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ON CONFLUENT HYPERGEOMETRIC FUNCTIONS AND REAL ANALYTIC SIEGEL MODULAR FORMS OF DEGREE 2

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We consider a vector-valued version of the confluent hypergeometric functions on the real symplectic groups, [11]. We characterize their vanishing in certain cases in Section 1, and give them another expressions of Fourier-Jacobi type in Section 2. They are applied to study Fourier-Jacobi expansions of certain real analytic Eisenstein series and also to construct a real analytic Siegel modular form.

1. VANISHING OF INTEGRALS

Let G be the real symplectic group of degree n with a maximal compact subgroup $K \simeq \mathrm{U}(n)$. We put $\mu_g(\mathbf{i}) = C\mathbf{i} + D$ for $g = \left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right) \in G$ with $\mathbf{i} = \sqrt{-1}1_n$. Let $\varphi(x)$ be a polynomial on complex symmetric matrices $x \in S(\mathbb{C})$ of size n, and let ℓ be an even integer. Then we define a function $\varphi_\ell(g,s) := \det(\mu_g(\mathbf{i}))^{-\frac{s-\ell}{2}} \det(\overline{\mu_g(\mathbf{i})})^{-\frac{s+\ell}{2}} \varphi(\mu_g(\mathbf{i})^{-1}\overline{\mu_g(\mathbf{i})})$ of $g \in G$ and $s \in \mathbb{C}$. A natural action of K on $S(\mathbb{C})$, and hence on $\varphi(x)$, shows that $\varphi_\ell(g,s)$ defines a K-finite vector in $I_P(s)$, a degenerate principal series representation induced from the Siegel maximal parabolic subgroup P of G.

For a real symmetric nonsingular n by n matrix $B \in S(\mathbb{R})$ we define an integral

$$W_B(g,s)(\varphi_\ell) := \int_{S(\mathbb{R})} \mathbf{e}(-\mathrm{tr}(Bx))\varphi_\ell(w_2 n(x)g,s)dx \tag{1.1}$$

with $\mathbf{e}(t) = e^{2\pi i t}$, $w_2 := \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}$ and $n(x) := \begin{pmatrix} 1_n & x \\ 0_n & 1_n \end{pmatrix}$. This is the confluent hypergeometric function associated with $\varphi_\ell(g,s) \in I_P(s)$.

If the parameter s is specialized to an integer, then $I_P(s)$ will become reducible. In that case we can obtain a vanishing criterion of (1.1) depending on each $\varphi_\ell(g,s) \in I_P(s)$ and the signature (p,q) of $B \in S(\mathbb{R})$. A typical example of this can be stated as following. Assume that $n=2$ and that $\varphi(x)$ belongs to an irreducible $\mathrm{U}(2)$-module of highest weight $(r,0)$ with an even integer $r \geq 0$. We understand $\det(x)$ is of weight $(2,2)$. In particular $\varphi_\ell(g,s)$ is of weight $(r-\ell,-\ell)$.

Proposition 1.1 ([6], [14]). *Let $n=2$ and $s=d+1$ with a positive even integer d. Assume that φ is of weight $(r,0)$. Then $W_B(g,d+1)(\varphi_\ell)$ is vanishing in the following cases.*

(i) *$r-\ell < d$ and $-\ell \leq -d$, and $(p,q) = (2,0)$.*
(ii) *$r-\ell \geq d$ and $-\ell \leq -d$, and $(p,q) = (0,2)$ or $(2,0)$.*
(iii) *$r-\ell \geq d$ and $-\ell > -d$, and $(p,q) = (0,2)$.*

As the complements we can prove that $W_B(g,d+1)(\varphi_\ell) \neq 0$ if $r-\ell \geq d$ and $-\ell \leq -d$, and $(p,q) = (1,1)$, for example. For results in higher degrees, see [7].

Proof. The proof proceeds as follows. It suffices to discuss the vanishing of

$$(1.2)\quad \int_{S(\mathbb{R})} \mathbf{e}(-\mathrm{tr}(Bx))\det(\varepsilon(x))^{-\frac{d-\ell+1}{2}}\det(\overline{\varepsilon(x)})^{-\frac{d+\ell+1}{2}}\varphi(\varepsilon(x)^{-1}\overline{\varepsilon(x)})dx, \quad \varepsilon(x)=1_2-ix.$$

Here we remark $\varepsilon(x)^{-1}\overline{\varepsilon(x)} = 2\varepsilon(x)^{-1}-1_2$. Then the following lemma is crucial.

Lemma 1.2 (A generalized binomial expansion formula). *Assume φ is of weight (r_1,r_2). Then*

$$\varphi(1_2+x)=\sum_{(r_1',r_2')}\varphi_{r_1',r_2'}(x),$$

where $\varphi_{r_1',r_2'}(x)$ is a polynomial belonging to the U(2)*-module of weight (r_1',r_2') with $0\le r_1'\le r_1$ and $0\le r_2'\le r_2$.*

This can be proved by constructing a basis of U(2)-modules by using Jack polynomials of two variables. Then the above binomial expansion is reduced to the corresponding property of Jack polynomials which was established by Lassalle [5], Kaneko [3]. See Yokokawa [14] for details, and [7] for the proof in higher degree case.

According to the lemma, (1.2) can be written as a sum of

$$(1.3)\quad \int_{S(\mathbb{R})} \mathbf{e}(-\mathrm{tr}(Bx))\det(\overline{\varepsilon(x)})^{-\frac{d+\ell+1}{2}}\det(\varepsilon(x))^{-\frac{d-\ell+1}{2}}\varphi_{r',0}(\varepsilon(x)^{-1})dx$$

with $r'\le r$. Each of these integrals can be studied by following the arguments by Shimura [11] and [12], Proposition 3.1. It implies indeed that (1.3) are vanishing for all $r'\le r$, if $r-\ell\ge d$ and $-\ell\le -d$ and $(p,q)=(2,0)$, for example. Thus the vanishing of (1.2) is concluded in this case. On the other hand, (1.2) can be rewritten in another form as

$$(1.2)=\int_{S(\mathbb{R})} \mathbf{e}(-\mathrm{tr}(Bx))\det(\varepsilon(x))^{-\frac{d+r-\ell+1}{2}}\det(\overline{\varepsilon(x)})^{-\frac{d-r+\ell+1}{2}}\psi(\overline{\varepsilon(x)}^{-1}\varepsilon(x))dx$$

with an appropriate ψ of weight $(r,0)$. By repeating the previous arguments, this expression yields that (1.2) is vanishing if $r-\ell\ge d$ and $-\ell\le -d$ and $(p,q)=(0,2)$. This combined with the above gives the assertion in (ii) of the proposition. □

2. Expressions of Fourier-Jacobi type

Let us take $\varphi=1$ of weight $(0,0)$ for brevity, and put $s=d+1$ and $\ell=d$ in (1.1). Then we have

$$(2.1)\quad (1.1)=\det(a)^{2-d}\int_{S(\mathbb{R})} \mathbf{e}(-\mathrm{tr}(B[a]x))\det(\varepsilon(x))^{-\frac{1}{2}}\det(\overline{\varepsilon(x)})^{-d-\frac{1}{2}}dx$$

when $g=m(a):=\begin{pmatrix} a & 0_2 \\ 0_2 & {}^ta^{-1}\end{pmatrix}$, $a=\begin{pmatrix}\sqrt{v'} & q/\sqrt{v} \\ 0 & \sqrt{v}\end{pmatrix}\in \mathrm{GL}_2(\mathbb{R})$, $v,v'>0$ and $q\in\mathbb{R}$. Also let us put coordinates on $x\in S(\mathbb{R})$ as $x=\begin{pmatrix} u' & p \\ p & u\end{pmatrix}$.

Assume that B is of the form $B=\begin{pmatrix}1 & 0\\ \ell & 1\end{pmatrix}\begin{pmatrix}1 & \lambda \\ \lambda & n\end{pmatrix}\begin{pmatrix}1 & \ell \\ 0 & 1\end{pmatrix}$ with $\lambda=0$ or $\frac{1}{2}$ and $\ell,n\in\mathbb{Z}$ (index 1) and is nondegenerate.

Proposition 2.1. *With the above setting (2.1) is equal to*

$$(2\pi v')^{\frac{d+1}{2}} e^{-2\pi v'} (2\pi v)^{\frac{d+1}{2}} \int_0^\infty e^{-4\pi v' t} \Omega\left(4\pi|\det(B)|v, 4\pi(\tfrac{q}{v}+\ell+\lambda)^2 v; \tfrac{t}{1+t}\right)(1+t)^{-1\frac{1}{2}} t^{d-\frac{1}{2}} dt,$$

when $\det(B) = n - \lambda^2 < 0$. *On the other hand, it is vanishing, when* $\det(B) > 0$. *Here we are defining*

$$\Omega(x,y;w) := (1-w)^{\frac{1}{2}} \exp\left(-\frac{x+y}{2}\right) \sum_{\kappa=0}^{\infty} \frac{\Gamma(\kappa+1)}{\Gamma(d+\frac{1}{2}+\kappa)} L_\kappa^{d-\frac{1}{2}}(x) L_\kappa^{-\frac{1}{2}}(y) w^\kappa$$

with $|w| < 1$ *using the Laguerre polynomials* $L_\kappa^\nu(z)$.

We note that $v^{\frac{d}{2}+\frac{1}{4}} e^{-2\pi|\det(B)|v} L_\kappa^{d-\frac{1}{2}}(4\pi|\det(B)|v)$ is the Whittaker functions of the antiholomorphic discrete series representation $\overline{\pi}_{d+\frac{1}{2}}$ of $\widetilde{SL_2}(\mathbb{R})$ of SO(2)-type (= weight) $-d-\frac{1}{2}-2\kappa$, and its product with $v^{\frac{1}{4}} e^{-2\pi(\frac{q}{v}+\ell+\lambda)^2 v} L_\kappa^{-\frac{1}{2}}(4\pi(\frac{q}{v}+\ell+\lambda)^2 v)$, which is of weight $\frac{1}{2}+2\kappa$, gives the Whittaker function of weight $-d$ belonging to a discrete series representation of the real Jacobi group. This means that $(2\pi v)^{\frac{d+1}{2}} \Omega(4\pi|\det(B)|v, 4\pi(\frac{q}{v}+\ell+\lambda)^2 v; \frac{t}{1+t})$ is a generating series of Whittaker functions of weight $-d$ on the real Jacobi group. Moreover, we should remark the generalized Hille-Hardy formula (Erdélyi [1], Rangarajan [9], and Srivastava [13]):

$$\Omega(x,y;w) = \Gamma(d+\tfrac{1}{2})^{-1} \exp\left(-\frac{x+y}{2}\cdot\frac{1+w}{1-w}\right) \Phi_3\left(d, d+\frac{1}{2}; \frac{xw}{1-w}, \frac{xyw}{(1-w)^2}\right),$$

where $\Phi_3(\beta,\gamma,X,Y)$ is an Humbert's confluent hypergeometric function, [2], Vol. I, p.225, (22). Then we can estimate the right hand side, cf. Shimomura [10], which is essential to verify the convergence of the integral expression in the proposition.

3. A SCALAR VALUED EISENSTEIN SERIES

We can apply the local formula in Proposition 2.1 to study the Fourier-Jacobi expansion of a scalar-valued Eisenstein series. Define at every finite prime p

$$\Lambda_p(n(x_p)m(a_p)k_p) := |\det(a_p)|_p^{d+1}$$

with $n(x_p)m(a_p) \in P(\mathbb{Q}_p)$ and $k_p \in G(\mathbb{Z}_p)$, and

$$\Lambda_\infty(g_\infty) := \det(\mu_{g_\infty}(\mathbf{i}))^{-\frac{1}{2}} \det(\overline{\mu_{g_\infty}(\mathbf{i})})^{-d-\frac{1}{2}}$$

with an even integer $d \geq 4$. We set $\Lambda(g) := \Lambda(g_\infty)\prod_p \Lambda_p(g_p)$, $g \in G(\mathbb{A})$, and define

$$E(g) := \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \Lambda(\gamma g).$$

It is a scalar valued Eisenstein series. We set $g = n(x_\infty)m(a_\infty)\prod_p k_p$ with $x_\infty = \begin{pmatrix} u' & p \\ p & u \end{pmatrix}$ and $a_\infty = \begin{pmatrix} \sqrt{v'} & q/\sqrt{v} \\ 0 & \sqrt{v'} \end{pmatrix}$, and consider the Fourier-Jacobi expansion

$$E(g) = \sum_{m\in\mathbb{Z}} \mathrm{FJ}_m(\tau, z; v' + \tfrac{q^2}{v}) \mathbf{e}(mu'), \quad \tau = u + iv, \quad z = p + iq.$$

Proposition 3.1. *Let $m=1$. Then there exists a family $\{\phi_1^{\kappa}(\tau,z) \mid \kappa=0,1,2,\dots\}$ of real analytic Jacobi form of index 1 and weight $-d$ satisfying the following properties.*

(i) *$\phi_1^0(\tau,z)$ is a skew holomorphic Jacobi Eisenstein series of index 1 and weight $-d$.*

(ii) *$\phi_1^{\kappa}(\tau,z)$ is obtained by differentiating $\phi_1^0(\tau,z)$ by k times.*

(iii) *The generating series*

$$\phi_1^{\Sigma}(\tau,z;w) := (1-w)^{\frac{1}{2}} \sum_{\kappa=0}^{\infty} \frac{\Gamma(\kappa+1)}{\Gamma(d+\frac{1}{2}+\kappa)} \phi_1^{\kappa}(\tau,z) w^{\kappa}, \qquad |w|<1$$

converges absolutely.

(iv) *The coefficient $\mathrm{FJ}_1(\tau,z;v'+\frac{q^2}{v})$ of index 1 is equal to*

$$(2\pi v')^{\frac{d+1}{2}} e^{-2\pi v'} \int_0^{\infty} e^{-4\pi v' t} \phi_1^{\Sigma}\left(\tau,z;\frac{t}{1+t}\right)(1+t)^{-\frac{1}{2}} t^{d-\frac{1}{2}} dt. \tag{3.1}$$

This result refines Kohnen's limit formula, [4]. Also by applying suitable operator, (3.1) yields a description of every coefficient of a positive index. As concerns the coefficients of negative indices we will meet another ingredient that did not appear in the case of positive index.

4. Vector-valued Siegel modular forms

One can generalize the results in Section 3 to a vector-valued Eisenstein series. We take a polynomial belonging to the $\mathrm{U}(2)$-module $V(d)$ of weight $(2d,0)$ and put

$$\Lambda_\infty(g_\infty)(\varphi) := \varphi_d(g_\infty, d+1)$$

using the notation in Section 1. Then we set $\Lambda(g)(\varphi) := \Lambda_\infty(g_\infty)(\varphi)\prod_p \Lambda_p(g_p)$ and define

$$E(g)(\varphi) := \sum_{\gamma\in P(\mathbb{Q})\backslash G(\mathbb{Q})} \Lambda(\gamma g)(\varphi). \tag{4.1}$$

This belongs to the $\mathrm{U}(2)$-module of weight $(d,-d)$ according to the right K-translation.

Proposition 1.1 implies that the Siegel-Fourier expansion of (4.1) is supported on those B of signature $(1,1)$, and besides, $(1,0)$, $(0,1)$, and $B=0_2$. Now we are concerned with the Fourier-Jacobi expansion. Then it turns out that this vector valued Eisenstein series has suitable symmetry for its coefficients of positive and negative indices and that we can treat them in a parallel way. Indeed, the coefficient of indices ± 1 can be described by suitably modifying the expressions (3.1). Besides these, we can also describe the coefficient of index 0, thus the Fourier-Jacobi expansion of $E(g)(\varphi)$ is understood well explicitly. See [8] for the details.

Our method can be extended to study other Siegel-type Fourier series of degree 2. Keep that φ varies in $V(d)$ and consider $W_B(g)(\varphi) := W_B(g,d+1)(\varphi_d)$ defined in (1.1). Besides it, let $h(\tau)$ be a cusp form of weight $d+\frac{1}{2}$ for $\Gamma_0(4)$ that corresponds to a normalized cuspidal eigenform of weight $2d$ for $\mathrm{SL}_2(\mathbb{Z})$ by Shimura correspondence. Consider its Fourier expansion

$$h(\tau) = \sum_{\ell=1}^{\infty} c(\ell)\mathbf{e}(\ell\tau).$$

Let us define

$$F(g_\infty k;\varphi) := \sum_B F_B(g_\infty k)(\varphi) \text{ for } g_\infty k \in G(\mathbb{R})\prod_p G(\mathbb{Z}_p), \tag{4.2}$$

where the coefficients $F_B(g_\infty k;\varphi)$ are determined by

(i) If $\mathbf{D}_B := -\det(2B) > 0$, then

$$F_B(g_\infty k;\varphi) := \left(\sum_{t|e_B} t^d c\left(\frac{\mathbf{D}_B}{t^2}\right)\right)\mathbf{D}_B^{\frac{1}{2}-d} W_B(g_\infty)(\varphi),$$

where $e_B := \gcd(m,r,n)$ for $B = \begin{pmatrix} m & r/2 \\ r/2 & n\end{pmatrix}$ with $m,n,r \in \mathbb{Z}$.

(ii) If $\mathbf{D}_B < 0$, or if $\operatorname{rank}(B) = 1$, then $F_B(g_\infty k;\varphi) := 0$.

(iii) If $B = 0_2$, then

$$F_{0_2}(g_\infty k;\varphi) := \sum_{0\neq \ell\in\mathbb{Z}} \left(\sum_{t|\ell} t^{d-1} c\left(\frac{\ell^2}{t^2}\right)\right)|\ell|^{1-2d} W_\ell^P(g_\infty)(\varphi)$$

where we put

$$W_\ell^P(g_\infty)(\varphi) := \int_0^\infty \mathbf{e}(-\ell s)\int_0^\infty \Lambda_\infty\left(w_1 n\left(\begin{pmatrix}0 & 0\\ 0 & t\end{pmatrix}\right) m\left(\begin{pmatrix}0 & 1\\ 1 & s\end{pmatrix}\right) g_\infty\right) dtds$$

with $w_1 = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\end{pmatrix}$.

The compact group $K \simeq \mathrm{U}(2)$ acts on $\{F(g_\infty k;\varphi) \mid \varphi \in V(d)\}$ by the right translation, which has the weight $(d,-d)$. Using our local formulas we can rewrite (4.2) into a series of Fourier-Jacobi type and study its transformation property for the action of Jacobi group. Then we get the following result by repeating the argument in the holomorphic case, [15].

Theorem 4.1 ([8], Theorem 9.4). *For every $\varphi \in V(d)$ (4.2) satisfies*

$$F(\gamma g_\infty k;\varphi) = F(g_\infty k;\varphi)$$

for all $\gamma \in \mathrm{Sp}(2,\mathbb{Z})$, thus it defines a real analytic Siegel modular form of degree 2.

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