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# Local theory of singularities of two functions and the product map

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## 1 Introduction

In this paper we prove some fundamental lemmas used in the author's talk at the workshop "Pursuit of the essence of singularity theory", while we refer the reader to [3] for the main results. The subject of the talk was estimating distances, naturally defined from a topological viewpoint, between two Morse functions on a manifold. The method for it was reading information of the two functions from the discriminant set of the product map, based on the lemmas proved in this paper.

We use the following notation. Suppose $M$ is a smooth $n$-manifold with $n \geq 2$, and $P, Q$ are oriented smooth 1-manifolds. Let $F : M \to P$ and $G : M \to Q$ be smooth functions, and let $\varphi : M \to P \times Q$ denote the product map of $F, G$, that is to say, $\varphi(x) = (F(x), G(x))$ for $x \in M$. Suppose $p \in M$ is either a fold point or a cusp point of $\varphi$, and $U \subset M$ is a small neighborhood of $p$. Let $\sigma \subset U$ denote the singular set of $\varphi|_U$, namely the set of singular points of $\varphi$ in $U$.

We postpone detailed descriptions of fold points and cusp points until Section 2, but just note that the discriminant set $\varphi(\sigma) \subset P \times Q$ is a smooth curve possibly with an ordinary cusp.

We analyze the curve $\varphi(\sigma)$. Note that the product structure of $P \times Q$ gives a local coordinate system $(f, g)$ at $\varphi(p)$. It allows us to define the *slope* of $\varphi(\sigma)$ at $\varphi(p)$. In particular, $\varphi(p)$ is called a *horizontal* (resp. *vertical*) point of $\varphi(\sigma)$ if the slope is zero (resp. infinity). We can also define the second derivative of $\varphi(\sigma)$ at $\varphi(p)$ if $\varphi(p)$ is not a vertical point nor a cusp. In particular, $\varphi(p)$ is called an *inflection* point of $\varphi(\sigma)$ if the second derivative is zero. Since zero or non-zero of the second derivative is preserved by rotating the coordinate system, the notion of inflection can be defined even if $\varphi(p)$ is a vertical point.

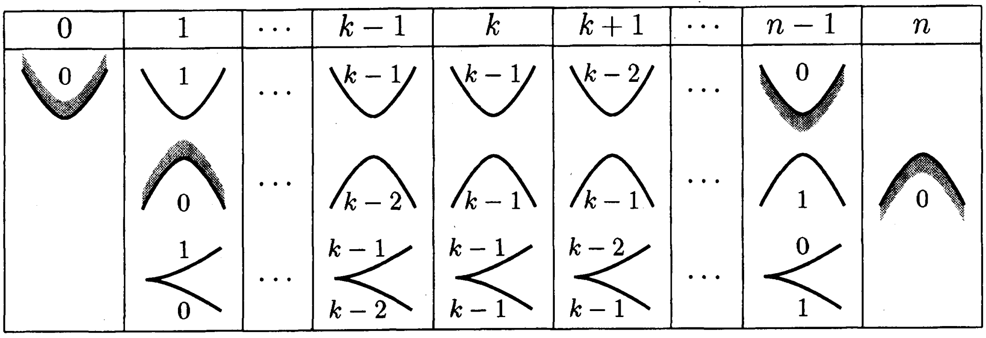
There is a correspondence between properties of the curve $\varphi(\sigma)$ at $\varphi(p)$ and properties of the functions $F, G$ at $p$ as the following.

**Lemma 1.** *The point $p$ is a critical point of $G$ (resp. $F$) if and only if $\varphi(p)$ is a horizontal (resp. vertical) point of $\varphi(\sigma)$.*

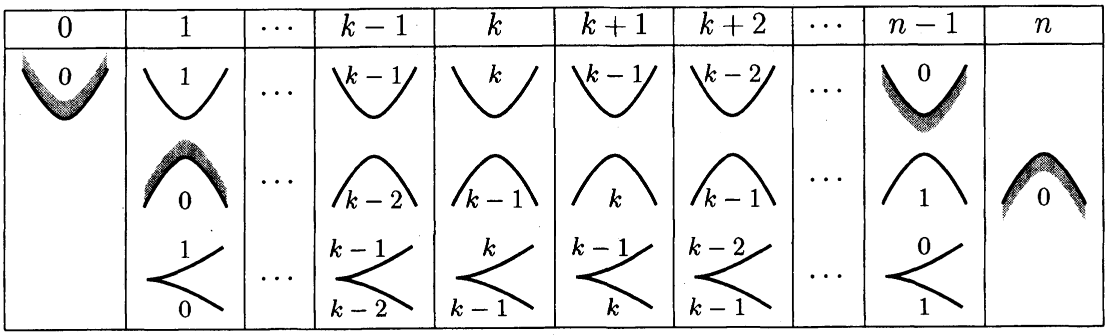
**Lemma 2.** *The point $p$ is a degenerate critical point of $G$ (resp. $F$) if and only if $p$ is a fold point of $\varphi$ and $\varphi(p)$ is a horizontal (resp. vertical) inflection point of $\varphi(\sigma)$.*

**Lemma 3.** *Suppose $p$ is a non-degenerate critical point of $G$. The index of $p$ is related to the type of the horizontal point $\varphi(p)$ of $\varphi(\sigma)$ as the following tables. The symmetrical holds for $F$.*

if $n = 2k$ for an integer $k \geq 1$




if $n = 2k + 1$ for an integer $k \geq 1$




In each of these tables, the first row shows the index of the non-degenerate critical point $p$, and the second row shows possible local pictures of $\varphi(\sigma)$ near the horizontal point $\varphi(p)$. We draw them so that the $f$-axis is horizontal and the coordinate $g$ increases from bottom to top. The number noted to each branch of $\varphi(\sigma)$ is the absolute index of the corresponding fold points. When $p$ is a fold point of absolute index 0, the image $\varphi(U)$ is shown in gray.

These lemmas are generalizations of what described by Johnson in [1, Section 6]. Johnson considered the case where $M$ is a closed orientable 3-manifold and $P = Q = \mathbb{R}$, and used it for comparing two Heegaard splittings of $M$. The author [2, Section 5] gave simple analytic proofs of Johnson's assertions, and we straightforwardly generalize them for the proofs of Lemmas 1, 2 and 3.

## 2 Folds and cusps

In this section, we review standard facts about fold points and cusp points of a smooth map $\varphi : M \to S$. Here, $M$ is a smooth $n$-manifold with $n \geq 2$, and $S$ is a smooth 2-manifold. In fact, generic singular points of $\varphi$ are classified into fold points and cusp points.

A *fold point* of $\varphi$ is a singular point $p \in M$ with the form

$$\begin{cases} (s \circ \varphi)(x_1, x_2, \dots, x_n) = x_1 \\ (t \circ \varphi)(x_1, x_2, \dots, x_n) = -x_2^2 - \cdots - x_{\lambda+1}^2 + x_{\lambda+2}^2 + \cdots + x_n^2 \end{cases} \tag{1}$$

for a coordinate system $(x_1, x_2, \dots, x_n)$ of a neighborhood $U$ of $p$ and a local coordinate system $(s, t)$ at $\varphi(p)$. The minimum of $\{\lambda,\ n - \lambda - 1\}$ does not depend on the choice of coordinate systems, and is called the *absolute index* of $p$. We can assume that $\lambda$ is the absolute index, namely $\lambda \leq \frac{n-1}{2}$, by reversing the coordinates if necessary. The singular set $\sigma$ of $\varphi|_U$ is the $x_1$-axis as the Jacobian matrix says. We can see that every singular point on $\sigma$ is also a fold point of absolute index $\lambda$ by translating the coordinates. The discriminant set $\varphi(\sigma)$ is the image of the $x_1$-axis in $\varphi$, that is, the $s$-axis. In particular, if $\lambda = 0$, the image $\varphi(U)$ is contained in the upper half $\{(s, t) \mid t \geq 0\}$.

A *cusp point* of $\varphi$ is a singular point $p \in M$ with the form

$$\begin{cases} (s \circ \varphi)(x_1, x_2, \dots, x_n) = x_1 \\ (t \circ \varphi)(x_1, x_2, \dots, x_n) = x_1 x_2 - x_2^3 - x_3^2 - \cdots - x_{\lambda+2}^2 + x_{\lambda+3}^2 + \cdots + x_n^2. \end{cases} \tag{2}$$

The minimum of $\{\lambda,\ n - \lambda - 2\}$ does not depend on the choice of coordinate systems. We can assume that $\lambda$ is the minimum, namely $\lambda \leq \frac{n-2}{2}$. The singular set $\sigma$ is the smooth regular curve $\{(3x_2^2, x_2, 0, \dots, 0)\}$. The branch $\sigma_- = \{(3x_2^2, x_2, 0, \dots, 0) \mid x_2 < 0\}$ consists of fold points of absolute index $\lambda$. The other branch $\sigma_+ = \{(3x_2^2, x_2, 0, \dots, 0) \mid x_2 > 0\}$ consists of fold points of absolute index $\lambda + 1$ except when $n$ is even and $\lambda = \frac{n-2}{2}$. In the exceptional case, both $\sigma_-$ and $\sigma_+$ consist of fold points of absolute index $\lambda$. The discriminant set $\varphi(\sigma)$ is the smooth curve $\{(s, t) = (3x_2^2,\ 2x_2^3)\}$. It has an ordinary cusp at $\varphi(p) = (0, 0)$, and the tangent line of $\varphi(\sigma)$ at the cusp is the $s$-axis. Separated by the $s$-axis, the lower side $\{(s, t) \mid t < 0\}$ contains the branch $\varphi(\sigma_-)$, and the upper side $\{(s, t) \mid t > 0\}$ contains the other branch $\varphi(\sigma_+)$.

# 3 Proofs

For the proofs of Lemmas 1, 2 and 3, we calculate the gradient vector and the Hessian matrix of $G$ from local forms of $\varphi$. On one hand, $\varphi$ has the form (1) or (2) for a coordinate system $(x_1, x_2, \dots, x_n)$ of a neighborhood $U$ of $p$ and a local coordinate system $(s, t)$ at $\varphi(p)$. On the other hand, by the definition of the product map, $\varphi$ has the form

$$\begin{cases} (f \circ \varphi)(x_1, x_2, \dots, x_n) = F(x_1, x_2, \dots, x_n) \\ (g \circ \varphi)(x_1, x_2, \dots, x_n) = G(x_1, x_2, \dots, x_n) \end{cases} \tag{3}$$

for the coordinate system $(f, g)$ given by the product structure of $P \times Q$. Note that there is a smooth regular coordinate transformation

$$\begin{cases} f = f(s, t) \\ g = g(s, t). \end{cases} \tag{4}$$

## 3.1 Cusp Case

We first deal with the case where $p$ is a cusp point of $\varphi$. The forms (2), (3), (4) and the chain rule gives

$$\begin{aligned}
\frac{\partial G}{\partial x_1} &= \frac{\partial s}{\partial x_1}\frac{\partial g}{\partial s} + \frac{\partial t}{\partial x_1}\frac{\partial g}{\partial t} \\
&= \frac{\partial}{\partial x_1}(x_1)\frac{\partial g}{\partial s} + \frac{\partial}{\partial x_1}(x_1x_2 - x_2^3 - x_3^2 - \cdots - x_{\lambda+2}^2 + x_{\lambda+3}^2 + \cdots + x_n^2)\frac{\partial g}{\partial t} \\
&= \frac{\partial g}{\partial s} + x_2\frac{\partial g}{\partial t}, \\
\frac{\partial}{\partial x_2} &= \frac{\partial s}{\partial x_2}\frac{\partial}{\partial s} + \frac{\partial t}{\partial x_2}\frac{\partial}{\partial t} \\
&= \frac{\partial}{\partial x_2}(x_1)\frac{\partial}{\partial s} + \frac{\partial}{\partial x_2}(x_1x_2 - x_2^3 - x_3^2 - \cdots - x_{\lambda+2}^2 + x_{\lambda+3}^2 + \cdots + x_n^2)\frac{\partial}{\partial t} \\
&= (x_1 - 3x_2^2)\frac{\partial}{\partial t}, \\
\frac{\partial^2 G}{\partial x_1 \partial x_2} &= \frac{\partial}{\partial x_2}\frac{\partial G}{\partial x_1} \\
&= \frac{\partial}{\partial x_2}\left(\frac{\partial g}{\partial s} + x_2\frac{\partial g}{\partial t}\right) \\
&= \frac{\partial}{\partial x_2}\left(\frac{\partial g}{\partial s}\right) + \frac{\partial}{\partial x_2}(x_2)\frac{\partial g}{\partial t} + x_2\frac{\partial}{\partial x_2}\left(\frac{\partial g}{\partial t}\right) \\
&= (x_1 - 3x_2^2)\frac{\partial}{\partial t}\left(\frac{\partial g}{\partial s}\right) + \frac{\partial g}{\partial t} + x_2(x_1 - 3x_2^2)\frac{\partial}{\partial t}\left(\frac{\partial g}{\partial t}\right) \\
&= \frac{\partial g}{\partial t} + (x_1 - 3x_2^2)\frac{\partial^2 g}{\partial s \partial t} + x_2(x_1 - 3x_2^2)\frac{\partial^2 g}{\partial t^2}.
\end{aligned}$$

By similar calculations,

$$\frac{\partial G}{\partial x_i} = \begin{cases} \dfrac{\partial g}{\partial s} + x_2\dfrac{\partial g}{\partial t} & (i = 1) \\ (x_1 - 3x_2^2)\dfrac{\partial g}{\partial t} & (i = 2) \\ -2x_i\dfrac{\partial g}{\partial t} & (3 \leq i \leq \lambda + 2) \\ 2x_i\dfrac{\partial g}{\partial t} & (\lambda + 3 \leq i \leq n), \end{cases}$$

$$\frac{\partial^2 G}{\partial x_i^2} = \begin{cases} \dfrac{\partial^2 g}{\partial s^2} + 2x_2\dfrac{\partial^2 g}{\partial s \partial t} + x_2^2\dfrac{\partial^2 g}{\partial t^2} & (i = 1) \\ -6x_2\dfrac{\partial g}{\partial t} + (x_1 - 3x_2^2)^2\dfrac{\partial^2 g}{\partial t^2} & (i = 2) \\ -2\dfrac{\partial g}{\partial t} + 4x_i^2\dfrac{\partial^2 g}{\partial t^2} & (3 \leq i \leq \lambda + 2) \\ 2\dfrac{\partial g}{\partial t} + 4x_i^2\dfrac{\partial^2 g}{\partial t^2} & (\lambda + 3 \leq i \leq n), \end{cases}$$

$$\frac{\partial^2 G}{\partial x_i \partial x_j} = \begin{cases} \dfrac{\partial g}{\partial t} + (x_1 - 3x_2^2)\dfrac{\partial^2 g}{\partial s \partial t} + x_2(x_1 - 3x_2^2)\dfrac{\partial^2 g}{\partial t^2} & (i = 1,\ j = 2) \\ -2x_j \dfrac{\partial^2 g}{\partial s \partial t} - 2x_2 x_j \dfrac{\partial^2 g}{\partial t^2} & (i = 1,\ 3 \le j \le \lambda + 2) \\ 2x_j \dfrac{\partial^2 g}{\partial s \partial t} + 2x_2 x_j \dfrac{\partial^2 g}{\partial t^2} & (i = 1,\ \lambda + 3 \le j \le n) \\ -2x_j(x_1 - 3x_2^2)\dfrac{\partial^2 g}{\partial t^2} & (i = 2,\ 3 \le j \le \lambda + 2) \\ 2x_j(x_1 - 3x_2^2)\dfrac{\partial^2 g}{\partial t^2} & (i = 2,\ \lambda + 3 \le j \le n) \\ 4x_i x_j \dfrac{\partial^2 g}{\partial t^2} & (3 \le i < j \le \lambda + 2) \\ -4x_i x_j \dfrac{\partial^2 g}{\partial t^2} & (3 \le i \le \lambda + 2 < j \le n) \\ 4x_i x_j \dfrac{\partial^2 g}{\partial t^2} & (\lambda + 3 \le i < j \le n). \end{cases}$$

The gradient vector of $G$ at $p = (0, 0, \dots, 0)$ is

$$\left(\left(\frac{\partial G}{\partial x_1}\right)_p, \left(\frac{\partial G}{\partial x_2}\right)_p, \dots, \left(\frac{\partial G}{\partial x_n}\right)_p\right) = \left(\left(\frac{\partial g}{\partial s}\right)_{\varphi(p)}, 0, \dots, 0\right).$$

The point $p$ is a critical point of $G$ if and only if this vector is zero, namely $\left(\frac{\partial g}{\partial s}\right)_{\varphi(p)} = 0$. It means that the $s$-axis is parallel to the $f$-axis at $\varphi(p)$. Recall that the $s$-axis is the tangent line of $\varphi(\sigma)$ at the cusp $\varphi(p)$. This finishes the proof of Lemma 1 in the case where $p$ is a cusp point.

The Hessian matrix of $G$ at $p = (0, 0, \dots, 0)$ is

$$\begin{pmatrix} \left(\frac{\partial^2 G}{\partial x_1^2}\right)_p & \left(\frac{\partial^2 G}{\partial x_1 \partial x_2}\right)_p & \cdots & \left(\frac{\partial^2 G}{\partial x_1 \partial x_n}\right)_p \\ \left(\frac{\partial^2 G}{\partial x_2 \partial x_1}\right)_p & \left(\frac{\partial^2 G}{\partial x_2^2}\right)_p & \cdots & \left(\frac{\partial^2 G}{\partial x_2 \partial x_n}\right)_p \\ \vdots & \vdots & & \vdots \\ \left(\frac{\partial^2 G}{\partial x_n \partial x_1}\right)_p & \left(\frac{\partial^2 G}{\partial x_n \partial x_2}\right)_p & \cdots & \left(\frac{\partial^2 G}{\partial x_n^2}\right)_p \end{pmatrix}$$

$$= \begin{pmatrix} \left(\frac{\partial^2 g}{\partial s^2}\right)_{\varphi(p)} & \left(\frac{\partial g}{\partial t}\right)_{\varphi(p)} & & & & & \\ \left(\frac{\partial g}{\partial t}\right)_{\varphi(p)} & 0 & & & & & \\ & & -2\left(\frac{\partial g}{\partial t}\right)_{\varphi(p)} & & & & \\ & & & \ddots & & & \\ & & & & -2\left(\frac{\partial g}{\partial t}\right)_{\varphi(p)} & & \\ & & & & & 2\left(\frac{\partial g}{\partial t}\right)_{\varphi(p)} & \\ & & & & & & \ddots \\ & & & & & & & 2\left(\frac{\partial g}{\partial t}\right)_{\varphi(p)} \end{pmatrix}$$

and its determinant is $(-1)^{\lambda+1}2^{n-2}\left(\frac{\partial g}{\partial t}\right)^n_{\varphi(p)}$. We suppose that $p$ is a critical point of $G$, and hence $\left(\frac{\partial g}{\partial s}\right)_{\varphi(p)} = 0$. It requires $\left(\frac{\partial g}{\partial t}\right)_{\varphi(p)} \neq 0$ since the regular coordinate transformation

(4) satisfies $\frac{\partial f}{\partial s}\frac{\partial g}{\partial t} - \frac{\partial f}{\partial t}\frac{\partial g}{\partial s} \neq 0$. It follows that the determinant $(-1)^{\lambda+1}2^{n-2}\left(\frac{\partial g}{\partial t}\right)^n_{\varphi(p)}$ is not zero, that is to say, the critical point $p$ is non-degenerate. This finishes the proof of Lemma 2 in the case where $p$ is a cusp point.

We consider the index of $p$, which is the sum of the multiplicities of negative eigenvalues of the Hessian matrix. The first two eigenvalues are the solutions of $\alpha$ for the equation $\alpha\left\{\alpha - \left(\frac{\partial^2 g}{\partial s^2}\right)_{\varphi(p)}\right\} = \left(\frac{\partial g}{\partial t}\right)^2_{\varphi(p)}$. Noting that $\left(\frac{\partial g}{\partial t}\right)_{\varphi(p)} \neq 0$, the two eigenvalues have opposite signs. The rest eigenvalues are $-2\left(\frac{\partial g}{\partial t}\right)_{\varphi(p)}$ and $2\left(\frac{\partial g}{\partial t}\right)_{\varphi(p)}$, whose multiplicities are $\lambda$ and $n-\lambda-2$, respectively. The index of $p$ is $\lambda+1$ if $\left(\frac{\partial g}{\partial t}\right)_{\varphi(p)}$ is positive, and the index of $p$ is $n-\lambda-1$ if $\left(\frac{\partial g}{\partial t}\right)_{\varphi(p)}$ is negative. In particular, when $n$ is even and $\lambda = \frac{n-2}{2}$, the index of $p$ is $\lambda+1$ regardless of the sign of $\left(\frac{\partial g}{\partial t}\right)_{\varphi(p)}$. Recall that in this case, both the two branches of $\sigma$ are of absolute index $\lambda$. In the other cases, we can assume $\lambda < \frac{n-2}{2}$ by reversing the coordinates if necessary. Recall that in these cases, one branch $\sigma_-$ of $\sigma$ is of absolute index $\lambda$ and the other $\sigma_+$ is of absolute index $\lambda+1$. Recall also that, separated by the tangent line of $\varphi(\sigma)$ at the cusp, $\varphi(\sigma_-)$ lies in the lower side and $\varphi(\sigma_+)$ lies in the upper side with respect to the coordinate $t$. With respect to the coordinate $g$, the same holds if $\left(\frac{\partial g}{\partial t}\right)_{\varphi(p)}$ is positive, and the opposite holds if $\left(\frac{\partial g}{\partial t}\right)_{\varphi(p)}$ is negative. This finishes the proof of Lemma 3 in the case where $p$ is a cusp point.

## 3.2 Fold Case

If $p$ is a fold point of $\varphi$, the forms (1), (3), (4) and the chain rule gives

$$\frac{\partial G}{\partial x_i} = \begin{cases} \dfrac{\partial g}{\partial s} & (i=1) \\ -2x_i\dfrac{\partial g}{\partial t} & (2 \leq i \leq \lambda+1) \\ 2x_i\dfrac{\partial g}{\partial t} & (\lambda+2 \leq i \leq n), \end{cases} \qquad \frac{\partial^2 G}{\partial x_i^2} = \begin{cases} \dfrac{\partial^2 g}{\partial s^2} & (i=1) \\ -2\dfrac{\partial g}{\partial t} + 4x_i^2\dfrac{\partial^2 g}{\partial t^2} & (2 \leq i \leq \lambda+1) \\ 2\dfrac{\partial g}{\partial t} + 4x_i^2\dfrac{\partial^2 g}{\partial t^2} & (\lambda+2 \leq i \leq n), \end{cases}$$

$$\frac{\partial^2 G}{\partial x_i \partial x_j} = \begin{cases} -2x_j\dfrac{\partial^2 g}{\partial s\partial t} & (i=1,\ 2 \leq j \leq \lambda+1) \\ 2x_j\dfrac{\partial^2 g}{\partial s\partial t} & (i=1,\ \lambda+2 \leq j \leq n) \\ 4x_ix_j\dfrac{\partial^2 g}{\partial t^2} & (2 \leq i < j \leq \lambda+1) \\ -4x_ix_j\dfrac{\partial^2 g}{\partial t^2} & (2 \leq i \leq \lambda+1 < j \leq n) \\ 4x_ix_j\dfrac{\partial^2 g}{\partial t^2} & (\lambda+2 \leq i < j \leq n). \end{cases}$$

The gradient vector of $G$ at $p$ is $\left(\left(\frac{\partial g}{\partial s}\right)_{\varphi(p)}, 0, \ldots, 0\right)$. The point $p$ is a critical point of $G$ if and only if $\left(\frac{\partial g}{\partial s}\right)_{\varphi(p)} = 0$. It means that the $s$-axis, which is just $\varphi(\sigma)$, is parallel to the $f$-axis at $\varphi(p)$. This finishes the proof of Lemma 1 in the case where $p$ is a fold point.

The Hessian matrix of $G$ at $p$ is

$$\begin{pmatrix} \left(\frac{\partial^2 g}{\partial s^2}\right)_{\varphi(p)} & & & & & & \\ & -2\left(\frac{\partial g}{\partial t}\right)_{\varphi(p)} & & & & & \\ & & \ddots & & & & \\ & & & -2\left(\frac{\partial g}{\partial t}\right)_{\varphi(p)} & & & \\ & & & & 2\left(\frac{\partial g}{\partial t}\right)_{\varphi(p)} & & \\ & & & & & \ddots & \\ & & & & & & 2\left(\frac{\partial g}{\partial t}\right)_{\varphi(p)} \end{pmatrix}$$

and its determinant is $(-1)^{\lambda}2^{n-1}\left(\frac{\partial g}{\partial t}\right)^{n-1}_{\varphi(p)}\left(\frac{\partial^2 g}{\partial s^2}\right)_{\varphi(p)}$. We suppose that $p$ is a critical point of $G$, and hence $\left(\frac{\partial g}{\partial s}\right)_{\varphi(p)} = 0$. It requires $\left(\frac{\partial f}{\partial s}\right)_{\varphi(p)} \neq 0$ and $\left(\frac{\partial g}{\partial t}\right)_{\varphi(p)} \neq 0$ since the regular coordinate transformation (4) satisfies $\frac{\partial f}{\partial s}\frac{\partial g}{\partial t} - \frac{\partial f}{\partial t}\frac{\partial g}{\partial s} \neq 0$. It follows that the critical point $p$ degenerates if and only if $\left(\frac{\partial^2 g}{\partial s^2}\right)_{\varphi(p)} = 0$.

We calculate the second derivative of $\varphi(\sigma)$ at $\varphi(p)$. The discriminant set $\varphi(\sigma) = \{(s,t) = (x_1, 0)\}$ is regarded as the graph of a function $g = \theta(f)$ near the horizontal point $\varphi(p)$. Its first and second derivatives are

$$\frac{d\theta}{df} = \frac{d}{df}g(x_1,0) = \frac{\frac{d}{dx_1}g(x_1,0)}{\frac{d}{dx_1}f(x_1,0)} = \frac{\frac{d}{dx_1}(x_1)\frac{\partial g}{\partial s}(x_1,0) + \frac{d}{dx_1}(0)\frac{\partial g}{\partial t}(x_1,0)}{\frac{d}{dx_1}(x_1)\frac{\partial f}{\partial s}(x_1,0) + \frac{d}{dx_1}(0)\frac{\partial f}{\partial t}(x_1,0)} = \frac{\frac{\partial g}{\partial s}(x_1,0)}{\frac{\partial f}{\partial s}(x_1,0)},$$

$$\begin{aligned} \frac{d^2\theta}{df^2} &= \frac{d}{df}\frac{\frac{\partial g}{\partial s}(x_1,0)}{\frac{\partial f}{\partial s}(x_1,0)} \\ &= \left\{\frac{d}{df}\left(\frac{\partial g}{\partial s}(x_1,0)\right)\frac{\partial f}{\partial s}(x_1,0) - \frac{\partial g}{\partial s}(x_1,0)\frac{d}{df}\left(\frac{\partial f}{\partial s}(x_1,0)\right)\right\} \Big/ \left(\frac{\partial f}{\partial s}(x_1,0)\right)^2 \\ &= \left\{\frac{\frac{\partial^2 g}{\partial s^2}(x_1,0)}{\frac{\partial f}{\partial s}(x_1,0)}\frac{\partial f}{\partial s}(x_1,0) - \frac{\partial g}{\partial s}(x_1,0)\frac{\frac{\partial^2 f}{\partial s^2}(x_1,0)}{\frac{\partial f}{\partial s}(x_1,0)}\right\} \Big/ \left(\frac{\partial f}{\partial s}(x_1,0)\right)^2. \end{aligned}$$

Noting that $\left(\frac{\partial g}{\partial s}\right)_{\varphi(p)} = 0$ and $\left(\frac{\partial f}{\partial s}\right)_{\varphi(p)} \neq 0$, the second derivative of $\theta$ at $\varphi(p) = (0,0)$ is $\left(\frac{\partial^2 g}{\partial s^2}\right)_{\varphi(p)} / \left(\frac{\partial f}{\partial s}\right)^2_{\varphi(p)}$. It follows that the horizontal point $\varphi(p)$ is an inflection point if and only if $\left(\frac{\partial^2 g}{\partial s^2}\right)_{\varphi(p)} = 0$.

By the results in the previous two paragraphs, the critical point $p$ degenerates if and only if the horizontal point $\varphi(p)$ is an inflection point. This finishes the proof of Lemma 2 in the case where $p$ is a fold point.

We consider the index of $p$ assuming that $p$ is a non-degenerate critical point of $G$. The eigenvalues of the Hessian matrix are $\left(\frac{\partial^2 g}{\partial s^2}\right)_{\varphi(p)}$, $-2\left(\frac{\partial g}{\partial t}\right)_{\varphi(p)}$ and $2\left(\frac{\partial g}{\partial t}\right)_{\varphi(p)}$, whose multiplicities are 1, $\lambda$ and $n-\lambda-1$, respectively. The sign of the first eigenvalue $\left(\frac{\partial^2 g}{\partial s^2}\right)_{\varphi(p)}$

is equal to the sign of the second derivative $\left(\frac{\partial^2 g}{\partial s^2}\right)_{\varphi(p)} / \left(\frac{\partial f}{\partial s}\right)^2_{\varphi(p)}$ of $\varphi(\sigma)$ at $\varphi(p)$. Noting that $\varphi(p)$ is a horizontal point but not an inflection point, the sign of the second derivative corresponds to whether $\varphi(\sigma)$ is downward or upward convex at $\varphi(p)$. For instance, if $\varphi(\sigma)$ is downward convex horizontal point, the index of $p$ is $\lambda$ or $n-\lambda-1$ according to the sing of $\left(\frac{\partial g}{\partial t}\right)_{\varphi(p)}$. Though we do not know the sing of $\left(\frac{\partial g}{\partial t}\right)_{\varphi(p)}$ in general, if the absolute index of $p$ is 0, it corresponds to whether $\varphi(U)$ lies in the lower or upper half. This finishes the proof of Lemma 3 in the case where $p$ is a fold point.

## References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