where $\epsilon > 0$ is sufficiently small and $\boldsymbol{n}$ is the principal normal vector field of γ.

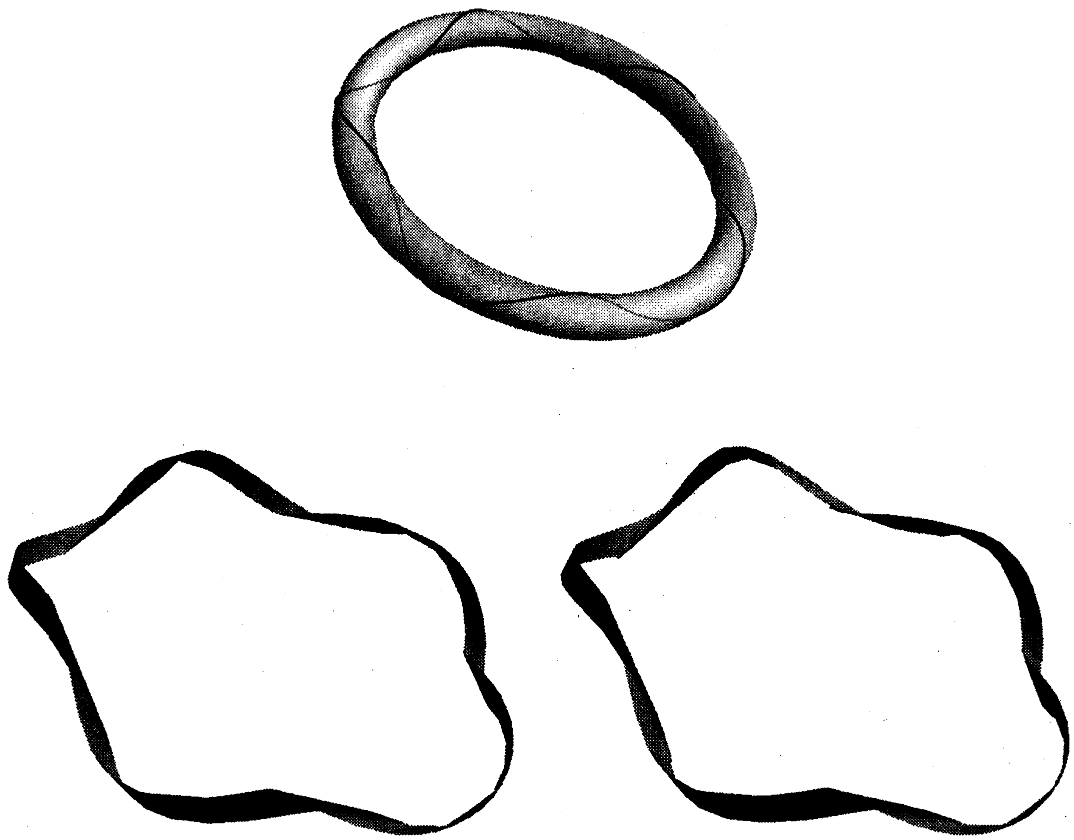

FIGURE 2. The first figure is the image of γ. The second and third figures are an orientable strip and a Möbius strip along γ which have Gaussian curvature -1 at each point on γ and twisting numbers -5 and $-9/2$, respectively. We can see a joint of the Möbius strip in the top center of the third figure.

Now, we set $K(s) = -1$ for $s \in \mathbb{S}^1$. Then, the function $b_K^{\pm}(s)$ is positive everywhere. Hence, the pair of γ and K satisfies $I_K \subset (0, \infty)$. By the item (1)–(i) in the Main Theorem, for each $n \geq \mathrm{SL}(\gamma) = -5$ ($n \in (1/2)\mathbb{Z}$), there exists a closed strip F along γ such that the Gaussian curvature of F on $\gamma(s)$ is equal to -1 and the twisting number $\mathrm{Mtn}(F)$ is n. For example, the second and third surfaces of Figure 2 are an orientable strip and a Möbius strip along γ in the cases of $n = -5$ and $n = -9/2$, respectively. On the other hand, by (1)–(ii), for each $n < -5$ ($n \in (1/2)\mathbb{Z}$), there does not exist such a strip along γ whose Gaussian curvature is -1 at each point on γ.

4. Outline of the proof

Let γ be a given C^ω-knot with arc-length parameter s whose curvature function $\kappa(s)$ never vanishes. Any closed strip $F : M \longrightarrow \mathbb{R}^3$ along γ can be expressed by

$$F(s,u) = \gamma(s) + u\xi(s) + u^2\eta(s,u) \quad (s \in \mathbb{R},\ |u| < \epsilon), \tag{4.1}$$

where $\xi(s)$ is a C^ω-vector field along γ and $\eta(s,u)$ is a C^ω-vector field along F. The strip F is ruled if $\eta(s,u)$ vanishes identically. The periodicity condition that the strip F is closed is equivalent to

$$\xi(s+l) = \pm\xi(s), \quad \eta(s+l,u) = \eta(s,\pm u) \qquad (s \in \mathbb{R},\ |u| < \epsilon) \tag{4.2}$$

for $M = M_\pm$, respectively. The condition that F gives an embedding is equivalent to

$$\gamma'(s) \times \xi(s) \neq 0 \qquad (s \in \mathbb{R}), \tag{4.3}$$

where $\times$ is the vector product in $\mathbb{R}^3$. Given $\xi(s)$ and $\eta(s,u)$ satisfying (4.2) and (4.3), a closed strip can be constructed by (4.1).

From now on, we may assume that any strip F is expressed as in the form in (4.1). By a suitable parameter change, we can normalize $\xi(s)$ so that

$$|\gamma'(s) \times \xi(s)| = 1 \qquad (s \in \mathbb{R}). \tag{4.4}$$

Then, the Gaussian curvature $K(s)$ of F along $\gamma(s)$ can be calculated to be

$$K = 2\det(\gamma',\xi,\gamma'')\det(\gamma',\xi,\eta_0) - \det(\gamma',\xi,\xi')^2, \tag{4.5}$$

where $\eta_0(s) := \eta(s,0)$ for $s \in \mathbb{R}$. Let $\{\boldsymbol{e}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)\}$ be the Frenet frame of γ. The vector field $\xi(s)$ can be expressed as a linear combination of $\{\boldsymbol{e}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)\}$ by

$$\xi(s) = p(s)\boldsymbol{e}(s) + \cos\theta(s)\boldsymbol{n}(s) + \sin\theta(s)\boldsymbol{b}(s) \quad (s \in \mathbb{R}), \tag{4.6}$$

where $p(s)$ and $\theta(s)$ are certain C^ω-functions satisfying

$$p(s+l) = (-1)^m p(s), \quad \theta(s+l) = \theta(s) + m\pi \qquad (s \in \mathbb{R}) \tag{4.7}$$

for some $m \in \mathbb{Z}$. The number

$$\mathrm{Gtn}(F) := m/2 \in (1/2)\mathbb{Z}$$

is called the *geometric twisting number* of F. The topological twisting number $\mathrm{Mtn}(F)$ and geometric twisting number $\mathrm{Gtn}(F)$ satisfy

$$\mathrm{Mtn}(F) = \mathrm{SL}(\gamma) + \mathrm{Gtn}(F). \tag{4.8}$$

Namely, we can use the geometric twisting number $\mathrm{Gtn}(F)$, instead of the topological twisting number $\mathrm{Mtn}(F)$ when we consider the isotopy type of F. Using (4.6), the right-hand side of (4.5) can be calculated to be

$$K = -q\kappa \sin\theta - (\theta' - p\kappa\sin\theta + \tau)^2, \tag{4.9}$$

where

$$q(s) := 2\det(\gamma'(s), \xi(s), \eta_0(s)). \tag{4.10}$$

By using (4.9), we can reduce the proof of the Main Theorem into considering the existence and non-existence of C^ω-functions $p(s)$, $\theta(s)$ and $q(s)$ satisfying the following conditions:

$$q(s)\kappa(s)\sin\theta(s) + K(s) \le 0 \quad (s \in \mathbb{R}) \tag{4.11}$$

is satisfied, and for each $s \in \mathbb{R}$, either

$$\theta'(s) = p(s)\kappa(s)\sin\theta(s) - \tau(s) + \sqrt{|q(s)\kappa(s)\sin\theta(s) + K(s)|} \tag{4.12}$$

or

$$\theta'(s) = p(s)\kappa(s)\sin\theta(s) - \tau(s) - \sqrt{|q(s)\kappa(s)\sin\theta(s) + K(s)|} \tag{4.13}$$

holds. In fact, for a pair of $(\gamma, K) : \mathbb{S}^1 \longrightarrow \mathbb{R}^3 \times (-\infty, 0)$ and a prescribed integer m, if we take C^ω-functions $p(s)$, $q(s)$ and $\theta(s)$ satisfying (4.7), (4.11), (4.12) (or (4.13)), and define $\xi(s)$ by (4.6) and $\eta(s,u) := q(s)\ \boldsymbol{e}(s) \times \xi(s)$, then these three functions satisfy (4.9). Hence, the strip F defined by (4.1) is a closed strip along γ so that its Gaussian curvature at each point on $\gamma(s)$ is equal to $K(s)$ and the geometric twisting number $\operatorname{Gtn}(F)$ is $m/2$. In order to construct such functions, we use a special technique to approximate a C^∞-function by a Fourier polynomial while we fix the jet of the function at finitely many points. This technique was given in the author's previous paper [5, Lemma A.4]. Fortunately, it can be applied also for negatively curved closed strips along γ.

5. Remaining problems

Let F_1 and F_2 be two negatively curved strips. We write $F_1 \approx_- F_2$ if they can be deformed into one another through negatively curved surfaces, and write $F_1 \approx F_2$ if they are isotopic. Although $F_1 \approx_- F_2$ implies $F_1 \approx F_2$, its converse is not true. In fact, the following examples are two negatively curved strips which have the same isotopy type, but they cannot be deformed into one another through negatively curved surfaces:

Example 5.1 ([6]). Two negatively curved strips F_1, F_2 are defined by

$$F_1(s,u) := f_0(s,u) \quad (|s| \leq \pi,\ |u| < 1/2),$$
$$F_2(s,u) := f_0(u\cos s, u\sin s) \quad (|s| \leq \pi,\ 1/2 < u < 1),$$

respectively, where $f_0(x,y) := (\sqrt{1+y^2}\cos x, \sqrt{1+y^2}\sin x, y)$ is a one-sheeted hyperboloid. Figure 3 shows the images of F_1 and F_2, which are described with curvature lines of f_0.

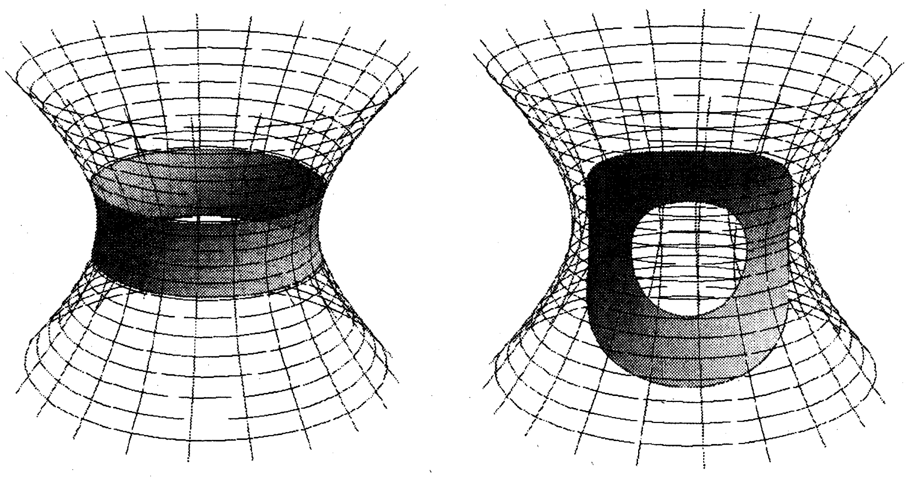

FIGURE 3. An example of $F_1 \approx F_2$ and $F_1 \not\approx_- F_2$

If a negatively curved strip F is orientable, then we can define the *rotational index* of the principal direction of curvature of F along γ in the tangent planes of F when γ is traversed once (cf. Røgen [8]). The rotational index does not change under deformations of F keeping negatively curved surfaces. On the other hand, F_1 and F_2 are isotopic, but their rotational indices are 0 and ± 1. Therefore, they cannot be deformed into one another through negatively curved surfaces.

Røgen [8] conjectured that two negatively curved surfaces having the same isotopy type can be deformed into one another through negatively curved surfaces if and only if their rotational indices are equal. Recently, Ghomi-Kossowski [2] solved this conjecture affirmatively by using the h-principle. Therefore, it is interesting to consider the following problem:

Problem. Give a necessary and sufficient condition for the existence of negatively curved closed strips along a given knot which have prescribed twisting numbers and rotational indices of principal curvatures.

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