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On a relation between certain character values of symmetric groups and its connection with creation operators of symmetric functions

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Abstract. We give a relation of new kind between certain character values of symmetric groups, which was found in investigating a curious phenomenon in irreducible characters of symmetric groups. We also investigate a relation between our result and Bernstein's creation operators for Schur functions, and consider analogous relations for projective characters of symmetric groups through creation operators for Schur Q-functions.

概要. 対称群の指標についてのとある現象を説明しようという試みから発見された, 指標の間の新しい関係式について説明する. また, その関係式と対称関数の生成作用素との関連を示し, Schur Q-関数の生成作用素を通して我々の得た関係式での射影指標での類似が得られることをみる.

1 Introduction

Irreducible characters of symmetric groups are an interesting subject in combinatorial representation theory. In examining the character tables of symmetric groups, we noticed a curious correspondence between the irreducible character values of S_n at transpositions and the degrees of the irreducible characters of S_{n-2} for $n \leq 7$, which is exhibited in §2.2. As is mentioned there, this phenomenon is not, at least directly, explained by applying the Murnaghan-Nakayama formula to remove a 2-cycle. We found a different formula (see Theorem 2.2) which explains this phenomenon. Our formula, like Murnaghan-Nakayama formula, "removes a cycle" from an irreducible character value; namely, given a partition λ of n and a partition μ of $n-m$, it expresses $\chi_\lambda(\mu \cup (m))$ as a linear combination of $\chi_\kappa(\mu)$ ($|\kappa| = n-m$) with coefficients ± 1; but it only works if μ has no parts divisible by m. Here $\chi_\lambda(\nu)$ denotes the value of the irreducible character of S_n indexed by λ at a partition of cycle type ν. This formula can be further generalized as in Remark at the end of §2. We also found that the operators describing the linear combinations in our formula can be written in a form similar to Bernstein's creation operators for Schur functions (see Theorem 5) and, by replacing these creation operators by those for Schur Q-functions, we can obtain an analogue of our formula for "projective characters" of symmetric groups.

These results form a part of master's thesis [5] by the author. In this abridged presentation, details of the proof are omitted, but some explanations are added. For the omitted proofs and additional results for Brauer algebras and walled Brauer algebras, please refer to [5] for the moment (the full version may be published elsewhere).

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2 Irreducible characters of symmetric groups

2.1 Preliminaries

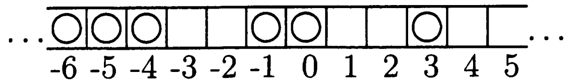
A sequence of positive integers $\lambda = (\lambda_1, \dots, \lambda_l)$ with $\lambda_1 \geq \cdots \geq \lambda_l > 0$ is called a *partition*. l is called the *length* of λ and denoted by $\ell(\lambda)$. The terms λ_i are called the *parts* of this partition. We write $|\lambda|$ for $\lambda_1 + \cdots + \lambda_l$, and if $|\lambda| = n$ then λ is called a *partition of n* and n is called the *size* of λ. The empty sequence is the unique partition of 0 (or of length 0) and is denoted by $\varnothing$. We sometimes write nonempty partitions in multiplicative form: $4^2 3 1^3$ stands for the partition $(4, 4, 3, 1, 1, 1)$, for example.

Let S_n denote the symmetric group over n letters. It is well known that, over a field of characteristic zero, the isomorphism classes of irreducible linear representations of S_n are indexed by the partitions of n. In this paper, we use objects called *maya diagrams* rather than partitions to index irreducible representations of symmetric groups.

Definition 2.1. *A maya diagram is a sequence of integers* $[x_1, x_2, \dots]$ *with* $x_1 > x_2 > \cdots$ *and* $x_i = -i$ *for* $i \gg 1$.

We write maya diagrams with [] and partitions with () (or with no parentheses if they are in multiplicative form), in order to avoid confusion between these two kinds of objects.

There is an easy correspondence between maya diagrams and partitions, by $(\lambda_1, \cdots, \lambda_l) \mapsto [\lambda_1 - 1, \lambda_2 - 2, \dots]$ and $[x_1, x_2, \dots] \mapsto (x_1 + 1, x_2 + 2, \cdots)$



here partitions are identified with infinite sequences obtained from them
/ attaching infinitely many zeroes. The maya diagram corresponding to partition λ under these bijections is also denoted by λ, but we believe iat this causes no confusion. The notion of size is also defined for maya .agrams through the correspondence above: if $|\lambda| = n$, the corresponding .aya diagram $[x_1, x_2, \ldots]$ satisfies $\sum_i (x_i + i) = n$.

Let F be a $\mathbb{C}$-vector space having the set of all maya diagrams as a basis.

We also define a *charged maya diagram* as a sequence of integers $[x_1, x_2, \ldots]$ ith $x_1 > x_2 > \cdots$ and $x_{i+1} = x_i - 1$ for $i \gg 1$. Let $\tilde{F}$ be a vector space /er $\mathbb{C}$ having the set of all charged maya diagrams as a basis.

.emark. The spaces F and $\tilde{F}$ are sometimes called an infinite or semi-finite wedge space ([2, 3]): in that case $[x_1, x_2, \ldots]$ is denoted as $v_{x_1} \wedge v_{x_2} \wedge$ ..

(Charged) maya diagrams are often depicted by drawing infinitely many ɔxes numbered $\ldots, -2, -1, 0, 1, \ldots$ in a row and drawing circles in the ɔxes corresponding to $x_1, x_2, \ldots$. For example, $[3, 0, -1, -4, -5, -6, \ldots]$ $\to (4, 2, 2))$ corresponds to the following picture:

The partition corresponding to a maya diagram can be read directly from ie picture of the maya diagram: counting the number of blanks on the left each circle gives the parts of the partition.

We sometimes use sequences $[x_1, x_2, \ldots]$ which satisfy $x_i = -i$ (or $x_{i+1} = -1$) for $i \gg 1$ but are not necessarily decreasing. In such case we consider iem as elements of F (or $\tilde{F}$ respectively) by the rule $[\ldots, i, \ldots, j, \ldots] + \ldots, j, \ldots, i, \ldots] = 0$ (in particular, sequences with duplicate terms are equal › zero).

Let χ_λ denote the irreducible character of a symmetric group indexed by maya diagram or a partition λ.

We say an element $w \in S_n$ has *cycle type* $\mu = (\mu_1, \cdots, \mu_l)$ if w is a :oduct of disjoint cycles with length $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_l \geq 1$. It is well iown that two elements in S_n are conjugate if and only if they have the ıme cycle type. Let $\chi_\lambda(\mu) := \chi_\lambda(w)$ for $w \in S_n$ with cycle type μ.

Then well-known formulae of Frobenius' and Murnaghan's essentially ate the following, under the present setting:

'heorem 2.1. *For $r \in \mathbb{Z}$, define a $\mathbb{C}$-linear map $A_r : F \to F$ by $A_r[x_1, x_2, \ldots,] = \sum_{i \geq 1} [x_1, x_2, \ldots, x_i - r, \ldots]$ for a maya diagram $[x_1, x_2, \ldots]$ (note that only* *finitely many summands on the right-hand side are nonzero). Then for a maya diagram* λ *and a partition* $\mu = (\mu_1, \cdots, \mu_l)$ *with* $|\lambda| = |\mu| = n$,

$$A_\mu \lambda := A_{\mu_1} \cdots A_{\mu_l} \lambda = \chi_\lambda(\mu)\varnothing.$$

2.2 A motivating phenomenon

The following tables show the irreducible character values of symmetric groups S_n through $n = 2$ to 6. Here the rows correspond to irreducible characters and the columns correspond to conjugacy classes. (Here we indexed rows by partitions rather than maya diagrams, because they need much less space to write down.)

S_2	1^2	2
2	1	1
1^2	1	-1

S_3	1^3	21	3
3	1	1	1
21	2	0	-1
1^3	1	-1	1

S_4	1^4	21^2	2^2	31	4
4	1	1	1	1	1
31	3	1	-1	0	-1
2^2	2	0	2	-1	0
21^2	3	-1	-1	0	1
1^4	1	-1	1	1	-1

S_5	1^5	21^3	2^21	31^2	32	41	5
5	1	1	1	1	1	1	1
41	4	2	0	1	-1	0	-1
32	5	1	1	-1	1	-1	0
31^2	6	0	-2	0	0	0	1
2^21	5	-1	1	-1	-1	1	0
21^3	4	-2	0	1	1	0	-1
1^5	1	-1	1	1	-1	-1	1

S_6	1^6	21^4	2^21^2	2^3	31^3	321	3^2	41^2	42	51	6
6	1	1	1	1	1	1	1	1	1	1	1
51	5	3	1	-1	2	0	-1	1	-1	0	-1
42	9	3	1	3	0	0	0	-1	1	-1	0
41^2	10	2	-2	-2	1	-1	1	0	0	0	1
3^2	5	1	1	-3	-1	1	2	-1	-1	0	0
321	16	0	0	0	-2	0	-2	0	0	1	0
31^3	10	-2	-2	2	1	1	1	0	0	0	-1
2^3	5	-1	1	3	-1	-1	2	1	-1	0	0
2^21^2	9	-3	1	-3	0	0	0	1	1	-1	0
21^4	5	-3	1	1	2	0	-1	-1	-1	0	1
1^6	1	-1	1	-1	1	-1	1	-1	1	1	-1

Table 2.1: character tables of symmetric groups.

Examining these character tables, one may notice that column 1^n of the table for S_n "re-appears" in column 21^n of the table for S_{n+2}: for example, the irreducible characters of S_3 have degrees $1, 2, 1$ respectively, while the irreducible characters of S_5 have values $1, 2, 1, 0, -1, -2, -1$ on its conjugacy class consisting of transpositions. This is more obvious in (S_4, S_6) and (S_5, S_7): irreducible characters of S_4 and S_5 have degrees $1, 3, 2, 3, 1$ and

$1, 4, 5, 6, 5, 4, 1$ respectively, which clearly appear as irreducible character values of S_6 and S_7 on transpositions (the whole character table of S_7 is not presented here because it's too large, but Table 2.2 shows its column for the transpositions.).

S_7	21^5
7	1
61	4
52	6
51^2	5
43	4
421	5
41^3	0
3^21	1
32^2	-1
321^2	-5
31^4	-5
2^31	-4
2^21^3	-6
21^5	-4
1^7	-1

S_8	21^6
8	1
71	5
62	10
61^2	9
53	10
521	16
51^3	5
4^2	4
431	10
42^2	4
421^2	0
41^4	-5
3^22	0
3^21^2	-4
32^21	-10
321^3	-16
31^5	-9
2^4	-4
2^31^2	-10
2^21^4	-10
21^6	-5
1^8	-1

Table 2.2: "transposition columns" of character tables of S_7 and S_8.

However, as one can see, the relationship does not hold for S_6 and S_8: in fact, if the "re-appearance" above is to hold for S_n and S_{n+2}, then we must have $2p(n) \leq p(n+2)$, but this does not hold for sufficiently large n. Still, it is natural to seek for some explanation, since this correspondence certainly holds up to S_5 and S_7.

We easily see that Murnaghan's formula does not give, at least directly, the explanation we expect. For example, for the character values of S_6 on transpositions what Murnaghan's formula gives is as follows:

$$\begin{aligned}
\chi_6(21^4) &= \chi_4(1^4) = 1, \\
\chi_{51}(21^4) &= \chi_{31}(1^4) = 3, \\
\chi_{42}(21^4) &= \chi_4(1^4) + \chi_{2^2}(1^4) = 1 + 2 = 3, \\
\chi_{41^2}(21^4) &= \chi_{21^2}(1^4) - \chi_4(1^4) = 3 - 1 = 2, \text{and} \\
\chi_{3^2}(21^4) &= \chi_{31}(1^4) - \chi_{2^2}(1^4) = 3 - 2 = 1.
\end{aligned}$$

So the one-to-one correspondence is not nicely explained by Murnaghan's formula.

2.3 Main result

Here we are going to state our first main result, which explains the re-appearance phenomenon above.

Let $m > 1$ be an integer and k be an integer. We define a $\mathbb{C}$-linear operator $\phi_k^{(m)} : F \to F$ by

$$\phi_k^{(m)}([x_1, x_2, \ldots]) = \left(\sum_{a_0, \ldots, a_{m-1}} [a_{m-1}, a_{m-2}, \ldots, a_0, x_1, x_2, \ldots] \right) - m \qquad (1)$$

for a maya diagram $[x_1, x_2, \ldots]$, where the sum is over all m-tuples of integers $(a_0, \ldots, a_{m-1})$ with $a_i \equiv i \pmod m$ and $a_0 + \cdots + a_{m-1} = (0 + \cdots + (m-1)) - km$. Here the $-m$ on the right-hand side is defined as a $\mathbb{C}$-linear operator $-m : \tilde{F} \to \tilde{F}$ by $[x_1, x_2, \ldots] - m = [x_1 - m, x_2 - m, \ldots]$ for a maya diagram $[x_1, x_2, \ldots]$, which was applied to the sum in order that the right-hand side to lie in F. Note that the sum above is essentially finite, since if some of the integers a_i are too small then $[a_{m-1}, a_{m-2}, \ldots, a_0, x_1, x_2, \ldots]$ must have some duplicate terms.

Our first result can now be stated as follows:

Theorem 2.2. *Let λ be a maya diagram of size n and let μ be a partition of $n - m$ which does not have any multiple of m as its part. Then we have*

$$\chi_\lambda(\mu \cup (m)) = \chi_{-\phi(\lambda)}(\mu).$$

Here $\mu \cup (m)$ means the partition obtained by appending a part m to μ, $\phi = \phi_1^{(m)}$, and the notation $\chi_\kappa(\nu)$ has been extended for $\kappa \in F$ linearly in κ. □

Example. Let $\lambda = [3, 0, -1, -4, -5, -6, \ldots] \leftrightarrow (4, 2, 2)$ and $m = 2$. Then

$$\begin{aligned}
\phi(\lambda) &= \left(\sum_{\substack{a_0 \equiv 0, a_1 \equiv 1 \pmod 2 \\ a_0 + a_1 = -1}} [a_1, a_0, 3, 0, -1, -4, -5, -6, \ldots] \right) - 2 \\
&= \big([1, -2, 3, 0, -1, -4, -5, -6, \ldots] + [-3, 2, 3, 0, -1, -4, -5, -6, \ldots]\big) - 2 \\
&= \big([3, 1, 0, -1, -2, -4, -5, -6, \ldots] - [3, 2, 0, -1, -3, -4, -5, -6, \ldots]\big) - 2 \\
&= [1, -1, -2, -3, -4, -6, -7, -8, \ldots] - [1, 0, -2, -3, -5, -6, -7, -8, \ldots] \\
&\leftrightarrow (2, 1, 1, 1, 1) - (2, 2, 1, 1).
\end{aligned}$$

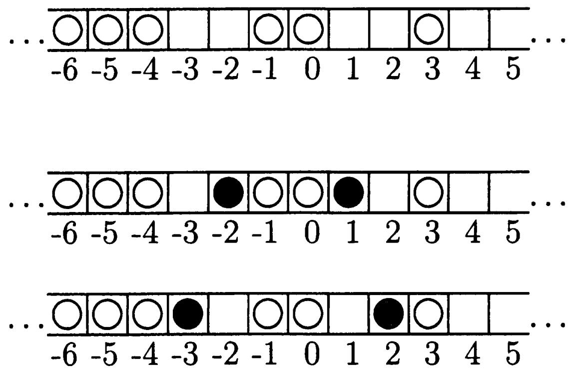


Figure 2.1: the original diagram λ and the nonzero terms in the sum in the definition of $\phi(\lambda)$ (the circles corresponding to the integers a_i are shaded).

Note that the summands with $(a_0, a_1) \neq (-2, 1), (2, -3)$ vanish: to avoid duplicate terms one must have $a_0, a_1 \geq -3$ so you only have to check $(a_0, a_1) = (-2, 1), (0, -1), (2, -3)$. From this we have $\chi_{42^2}(\mu \cup (2)) = -\chi_{21^4}(\mu) + \chi_{2^21^2}(\mu)$ for a partition μ with all parts odd. Note that this is different from the one given by Murnaghan's formula, i.e. $\chi_{42^2}(\mu \cup (2)) = \chi_{2^3}(\mu) - \chi_{41^2}(\mu) + \chi_{42}(\mu)$ (although the latter is valid for any partition μ).

Now, Theorem 2.2 with $m = 2$ and $\mu = (1, \ldots, 1)$ explains the phenomenon, as shown in Table 2.3. In fact, it can be shown that if $m = 2$ then the sum in $\phi(\lambda)$ has at most $r-1$ nonzero terms for $|\lambda| < 2r^2$. So with $r = 2$, it can be seen that $\chi_\lambda(21^*)$ can be expressed in the form $\pm\chi_{\bar{\lambda}}(1^*)$ (or zero) for a single diagram $\bar{\lambda}$ if $|\lambda| \leq 7$.

The proof of Theorem 2.2 depends on proving the following two things: (i) ϕ commutes with A_l $(m \nmid l)$, and (ii) $-\phi(\lambda) = A_m\lambda$ for maya diagrams λ of size m. In fact, with (i) and (ii) we can show the theorem as

$$\chi_\lambda(\mu \cup (m))\varnothing = A_m A_\mu \lambda = -\phi(A_\mu \lambda) = A_\mu(-\phi(\lambda)) = \chi_{-\phi(\lambda)}(\mu)\varnothing.$$

For the details of the proof, see [5].

Remark. It can also be shown that $[A_{rm}, \phi_k^{(m)}] = m\phi_{k+r}^{(m)}$ holds. This will yield a recurrence formula for $\chi_\lambda(\mu)$ for μ with more than one part divisible by m, in terms of operators $\phi_k^{(m)}$. For example, it can be shown that one has

$$\chi_\lambda(\mu \cup m^k) = -\sum_{i=0}^{k-1}(-m)^i \binom{k-1}{i} \chi_{\phi_{i+1}^{(m)}(\lambda)}(\mu \cup m^{k-1-i})$$

for a maya diagram λ of size n and a partition μ of size $n - km$ without any part divisible by m.

$$\chi_{2}(2) = f^{\varnothing}$$
$$\chi_{11}(2) = -f^{\varnothing}$$

$$\chi_{3}(21) = f^{1}$$
$$\chi_{21}(21) = 0$$
$$\chi_{111}(21) = -f^{1}$$

$$\chi_{4}(21^2) = f^{2}$$
$$\chi_{31}(21^2) = f^{11}$$
$$\chi_{22}(21^2) = 0$$
$$\chi_{211}(21^2) = -f^{11}$$
$$\chi_{1111}(21^2) = -f^{2}$$

$$\chi_{5}(21^3) = f^{3}$$
$$\chi_{41}(21^3) = f^{21}$$
$$\chi_{32}(21^3) = f^{111}$$
$$\chi_{311}(21^3) = 0$$
$$\chi_{221}(21^3) = -f^{111}$$
$$\chi_{2111}(21^3) = -f^{21}$$
$$\chi_{11111}(21^3) = -f^{3}$$

$$\chi_{6}(21^4) = f^{4}$$
$$\chi_{51}(21^4) = f^{31}$$
$$\chi_{42}(21^4) = f^{211}$$
$$\chi_{411}(21^4) = f^{22}$$
$$\chi_{33}(21^4) = f^{1111}$$
$$\chi_{321}(21^4) = 0$$
$$\chi_{3111}(21^4) = -f^{22}$$
$$\chi_{222}(21^4) = -f^{1111}$$
$$\chi_{2211}(21^4) = -f^{211}$$
$$\chi_{21111}(21^4) = -f^{31}$$
$$\chi_{111111}(21^4) = -f^{4}$$

$$\chi_{7}(21^5) = f^{5}$$
$$\chi_{61}(21^5) = f^{41}$$
$$\chi_{52}(21^5) = f^{311}$$
$$\chi_{511}(21^5) = f^{32}$$
$$\chi_{43}(21^5) = f^{2111}$$
$$\chi_{421}(21^5) = f^{221}$$
$$\chi_{4111}(21^5) = 0$$
$$\chi_{331}(21^5) = f^{11111}$$
$$\chi_{322}(21^5) = -f^{11111}$$
$$\chi_{3211}(21^5) = -f^{221}$$
$$\chi_{31111}(21^5) = -f^{32}$$
$$\chi_{2221}(21^5) = -f^{2111}$$
$$\chi_{22111}(21^5) = -f^{311}$$
$$\chi_{211111}(21^5) = -f^{41}$$
$$\chi_{1111111}(21^5) = -f^{5}$$

$$\chi_{8}(21^6) = f^{6}$$
$$\chi_{71}(21^6) = f^{51}$$
$$\chi_{62}(21^6) = f^{411}$$
$$\chi_{611}(21^6) = f^{42}$$
$$\chi_{53}(21^6) = f^{3111}$$
$$\chi_{521}(21^6) = f^{321}$$
$$\chi_{5111}(21^6) = f^{33}$$
$$\chi_{44}(21^6) = -f^{222} + f^{2211}$$
$$\chi_{431}(21^6) = f^{21111} + f^{222}$$
$$\chi_{422}(21^6) = -f^{21111} + f^{2211}$$
$$\chi_{4211}(21^6) = 0$$
$$\chi_{41111}(21^6) = -f^{33}$$
$$\chi_{332}(21^6) = 0$$
$$\chi_{3311}(21^6) = -f^{222} + f^{111111}$$
$$\chi_{3221}(21^6) = -f^{2211} - f^{111111}$$
$$\chi_{32111}(21^6) = -f^{321}$$
$$\chi_{311111}(21^6) = -f^{42}$$
$$\chi_{2222}(21^6) = -f^{21111} + f^{111111}$$
$$\chi_{22211}(21^6) = -f^{3111}$$
$$\chi_{221111}(21^6) = -f^{411}$$
$$\chi_{2111111}(21^6) = -f^{51}$$
$$\chi_{11111111}(21^6) = -f^{6}$$

Table 2.3: The expansions of the character values $\chi_\lambda(21^n)$ given by Theorem 2.2 with $m = 2$ and $\mu = (1, \ldots, 1)$. (Here f^λ stands for $\chi_\lambda(1^{|\lambda|})$.)

3 Relation with Bernstein operator

Let Λ denote the ring of symmetric functions in the variables $X_1, X_2, \ldots$: $\Lambda = \mathbb{Z}[e_1, e_2, \ldots] = \mathbb{Z}[h_1, h_2, \ldots]$ where $e_r = \sum_{i_1 < \ldots < i_r} X_{i_1} \cdots X_{i_r}$ and $h_r = \sum_{i_1 \leq \ldots \leq i_r} X_{i_1} \cdots X_{i_r}$. We define $h_0 = e_0 = 1$ and $h_r = e_r = 0$ for $r < 0$. Let $H(u) := \sum_{n \geq 0} h_n u^n = \prod_{i \geq 1} \frac{1}{1 - X_i u}$ and $E(u) := \sum_{n \geq 0} e_n u^n = \prod_{i \geq 1} (1 + X_i u)$ be their generating functions. Let $p_r = \sum_{i \geq 1} X_i^r$ $(r \geq 1)$. We denote the Schur function by s_λ ([4, §I.3]). In this paper, the index of a Schur function may be either a partition, written in parentheses, or a maya diagram, written in brackets.

We consider on Λ, as usual, the Hall inner product $\langle \cdot, \cdot \rangle : \Lambda \times \Lambda \to \mathbb{Z}$ by $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$ for partitions λ, μ. Let $f^\perp$ denote the adjoint operator of the multiplication by f with respect to this inner product, say: $\langle f^\perp(g), h \rangle = \langle g, fh \rangle$.

Using the Schur functions and the Hall inner product, Theorem 2.1 can be stated as $p_{\mu_1}^\perp \cdots p_{\mu_l}^\perp s_\lambda = \chi_\lambda(\mu)$ ([4, §I.7]), which is in fact much closer to the original Frobenius formula.

Bernstein's creation operator is defined as follows:

Definition 3.1. *For $n \in \mathbb{Z}$,*

$$B_n = \sum_{i \geq 0} (-1)^i h_{n+i} e_i^\perp \tag{2}$$

where h_{n+i} denotes the multiplication operator by h_{n+i}. Note that, even though the expression above is an infinite sum, only finitely many terms give nonzero image when applied to each $f \in \Lambda$.

Bernstein operators have the following property ([4, §I.5, Example 29]):

Proposition 3.1. *For a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$,*

$$B_{\lambda_1} \cdots B_{\lambda_l}(1) = s_\lambda.$$

This is still valid for a general integer sequence λ if we interpret Schur functions indexed by integer sequences which is nondecreasing or having nonpositive parts as $s_{(\ldots, i, j, \ldots)} = -s_{(\ldots, j-1, i+1, \ldots)}$. In our maya-diagram setting, this implies

$$B_n(s_{[x_1, x_2, \ldots]}) = s_{[n, x_1, x_2, \ldots] - 1}, \tag{3}$$

or, for λ a maya diagram,

$$B_n(s_\lambda) = s_{b_n \lambda},$$

where $b_n \lambda = \tilde{b}_n \lambda - 1$ with $\tilde{b}_n$ defined just before the proof of theorem 2.2. Thus we have $B_{a_{m-1}} \cdots B_{a_0} s_{[x_1, x_2, \ldots]} = s_{[a_{m-1} - 1, \ldots, a_0 - m, x_1 - m, x_2 - m, \ldots]} =$

$s_{[a_{m-1}+m-1,\ldots,a_0,x_1,x_2,\ldots]-m}$, and so if we identify F with $\Lambda_{\mathbb{C}} = \Lambda \otimes \mathbb{C}$ by identifying each maya diagram λ with the Schur function s_λ we get

$$\phi_k^{(m)} = \sum B_{a_{m-1}} \cdots B_{a_0} \tag{4}$$

where sum runs over all m-tuples $(a_0, \ldots, a_{m-1})$ with $a_i \equiv 0 \pmod{m}$ and $a_0 + \cdots + a_{m-1} = -km$.

We have the following Bernstein-operator like expression for $\phi_k^{(m)}$:

Theorem 3.1. *For $m \geq 1$ and $n \in \mathbb{Z}$ we have*

$$\phi_{-n}^{(m)} = \sum_{i \geq 0} (-1)^i (h_{n+i} \circ p_m)(e_i \circ p_m)^\perp \tag{5}$$

where $-\circ p_m$ denotes the plethysm with the m-th power sum: $(f \circ p_m)(\{X_i\}) = f(\{X_i^m\})$. □

This can be shown by using the equation $B(u_1)\cdots B(u_m) = \prod_{p<q}(1 - u_p^{-1}u_q) \cdot H(u_1)\cdots H(u_m)E^\perp(-u_1^{-1})\cdots E^\perp(-u_m^{-1})$ ([4, §I.5, Example 29]): using a primitive m-th root of unity ω, summing the equation for all $(u_1, \ldots, u_m) = (\omega^{j_1}u, \ldots, \omega^{j_m}u)$ $(j_1, \ldots, j_m \in \mathbb{Z}/m\mathbb{Z})$ gives the desired result. For the details of the proof, see [5].

Remark. In fact, the properties (i) and (ii) of ϕ in the proof of Theorem 2.2 can be also seen from the above expression for $\phi_n^{(m)}$.

4 An analog for projective characters

A *projective representation* $\pi = (V, \pi)$ of a group G is a group homomorphism π from G to $PGL(V)$, the projective general linear group of a vector space V. Two projective representations (V, π) and (W, ρ) are said to be *projectively equivalent* if there exists a vector space isomorphism $V \to W$ such that the induced group isomorphism $f : PGL(V) \to PGL(W)$ satisfies $f\pi(g) = \rho(g)f$ for all $g \in G$. Let us call a projective representation π *nontrivial* if π is not projectively equivalent to any representations obtained from a linear representation of G by composing with $GL(V) \twoheadrightarrow PGL(V)$.

Let $\tilde{S}_n$ be the group generated by the generators $s_1, \ldots, s_{n-1}, z$ bound by the relations:

- z is a central element with $z^2 = 1$,
- $s_i^2 = z$, $s_i s_j = z s_j s_i$ $(|i - j| \geq 2)$, $(s_i s_{i+1})^3 = z$.

Clearly one has a surjective group homomorphism $\theta : \tilde{S}_n \twoheadrightarrow S_n$ which sends z to 1 and s_i to $(i\ i+1)$, $i = 1, \ldots, n-1$. This homomorphism has kernel $\{1, z\}$. Since z is in the center of $\tilde{S}_n$ and has order 2, its action on an irreducible representation is either by 1 or by -1. Call an irreducible representation of $\tilde{S}_n$ *negative* if z acts by -1. Call two representations of $\tilde{S}_n$ being *associate* of each other if one can be obtained from the other by tensoring with $sgn \circ \theta$, where sgn is the sign representation of S_n. The following relationship between the projective representations of S_n and the linear representations of $\tilde{S}_n$ is known:

Proposition 4.1 ([1, Chap. 2]). *If $n \geq 4$, Projective isomorphism classes of nontrivial irreducible projective representations of S_n is in one-to-one correspondence with the associate classes of negative irreducible representations of $\tilde{S}_n$.*

It is known that the isomorphism classes of negative representations of $\tilde{S}_n$ are indexed by the strict partitions of n (a partition $(\lambda_1, \ldots, \lambda_l)$ is called *strict* if $\lambda_1 > \cdots > \lambda_l$). Let ψ_λ denote the irreducible character of $\tilde{S}_n$ indexed by λ.

If C is a conjugacy class of S_n, $\theta^{-1}(C)$ is either a single conjugacy class or the union of two conjugacy classes. In the former case, g and zg are conjugate for $g \in \theta^{-1}(C)$ and thus negative irreducible characters vanish there. In the latter case we say that C *splits*. Conjugacy class C_μ of S_n with cycle type μ splits iff: (i) all parts of μ are odd (in which case we call μ *all-odd*), or (ii) μ is strict and C_μ consists of odd permutations ([1, Theorem 3.8]). In fact it is easy to describe the character values explicitly in the case (ii), so we are interested in the case (i). Let $\psi_\lambda(\mu)$ denote the value of ψ_λ evaluated at an element g_μ, which is chosen from $\theta^{-1}(C_\mu)$ so that the character of the "basic representation" ([1, Chap. 6]) of $\tilde{S}_n$ takes a positive value at g_μ. We also let $\tilde{\psi}_\lambda(\mu) = 2^{\lceil \frac{\ell(\lambda)-\ell(\mu)}{2} \rceil}\psi_\lambda(\mu)$.

Let $q_n = \sum h_{n-i}e_i \in \Lambda$ and let Γ be the subring of Λ generated by $q_1, q_2, q_3, \ldots$. It is known that $\Gamma \otimes \mathbb{Q} = \mathbb{Q}[p_1, p_3, p_5, p_7, \ldots]$. We have a basis $\{Q_\lambda\}_{\lambda:\text{strict partition}}$ of Γ consisting of so-called *Schur Q-functions* ([1, Chap. 7], [4, §III.8]). As Schur functions carry information about irreducible characters of linear representations of symmetric groups, Schur Q-functions carry information about projective characters ψ_λ: in fact, if we define an inner product $\langle , \rangle$ on Γ by $\langle Q_\lambda, Q_\mu \rangle = \delta_{\lambda\mu}2^{\ell(\lambda)}$ for all strict partitions λ and μ and denote the adjoint of the multiplication by $f \in \Gamma$ as $f^\perp$, then for strict λ and all-odd μ we have $\tilde{\psi}_\lambda(\mu) = p_{\mu_1}^\perp \cdots p_{\mu_l}^\perp Q_\lambda$ ([1, Chap. 8]).

As in the Schur-funtion case, Schur Q-functions also have creation operators:

Proposition 4.2 ([1, Theorem 7.21]). *If we define* $\mathcal{B}_n = \sum_{i\geq 0}(-1)^i q_{n+i} q_i^{\perp}$ *for* $n \in \mathbb{Z}$, *then for a strict partition* λ *we have* $Q_\lambda = \mathcal{B}_{\lambda_1} \cdots \mathcal{B}_{\lambda_l}(1)$.

Let $Q_\alpha = \mathcal{B}_{\alpha_1} \cdots \mathcal{B}_{\alpha_r}(1)$ for all $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{Z}^r$. The operators $\mathcal{B}_r$ satisfy $\mathcal{B}_r\mathcal{B}_s + \mathcal{B}_s\mathcal{B}_r = 2(-1)^r\delta_{r,-s}$ ([1, Theorem 9.1]), and from this one has "reordering rules" for writing Q_α as a linear combination of the functions Q_λ with strict partitions λ:

- If, for some $i \geq 1$, the subsequence of α consisting of all occurrences of $\pm i$ is *not* of the form $i, -i, i, \dots, -i, i$ or $-i, i, \dots, -i, i$, then $Q_\alpha = 0$.
- Otherwise, there exists a permutation of the sequence α which has the form $\lambda, -a_1, a_1, \dots, -a_r, a_r, 0, \dots, 0$ for a strict partition λ and positive integers $a_1, \dots, a_r$. In this case $Q_\alpha = (-1)^{a_1+\dots+a_r} 2^r \epsilon Q_\lambda$, where ϵ is the sign of any permutation which permutes α into the form above while keeping the order of the terms $\pm i$ for each $i \geq 0$. (see the example below).

For example, $Q_{2,-3,0,4,-2,0,3,2} = (-1) \cdot Q_{4,2,-3,3,-2,2,0,0} = 4Q_{4,2}$ (see Figure 4.1).

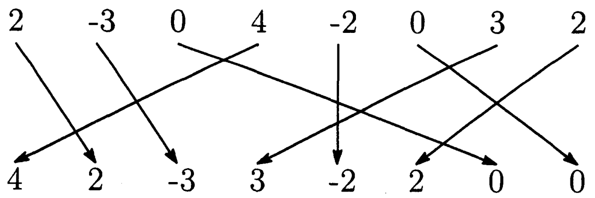


Figure 4.1: $Q_{2,-3,0,4,-2,0,3,2} = \mathrm{sgn}(23715846) \cdot Q_{4,2,-3,3,-2,2,0,0} = -Q_{4,2,-3,3,-2,2,0,0}$.

Since the definition of $\mathcal{B}_n$ is similar to the definition (2) of the Bernstein operator, we can consider a modification of $\mathcal{B}_n$ analogous to (5): let $\Phi_n^{(m)} = \sum_{i\geq 0}(-1)^i(q_{n+i} \circ p_m)(q_i \circ p_m)^{\perp} : \Gamma \to \Gamma$ for $m \geq 1$ odd and $n \in \mathbb{Z}$. Then we have the following formula for $\Phi_n^{(m)}$ analogous to (4):

Theorem 4.1 ([5]). *For any* $m \geq 1$ *odd, we have*

$$\sum_n \Phi_n^{(m)} u^{nm}$$
$$= \left(1 + \sum_{\substack{i,j\in\mathbb{Z} \\ i\equiv -2 \\ j\equiv 2}} \mathcal{B}_{i,j} u^{i+j}\right)\left(1 + \sum_{\substack{i,j\in\mathbb{Z} \\ i\equiv -4 \\ j\equiv 4}} \mathcal{B}_{i,j} u^{i+j}\right) \cdots \left(1 + \sum_{\substack{i,j\in\mathbb{Z} \\ i\equiv -m+1 \\ j\equiv m-1}} \mathcal{B}_{i,j} u^{i+j}\right)\left(\sum_{\substack{k\in\mathbb{Z} \\ k\equiv 0}} \mathcal{B}_k u^k\right)$$

where the congruences are modulo m *and* $\mathcal{B}_{i,j} = \begin{cases} \mathcal{B}_i\mathcal{B}_j & (i > j) \\ -\mathcal{B}_j\mathcal{B}_i & (i < j) \\ 0 & (i = j) \end{cases}$. $\square$

We note that in the product above the coefficient of each u^d is well-defined by the reordering rule above.

Since $\Phi_{-1}^{(m)}$ commutes with $p_l^{\perp}$ $(m \nmid l)$ and coincides with a constant multiple of $p_m^{\perp}$ (in fact $-2p_m^{\perp}$) on degree m part of Γ by the same reason as the remark after Theorem 3.1, we have a corresponding relation for $\tilde{\psi}_\lambda(\mu)$ as in Theorem 2.2. We give an example:

Example.

$$\begin{aligned} \Phi_{-1}^{(3)}Q_{4,3,2} &= Q_{\underline{1,-4,0},4,3,2} + Q_{\underline{4,-4,-3},4,3,2} - Q_{\underline{2,-2,-3},4,3,2} + Q_{\underline{-3},4,3,2} \\ &= -2Q_{3,2,1} + 4Q_{4,2} - 4Q_{4,2} + 2Q_{4,2} \\ &= -2Q_{3,2,1} + 2Q_{4,2}, \end{aligned}$$

so we have $\tilde{\psi}_{4,3,2}(\mu \cup (3)) = \tilde{\psi}_{3,2,1}(\mu) - \tilde{\psi}_{4,2}(\mu)$ for μ with no parts divisible by 3. The fact that no other terms appear in $\Phi_{-1}^{(3)}Q_{4,3,2}$ can be seen as follows. By Theorem 4.1, each term appearing in $\Phi_{-1}^{(m)}Q_\lambda$ is of the form $Q_{\alpha,\lambda}$ for an integer sequence α whose terms modulo m are $0, \pm c_1, \ldots, \pm c_p$ for some $c_1, \ldots, c_p \in \mathbb{Z}/m\mathbb{Z} \setminus \{0\}$ with $c_i \neq \pm c_j$ $(i \neq j)$. In order $Q_{\alpha,\lambda}$ to be nonzero, α must satisfy for each $i \geq 1$,

- if i appears in λ then the terms $\pm i$ appearing in α must be of the form $\ldots, i, -i$ (i and $-i$ alternate), and
- if i does not appear in λ then the terms $\pm i$ must appear in α must be of the form $\ldots, -i, i$ (i and $-i$ alternate).

Considering these conditions we get the four terms above.

Theorem 4.1 can be shown by the calculations similar to the ones in the proof of Theorem 3.1. The main difficulty here is that, unlike the Schur-function case, the product $\mathcal{B}(\omega^{j_1}u)\cdots\mathcal{B}(\omega^{j_m}u)$ is not well-defined. For the details of the proof, see [5].

Remark. As operators $\mathcal{B}_r$ almost anticommute, it is natural to try constructing, in the same way as the construction of $\phi_k^{(m)}$, an operator $\sum \mathcal{B}_{a_{m-1}}\cdots\mathcal{B}_{a_0}$ where the sum runs over all $(a_0, \ldots, a_{m-1}) \in \mathbb{Z}^m$ with $a_i \equiv i \pmod m$ $(0 \leq i \leq m-1)$ and $\sum a_i = (0 + \cdots + (m-1)) - km$. It can be shown, by the same calculation as Theorem 2.2, that if $m = 2$ this operator commutes with all $p_l^{\perp}$ (l : odd) and thus gives a relation of characters. However, this operator in fact coincides with $p_{2k-1}^{\perp}$, and the relation obtained thus coincides with ordinary recurrence formula given by expanding $p_l^{\perp}Q_{\lambda}$ by Q-functions (see eg. [1, Chap. 10]).

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