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THE REGULARIZED SIEGEL-WEIL FORMULA: A SECOND TERM IDENTITY

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In this talk, we describe a recent joint work [GQT] with Yannan Qiu and Shuichiro Takeda, which proves the second term identity in the theory of the regularised Siegel-Weil formula for all classical dual pairs. This expository paper is a summary of the results in [GQT]. We begin by introducing some notation.

1. Spaces and Groups

Let F be a number field with ring of adèles $\mathbb{A}$. Fix a non-trivial additive character $\psi = \otimes_v' \psi_v$ on $F\backslash\mathbb{A}$. We let E be either F or a quadratic extension of F. Moreover, let χ_E denote the (possibly trivial) quadratic character of $\mathbb{A}^\times$ associated to E/F by global class field theory.

1.1. **ϵ-Hermitian spaces.** Fix a sign

$$\epsilon = \pm 1.$$

Let V_0 be an m_0-dimensional vector space over E equipped with a nondegenerate *anisotropic* ϵ-Hermitian form $(-,-)$. With $\mathbb{H}$ denoting the hyperbolic plane, i.e. the split ϵ-Hermitian space of dimension 2, we set

$$V_r = V_0 \oplus \mathbb{H}^r,$$

and let

$$m = \dim_E V = m_0 + 2r.$$

The family of spaces $\{V_r : r \geq 0\}$ forms a Witt tower of ϵ-Hermitian spaces.

1.2. **The invariant ϵ_0.** Set

$$\epsilon_0 = \begin{cases} \epsilon, & \text{if } E = F; \\ 0, & \text{if } E \neq F, \end{cases}$$

and

$$d(n) = n + \epsilon_0$$

for $n \in \mathbb{Z}$.

1.3. **Isometry groups.** Let $H_r = H(V_r)$ be the associated isometry group, so that

$$H_r \cong \begin{cases} \mathrm{O}_m, & \text{if } \epsilon_0 = 1; \\ \mathrm{Sp}_m, & \text{if } \epsilon_0 = -1; \\ \mathrm{U}_m, & \text{if } \epsilon_0 = 0. \end{cases}$$

The family of groups $\{H_r : r \geq 0\}$ forms a Witt tower of classical groups.

The space V_r has a maximal isotropic space X_r of dimension r, so that

$$V_r = X_r \oplus V_0 \oplus X_r^*.$$

Let

$$P(X_r) = M(X_r) \cdot N(X_r)$$

be the maximal parabolic subgroup of H_r which stabilizes the space X_r. Then its Levi factor is

$$M(X_r) \cong \mathrm{GL}(X_r) \times H(V_0).$$

To simplify notation, we shall sometimes write P_r in place of $P(X_r)$.

The group H_r comes equipped with a family of maximal compact subgroups $\{K_{H_{r,v}}\}$, such that $K_{H_{r,v}}$ is hyperspecial for almost all places v of F. We may and do assume that $K_{H_{r,v}}$ is a special maximal compact for all finite v. Then

$$K_{H_r} = \prod_v K_{H_{r,v}} \subset H_r(\mathbb{A})$$

is a maximal compact subgroup of $H_r(\mathbb{A})$.

1.4. **$-\epsilon$-Hermitian Spaces.** Similarly, let W_n be a $2n$-dimensional vector space over E equipped with a nondegenerate $-\epsilon$-Hermitian form $\langle -, - \rangle$ and a maximal isotropic subspace Y_n of dimension n, so that

$$W_n = Y_n \oplus Y_n^*.$$

Let $G_n = G(W_n)$ be the associated isometry group or the unique two-fold cover thereof, according to

$$G_n = \begin{cases} \mathrm{Sp}_{2n}, & \text{if } \epsilon_0 = 1 \text{ and } m_0 \text{ is even;} \\ \mathrm{Mp}_{2n}, & \text{if } \epsilon_0 = 1 \text{ and } m_0 \text{ is odd;} \\ \mathrm{O}_{2n}, & \text{if } \epsilon_0 = -1; \\ \mathrm{U}_{2n}, & \text{if } \epsilon_0 = 0. \end{cases}$$

Thus, in addition to the space W_n, the group G_n depends on the space V_0 (or rather the parity of its dimension) in the first two cases.

For an r-dimensional subspace $Y_r \subset Y_n$, let

$$Q(Y_r) = L(Y_r) \cdot U(Y_r)$$

denote the maximal parabolic subgroup fixing Y_r. Then its Levi factor is

$$L(Y_r) \cong \mathrm{GL}(Y_r) \times G_{n-r},$$

though this is not entirely accurate in the metaplectic case. As before, we shall sometimes write Q_r in place of $Q(Y_r)$. Moreover, if there is a need to indicate that Q_r is a subgroup of G_n, we shall write Q_r^n. When $r = n$, Q_r is a Siegel parabolic subgroup of G_n.

1.5. **Summary.** The following table summarizes the groups discussed so far:

		$G_n = G(W_n)$	$H_r = H(V_r)$
$\epsilon_0 = 1$	m_0 even	Sp_{2n}	O_m
	m_0 odd	Mp_{2n}	O_m
$\epsilon_0 = -1$		O_{2n}	Sp_m
$\epsilon_0 = 0$		U_{2n}	U_m
Parabolic		$Q_r = Q(Y_r) = L(Y_r) \cdot U(Y_r)$	$P_r = P(X_r) = M(X_r) \cdot N(X_r)$
Levi factor		$L(Y_r) = \mathrm{GL}(Y_r) \times G_{n-r}$	$M(X_r) = \mathrm{GL}(X_r) \times H(V_0)$

1.6. **Complementary Spaces.** With W_n fixed, one may associate to each V_r a complementary space $V_{r'}$ in the same Witt tower, characterized by

$$m_0 + r + r' = d(n).$$

Of course, this only makes sense if $r' \geq 0$. Thus, the notion of complementary spaces is only relevant when

$$0 \leq m_0 + r \leq d(n).$$

When $V_{r'}$ exists, and $r \geq r'$, we may write:

$$V_r = X'_{r-r'} \oplus V_{r'} \oplus X'^{*}_{r-r'}$$

with $X'_{r-r'}$ isotropic of dimension $r - r'$. We set

$$m' = \dim V_{r'} = m_0 + 2r'.$$

1.7. **Measures.** Fix a nontrivial additive character ψ of $F\backslash\mathbb{A}$, This gives rise to the Haar measure dx_v on F_v (for all v) which is self-dual with respect to ψ_v. The product measure dx on $\mathbb{A}$ is independent of the choice of ψ and is the Tamagawa measure of $\mathbb{A}$. For any algebraic group G over F, we always use the Tamagawa measure on $G(\mathbb{A})$ when $G(\mathbb{A})$ is unimodular. This applies to the groups $G_n(\mathbb{A})$ and $H_r(\mathbb{A})$, as well as the Levi subgroups and unipotent radical of their parabolic subgroups. In addition, for any compact group K, we always use the Haar measure dk with respect to which K has volume 1. We ignore the metaplectic case here so as not to get overly technical.

In any case, we write $\tau(G)$ for the Tamagawa number of G. In particular, we have

$$\tau(H_r) = \begin{cases} 1, & \text{if } \epsilon_0 \neq 0; \\ 2, & \text{if } \epsilon_0 = 0, \end{cases}$$

except when $\epsilon_0 = 1$, $m_0 = 1$ and $r = 0$, in which case $H_r = \mathrm{O}_1$ has $\tau(\mathrm{O}_1) = 1/2$. (See [We2].)

2. Regularization of Theta Integral

In this section, we introduce the regularised theta integral which intervenes on one side of the Siegel-Weil formula.

2.1. **Weil Representation.** We need to fix some data to talk about the Weil representation of $H_r \times G_n$. Choose a Hecke character χ as follows:

$$\chi = \chi_{V_0} = \begin{cases} \text{the quadratic character of } \mathbb{A}^\times \text{ associated to } \mathrm{disc}(V_0), \text{ if } \epsilon_0 = 1; \\ \text{the trivial character of } \mathbb{A}^\times, \text{ if } \epsilon_0 = -1; \\ \text{a character of } \mathbb{A}_E^\times \text{ such that } \chi|_{\mathbb{A}^\times} = \chi_E^{m_0}, \text{ if } \epsilon_0 = 0. \end{cases}$$

Note that when $\epsilon_0 = \pm 1$, χ is completely determined by V_0.

Associated to the pair (ψ, χ), the group $G_n(\mathbb{A}) \times H_r(\mathbb{A})$ has a distinguished representation

$$\omega_{n,r} = \omega_{\psi,\chi,W_n,V_r}$$

known as the Weil representation. The Weil representation can be realized on the space $S((Y_n^* \otimes V_r)(\mathbb{A}_E))$ of Schwarz functions on the adelic space $(Y_n^* \otimes V_r)(\mathbb{A}_E)$. There is a natural $H_r \times G_n$-equivariant map

$$\theta_{n,r} : S((Y_n^* \otimes V_r)(\mathbb{A}_E)) \longrightarrow \{\text{Functions on } [H_r] \times [G_n].\}$$

This gives an automorphic realisation of the Weil representation.

2.2. **Theta Integral.** The theta integral we are interested in is:

$$I_{n,r}(\phi)(g) := \int_{[H_r]} \theta_{n,r}(\phi)(g,h)\,dh,$$

where we have written $[H_r]$ for $H(F)\backslash H(\mathbb{A})$ and $\phi \in S((Y_n^* \otimes V_r)(\mathbb{A}_E))$. It is an automorphic form on G_n if the integral converges absolutely. By Weil [Wel], it is known that the above integral converges if and only if

$$r = 0 \quad \text{or} \quad m - r = m_0 + r > d(n).$$

We call this the *Weil's convergent range.*

In particular, the pair (W_n, V_0) is in this range. Then for $\phi_0 \in \mathcal{S}(Y_n^* \otimes V_0)(\mathbb{A})$, we have the theta integral $I_{n,0}(\phi_0) \in \mathcal{A}(G_n)$. The classical Siegel-Weil formula, first formulated in [Wel] and extended by Kudla-Rallis in [KR1], determines the theta integral $I_{n,0}(\phi_0)$ as an Eisenstein series on G_n.

2.3. **Regularization.** In [KR3], Kudla and Rallis defined a regularization of the integral $I_{n,r}(\phi)$ beyond the Weil's convergent range. More precisely, we suppose that

$$r > 0, \quad \text{and} \quad m - r \leq d(n). \tag{2.1}$$

Note that the second inequality implies that

$$r \leq d(n) \leq n + 1.$$

We further impose (as did Kudla-Rallis) that

$$r \leq n, \tag{2.2}$$

or equivalently that the F-rank of G_n is at least that of H_r. Note that this is only an extra condition when $\epsilon_0 = 1$; when $\epsilon_0 = -1$ or 0, it follows automatically from (2.1) since $d(n) \leq n$. Indeed, the only case satisfying (2.1) but not (2.2) is when $\epsilon_0 = 1$ and $m_0 = 0$ so that $H_r = \mathrm{O}_{n+1,n+1}$ (split) and $G_n = \mathrm{Sp}_{2n}$. We also note that under (2.1):

$$0 < m \leq 2 \cdot d(n).$$

For a given place v, one can find an element z_G (resp. z_H) in the center of universal enveloping algebra of $\mathfrak{g}_v$ (resp. $\mathfrak{h}_v$) for v real ([KR3]) or in the spherical Hecke algebra of $G_n(F_v)$ (resp. $H_r(F_v)$) for v non-archimedean ([I1, I2, ?]) so that

$$\omega_{n,r}(z_G) = \omega_{n,r}(z_H),$$
$$\omega_{n,r}(g,h)\omega_{n,r}(z_G) = \omega_{n,r}(z_G)\omega_{n,r}(g,h)$$

(i.e. the action of z_G (and hence z_H) commutes with the action of $G_n(\mathbb{A}) \times H_r(\mathbb{A})$), and such that the function $\theta_{n,r}(\omega_{n,r}(z_G)\varphi)(g,-)$ is rapidly decreasing on a Siegel domain of $H_r(\mathbb{A})$. It follows that one can integrate $\theta_{n,r}(\omega_{n,r}(z_G)\phi)$ against any automorphic form on H_r; we shall integrate it against an auxiliary Eisenstein series on H_r to be defined next.

2.4. **Auxiliary Eisenstein series on H_r.** Recall P_r is the parabolic subgroup of H_r whose Levi is $M_r = \mathrm{GL}(X_r) \times H(V_0)$. Consider the family of (normalized) induced representation

$$I_{H_r}(s) := \mathrm{Ind}_{P_r(\mathbb{A})}^{H_r(\mathbb{A})} |\det|^s \boxtimes \mathbf{1}_{H_0}$$

where $|\det|^s$ is a character of $\mathrm{GL}(X_r)(\mathbb{A})$ and $\mathbf{1}_{H_0}$ is the trivial representation of $H_0(\mathbb{A})$. Let $f_s^0 \in I_{H_r}(s)$ be the K_{H_r}-spherical standard section with $f_s^0(1) = 1$. Then we define the Eisenstein series $E_{H_r}(s,-)$ on H_r by

$$E_{H_r}(s,h) = \sum_{\gamma \in P_r(F)\backslash H_r(F)} f_s^0(\gamma h)$$

for $h \in H_r(\mathbb{A})$ and $Re(s) \gg 0$. We call $E_{H_r}(s,-)$ the auxiliary Eisenstein series.

We are interested in the point

$$s = \rho_{H_r} := \frac{m-r-\epsilon_0}{2},$$

since at this point, the auxiliary Eisenstein series $E_{H_r}(s,-)$ has a pole of order 1 and its residue there is a constant function on $H_r(\mathbb{A})$. We set

$$\mathrm{Res}_{s=\rho_{H_r}} E_{H_r}(s,h) = \kappa_r$$

for all $h \in H_r(\mathbb{A})$. Furthermore, the regularizing element z_H acts on $E_{H_r}(s,-)$ by a scalar $P_{n,r}(s)$:

$$z_H * E_{H_r}(s,-) = P_{n,r}(s) \cdot E_{H_r}(s,-).$$

Here the scalar $P_{n,r}(s)$ depends on the choice of z_H, though we suppress the dependence from our notation. The function $P_{n,r}(s)$ can be explicitly computed; see [KR3, Lemma 5.5.3], [I1, p. 208], [I2, p. 249] and [T, Cor. 2.2.5] for the various cases.

2.5. **Regularized theta integral.** The **regularized theta integral** is defined to be the function

$$B^{n,r}(s,\phi)(g) = \frac{1}{\kappa_r \cdot P_{n,r}(s)} \cdot \int_{[H_r]} \theta_{n,r}(\omega_{n,r}(z)\phi)(g,h)\, E_{H_r}(s,h)\, dh.$$

The integral converges absolutely at all points s where $E_{H_r}(s,h)$ is holomorphic, and defines a meromorphic function of s (for fixed ϕ).

We can now explain why the extra condition $r \leq n$ as in (2.2) is necessary. Indeed, if this condition is not satisfied, one cannot hope to regularize the theta integral in the same way as above. Otherwise, one may integrate the regularized theta kernel against the Eisenstein series associated to the family of principal series representations induced from the minimal parabolic subgroup of H_r; this gives a meromorphic function in r complex variables whose iterated residue at a specific point is the regularized theta lift of the constant function of H_r. If this meromorphic function is not identically zero, then a Zariski open set of these minimal principal series representations will have a nonzero theta lift to G_n. For this to happen, it is necessary that $n \geq r$.

2.6. **First and Second term range.** The analytic behavior of $B^{n,r}(s,\phi)$ at the point of interest $s = \rho_H$ is described as follows:

Lemma 2.3. *(i) If $m \leq d(n)$, then $B^{n,r}(s,\phi)$ has a pole of order at most 1 at $s = \rho_{H_r}$.*

(ii) If $d(n) < m \leq 2 \cdot d(n)$, then $P_{n,r}(s)$ has a simple zero at $s = \rho_{H_r}$ and $B^{n,r}(s,\phi)$ has a pole of order at most 2 at $s = \rho_{H_r}$.

Because the analytic behavior of $B^{n,r}(s,\phi)$ at $s = \rho_{H_r}$ differs for different ranges of the pair (n,m), we introduce the following terminology outside the Weil's convergent range:

$$m < d(n) : \text{the first term range;}$$
$$m = d(n) : \text{the boundary case;}$$
$$d(n) < m \leq 2 \cdot d(n) : \text{the second term range.}$$

Observe that if V_r and $V_{r'}$ are complementary spaces, then (W_n, V_r) is in the first term range if and only if $(W_n, V_{r'})$ is in the second term range. Moreover, in the boundary case, we have $V_r = V_{r'}$. Note also that when $m > 2 \cdot d(n)$, one is automatically in the Weil's convergent range.

2.7. **Laurent expansion.** In the first term range and the boundary case, we may thus consider the Laurent expansion at $s = \rho_{H_r}$:

$$B^{n,r}(s,\phi) = \frac{B^{n,r}_{-1}(\phi)}{s-\rho_{H_r}} + B^{n,r}_0(\phi) + \cdots$$

whereas in the second term range, we have

$$B^{n,r}(s,\phi) = \frac{B^{n,r}_{-2}(\phi)}{(s-\rho_{H_r})^2} + \frac{B^{n,r}_{-1}(\phi)}{s-\rho_{H_r}} + B^{n,r}_0(\phi) + \cdots.$$

The functions $B^{n,r}_d(\phi)$ are automorphic forms on G_n and the linear map $\omega_{n,r} \to \mathcal{A}(G_n)$ (where $\mathcal{A}(G_n)$ is the space of automorphic forms on G_n) given by $\phi \mapsto B^{n,r}_d(\phi)$ is $G_n(\mathbb{A})$-equivariant. Moreover, if $B^{n,r}_d$ is the leading term in the Laurent expansion, it is $H_r(\mathbb{A})$-invariant.

The purpose of the regularized Siegel-Weil formula is to give an alternative construction of the automorphic forms $B^{n,r}_{-1}(\phi)$ and $B^{n,r}_{-2}(\phi)$.

3. Siegel Eisenstein Series

In this section, we introduce the other side of the Siegel-Weil formula, namely certain Siegel Eisenstein series.

3.1. **Siegel Eisenstein series.** We consider the global Siegel principal series representation $I^n_n(s,\chi)$ of $G_n(\mathbb{A})$.

$$I^n_n(s,\chi) := \mathrm{Ind}^{G_n(\mathbb{A})}_{Q_n(\mathbb{A})}(\chi \circ \det)\cdot|\det|^s.$$

Here, we take $\chi = \chi_{V_r} = \chi_{V_0}$ to be a Hecke character fixed earlier.

For a standard section $\Phi_s \in I^n_n(s,\chi)$, we consider the Siegel Eisenstein series defined for $\mathrm{Re}(s) \gg 0$ by

$$E(s,\Phi)(g) := \sum_{\gamma \in Q_n(F)\backslash G_n(F)} \Phi_s(\gamma g)$$

for $g \in G_n(\mathbb{A})$. Sometimes we write

$$E(s,\Phi) = E^{n,n}(s,\Phi)$$

when we want to emphasize the rank of the group. It admits a meromorphic continuation to $\mathbb{C}$.

3.2. **Global Siegel-Weil sections.** We are interested in certain special sections of $I^n_n(s,\chi)$ known as the Siegel-Weil sections. For an ϵ-Hermitian space V_r of dimension $m = m_0 + 2r$ over E, we have

$$\Phi^{n,r} : \mathcal{S}(Y_n^* \otimes V_r)(\mathbb{A}_E) \longrightarrow I^n_n(\frac{m-d(n)}{2},\chi)$$

defined by

$$\Phi^{n,r}(\phi)(g) = \omega_{n,r}(g)\phi(0).$$

The map $\Phi^{n,r}$ is $H_r(\mathbb{A})$-invariant and $G_n(\mathbb{A})$-equivariant. Its image is isomorphic to the maximal $H_r(\mathbb{A})$-invariant quotient of $\omega_{n,r}$, and the standard sections associated to its image is the space of Siegel-Weil sections (relative to V_r). Henceforth, we shall set

$$s_{m,n} = \frac{m-d(n)}{2}.$$

Observe that, outside the Weil's convergent range, we have:

$$\begin{cases} s_{m,n} > 0 \longleftrightarrow \text{second term range,} \\ s_{m,n} = 0 \longleftrightarrow \text{boundary case,} \\ s_{m,n} < 0 \longleftrightarrow \text{first term range.} \end{cases}$$

3.3. **The Laurent coefficients $A_d^{n,r}$.** Suppose that we are in the boundary or second term range, so that $0 \leq s_{m,n} \leq d(n)/2$. It is known that the Siegel Eisenstein series has at most a simple pole at $s = s_{m,n}$ which is attained for some standard section if $s_{m,n} > 0$. Considering the Siegel-Weil sections arising from V_r (with $\dim_E V_r = m$), we have the Laurent expansion

$$E(s, \Phi^{n,r}(\phi)) = \frac{A_{-1}^{n,r}(\phi)}{s - s_{m,n}} + A_0^{n,r}(\phi) + \cdots \quad \text{if } s_{m,n} > 0;$$

or

$$E(s, \Phi^{n,r}(\phi)) = A_0^{n,r}(\phi) + A_1^{n,r}(\phi) \cdot s + \cdots \quad \text{if } s_{m,n} = 0.$$

Thus $A_d^{n,r}$ is viewed as a linear map:

$$A_d^{n,r} = E_d^{n,m} \circ \Phi^{n,r} : \mathcal{S}(Y_n^* \otimes V_r)(\mathbb{A}) \longrightarrow \mathcal{A}(G_n).$$

4. The Siegel-Weil formula

The Siegel-Weil formula was discovered by Siegel in the context of classical modular forms and then cast in the representation theoretic language and considerably extended in an influential paper of Weil [Wel]. It identifies the global theta lift of the trivial representation of $H(V_r)$ to $G(W_n)$ as an Eisenstein series, at least when some convergence conditions are satisfied. In a series of 3 papers [KR1, KR2, KR3], Kudla and Rallis greatly extended the theory of the Siegel-Weil formula to situations where these convergence conditions are not satisfied. Their work culminates in a *regularized Siegel-Weil formula*, and they established what is now known as the *first term identity*, at least when $G(W_n)$ is symplectic and $H(V_r)$ orthogonal. Their work was subsequently refined and extended to other dual pairs by others ([Ik, I1, I2, I3, Y2, Y3, Mo, JS]), especially in the work of Ikeda, Ichino and Yamana.

In this section, we recall the known cases of the Siegel-Weil formula.

4.1. **Anisotropic case.** We first consider the case $r = 0$, i.e. the pair (W_n, V_0), so $H_0 = H(V_0)$ is anisotropic. In this case, the Siegel-Weil formula is due to Weil [Wel], Kudla-Rallis [KR1], Ichino [I3] and Yamana [Y2].

Theorem 4.1. *For $\phi \in \mathcal{S}(Y_n^* \otimes V_0)(\mathbb{A})$, the Eisenstein series $E(s, \Phi^{n,0}(\phi))$ is holomorphic at $s = s_{m,n} = (m - d(n))/2$ and*

$$c_{m,n} \cdot \tau(H_0)^{-1} \cdot I_{n,0}(\phi) = E(s_{m,n}, \Phi^{n,0}(\phi))$$

with

$$c_{m,n} = \begin{cases} 1, & \text{if } s_{m,n} > 0, \\ 2, & \text{if } s_{m,n} \leq 0; \end{cases}$$

and $\tau(H_0)$ denoting the Tamagawa number of H_0.

More precisely, Weil [Wel] established the case when $m > 2 \cdot d(n)$, Kudla-Rallis [KR1] established the case when $\epsilon_0 = 1$, Ichino showed the case when $\epsilon_0 = 0$ and $d(n) < m \leq 2 \cdot d(n)$, and Yamana completed the case $\epsilon_0 = 0$ and $m \leq d(n)$. There is nothing to check when $\epsilon_0 = -1$.

4.2. **Ikeda's map.** Before coming to the first term identity, we need to recall one last ingredient. Suppose that $V_r \supset V_{r'}$ are complementary spaces. Writing

$$V_r = X'_{r-r'} \oplus V_{r'} \oplus X'^*_{r-r'},$$

we define a map

$$\mathrm{Ik}^{n,r} : \mathcal{S}(Y_n^* \otimes V_r)(\mathbb{A}) \longrightarrow \mathcal{S}(Y_n^* \otimes V_{r'})(\mathbb{A}),$$

given by

$$\mathrm{Ik}^{n,r}(\phi)(a) = \int_{(Y_n^*\otimes X'_{r-r'})(\mathbb{A})} \phi(x,a,0)\,dx,$$

for $a \in (Y_n^* \otimes V_{r'})(\mathbb{A})$. Thus, $\mathrm{Ik}^{n,r}$ is the composite

$$\begin{array}{c} \mathcal{S}(Y_n^*\otimes V_r) = \mathcal{S}(Y_n^*\otimes V_{r'}) \otimes \mathcal{S}(Y_n^* \otimes (X'_{r-r'} + X'^*_{r-r'})) \\ \Big\downarrow {\scriptstyle Id\otimes\mathcal{F}_1} \\ \mathcal{S}(Y_n^*\otimes V_{r'}) \otimes \mathcal{S}(W_n\otimes X'_{r-r'}) \\ \Big\downarrow {\scriptstyle Id\otimes ev_0} \\ \mathcal{S}(Y_n^*\otimes V_{r'}) \end{array}$$

where

$$\mathcal{F}_1 : \mathcal{S}(Y_n^*\otimes (X'_{r-r'} + X'^*_{r-r'})) \longrightarrow \mathcal{S}(W_n\otimes X'_{r-r'})$$

is the partial Fourier transform in the subspace $(Y_n^*\otimes X'^*_{r-r'})(\mathbb{A})$, and ev_0 is evaluation at 0.

The map $\mathrm{Ik}^{n,r}$ was used by T. Ikeda in his refinement ([Ik]) of the first term identity of Kudla-Rallis ([KR3]). Thus we call $\mathrm{Ik}^{n,r}$ an Ikeda map.

4.3. **First term identity.** We can now recall the regularized Siegel-Weil formula in the first term range. Here the problem is to identify the automorphic form $B_{-1}^{n,r}(\phi)$ for $\phi \in \mathcal{S}(Y_n^*\otimes V_m)(\mathbb{A})$. Since $B_{-1}^{n,r}$ is the leading term of the Laurent expansion in this case, the resulting identity is called the first term identity.

The following theorem was shown by Kudla-Rallis [KR3], Moeglin [Mo], Ichino [I1, Theorem 3.1] and [I2, Theorems 4.1 and 4.2]), Jiang-Soudry [JS, Theorem 2.4] and Yamana [Y2, Proposition 5.8]:

Theorem 4.2. *Suppose that $m + m' = 2\cdot d(n)$ with $m > m'$, where recall $m' = m_0 + 2r'$. Then for $\phi \in \mathcal{S}(Y_n^*\otimes V_r)(\mathbb{A})$, we have:*

$$A_{-1}^{n,r}(\phi) = \kappa_{r,r'}\cdot \tau(H_r)^{-1}\cdot B_{-1}^{n,r'}(\mathrm{Ik}^{n,r}(\pi_{K_{H_r}}\phi)),$$

where $\kappa_{r,r'}$ is an explicit nonzero constant and $\pi_{K_{H_r}}$ is the projection onto the K_{H_r}-fixed subspace. In the boundary case, where $m = m' = d(n)$, one has:

$$A_0^{n,r}(\phi) = 2\cdot\tau(H_r)^{-1}\cdot B_{-1}^{n,r}(\phi).$$

It should be mentioned that the above statement of the first term identity is not quite natural, since the pair (n, r') is in the first term range, but we are considering $\phi \in \mathcal{S}(Y_n^*\otimes V_r)(\mathbb{A})$ which is the Weil representation in the second term range. In [Y2, Theorem 2.2], Yamana has shown the more natural statement below:

Theorem 4.3. *Suppose that (n, r') is in the first term range, so that $m' = m_0 + 2r' < d(n)$. Then for $\phi \in \mathcal{S}(Y_n^*\otimes V_{r'})(\mathbb{A})$, $E(s, \Phi^{n,r'}(\phi))$ is holomorphic at $s = s_{m',n} < 0$, and*

$$A_0^{n,r'}(\phi) = 2\cdot\tau(H_r)^{-1}\cdot B_{-1}^{n,r'}(\phi).$$

5. **The Second Term Identity**

Now we come to our main result. The goal is to relate the automorphic forms $B_{-1}^{n,r}(\phi)$ and $B_{-2}^{n,r}(\phi)$ with $A_0^{n,r}(\phi)$ and $A_{-1}^{n,r}(\phi)$. Earlier special cases were shown in [GT], [KRS], [K], [T] and [X].

5.1. **Main result.** The following is the main result of [GQT].

Theorem 5.1 (Siegel-Weil formula). *Suppose that $0 < r \leq n$ and $n + \epsilon_0 < m \leq n + \epsilon_0 + r$, so that we are in the second term range. Then one has:*

(i) (First term identity) For all $\phi \in \omega_{n,r}$, one has

$$A_{-1}^{n,r}(\phi) = \tau(H(V_r))^{-1} \cdot B_{-2}^{n,r}(\phi).$$

(ii) (Second term identity) For all $\phi \in \omega_{n,r}$, one has

$$A_0^{n,r}(\phi) = \tau(H(V_r))^{-1} \cdot B_{-1}^{n,r}(\phi) + c_{n,r} \cdot \tau(H(V_{r'}))^{-1} \cdot B_0^{n,r'}(\mathrm{Ik}^{n,r}(\pi_{K_{H_r}}\phi)) \mod \mathrm{Im}\, A_{-1}^{n,r}.$$

Here, $c_{n,r}$ is some explicit constant and $r' < r$ is such that

$$\dim V_r + \dim V_{r'} = 2 \cdot (n + \epsilon_0),$$

so that $V_{r'}$ is the complementary space to V_r with respect to W_n. Moreover,

$$\mathrm{Ik}^{n,r} : \omega_{n,r} \longrightarrow \omega_{n,r'}$$

is the Ikeda map which is $G(W_n) \times H(V_{r'})$-equivariant.

5.2. **Strategy.** We give a brief sketch of the main idea of the proof. The strategy of the proof has already been used in [GT] to prove a weak form of the theorem in the case when $\epsilon_0 = -1$. It is based on induction on the integer

$$2 \cdot s_{m,n} = m - d(n).$$

Let us illustrate how one starts the induction argument by going from the theorem in the boundary case ($s_{m,n} = 0$) to the first case in the second term range ($2 \cdot s_{m,n} = 1$). Thus suppose we are dealing with the Weil representation of $G(W_{n+1}) \times H(V_r)$ with $m = d(n+1)$. Then we are in the boundary case, and for $\phi \in \mathcal{S}(Y_{n+1}^* \otimes V_r)(\mathbb{A})$, we have the first term identity supplied by Theorem 4.2:

$$A_0^{n+1,r}(\phi) = 2 \cdot \tau(H_r)^{-1} \cdot B_{-1}^{n+1,r}(\phi).$$

We may take the constant term of both sides with respect to the maximal parabolic $Q^{n+1}(Y_1) = L^{n+1}(Y_1) \cdot U^{n+1}(Y_1)$ of G_{n+1}, which gives

$$A_0^{n+1,r}(\phi)_{U^{n+1}(Y_1)} = 2 \cdot \tau(H_r)^{-1} \cdot B_{-1}^{n+1,r}(\phi)_{U^{n+1}(Y_1)},$$

which is an identity of automorphic forms on $L(Y_1) = \mathrm{GL}(Y_1) \times G(W_n)$, where $W_n = Y_n' \oplus Y_n'^*$. (Note that the superscript $^{n+1}$ in the groups $Q^{n+1}(Y_1)$ etc indicates the rank of the ambient group G_{n+1}.)

Now note that both sides of this last equation are constant terms of the Laurent coefficients of some Eisenstein series. Thus we may compute them by first computing the constant term of the relevant Eisenstein series, followed by extracting the relevant Laurent coefficients. The constant terms along $U^{n+1}(Y_1)$ of Eisenstein series on G_{n+1} are simply the sum of Eisenstein series on $L(Y_1)$. We can compare terms on both sides with the same $\mathrm{GL}(Y_1)$ part, and thus obtain an identity of Eisenstein series on $G(W_n)$. It remains to identify these Eisenstein series and the sections where they are evaluated with the desired objects coming from the Weil representation of $G_n \times H_r$. The theorem will then follow by extracting the relevant Laurent coefficients (plus some extra technical work).

5.3. **Rallis inner product formula.** The main application of the second term identity is he following Rallis inner product formula for the theta lifting from $G(U_n)$ to $H(V_r)$, where U_n is any n-dimensional ϵ-Hermitian space. Since the Rallis inner product formula has been established in the convergent range and the first term range, we shall focus on the case

$$(n+\epsilon_0) < m \leq 2\cdot(n+\epsilon_0).$$

Then we have:

Theorem 5.2 (Rallis inner product formula). *Suppose that*

$$(n+\epsilon_0) < m \leq 2\cdot(n+\epsilon_0) \quad \text{and} \quad r \leq n$$

so that we are either in the second term range or the convergent range, depending on whether $m \leq (n+\epsilon_0)+r$ or not. Let π be a cuspidal representation of $G(U_n)$ and consider its global theta lift $\Theta_{n,r}(\pi)$ to $H(V_r)$.

(i) Assume that $\Theta_{n,j}(\pi) = 0$ for $j < r$, so that $\Theta_{n,r}(\pi)$ is cuspidal. Then for $\phi_1, \phi_2 \in \omega_{\psi,U_n,V_r}$ and $f_1, f_2 \in \pi$,

$$\langle \theta(\phi_1, f_1), \theta(\phi_2, f_2)\rangle = \tau(H(V_r))\,\mathrm{Val}_{s=s_{m,n}}\left(L(s+\frac{1}{2},\pi)\cdot Z^*(s,\phi_1\otimes\overline{\phi_2}, f_1, f_2)\right),$$

where

$$s_{m,n} = \frac{m-n-\epsilon_0}{2} > 0,$$

$L(s,\pi)$ is the standard L-function of π, and $Z^*(s,-)$ denotes the normalized doubling zeta integral as in [Y4].

(ii) Assume further that for all places v of F, the local theta lift $\Theta_{n,r}(\pi_v)$ is nonzero. Then $L(s+\frac{1}{2},\pi)$ is holomorphic at $s = s_{m,n}$, so that in the context of (i),

$$\langle \theta(\phi_1, f_1), \theta(\phi_2, f_2)\rangle = \tau(H(V_r))\cdot L(s_{m,n}+\frac{1}{2},\pi)\cdot Z^*(s_{m,n},\phi_1\otimes\overline{\phi_2}, f_1, f_2).$$

As a consequence, we are able to deduce the following local-global criterion for the nonvanishing of global theta lifts.

Theorem 5.3 (Local-Global nonvanishing criterion). *Assume the same conditions on (m,n) as in Theorem 5.2. Let π be a cuspidal representation of $G(U_n)$ and consider its global theta lift $\Theta_{n,r}(\pi)$ to $H(V_r)$. Assume that $\Theta_{n,j}(\pi) = 0$ for $j < r$, so that $\Theta_{n,r}(\pi)$ is cuspidal.*

(i) If $\Theta_{n,r}(\pi)$ is nonzero, then

- *(a) for all places v, $\Theta_{n,r}(\pi_v) \neq 0$, and*
- *(b) $L(s_{m,n}+\frac{1}{2},\pi) \neq 0$ i.e. nonzero holomorphic.*

(ii) The converse to (i) holds when one assumes one of the following conditions:

- *$\epsilon_0 = -1$;*
- *$\epsilon_0 = 0$ and $E_v = F_v \times F_v$ for all archimedean places v of F;*
- *$\epsilon_0 = 1$ and F is totally complex;*
- *$m = d(n)+1$.*

(iii) In general, under the conditions (a) and (b) in (i), there is an ϵ-Hermitian space V' over E such that

- *$V'\otimes F_v \cong V\otimes F_v$ for every finite or complex place of F;*
- *the global theta lift $\Theta_{U_n,V'}(\pi)$ of π to $H(V')$ is nonzero.*

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