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Some remarks on the automorphic spectrum of the inner forms of SL(N)

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Abstract

In this survey article, we start by reviewing Arthur's conjectures for the multiplicities of L^2-automorphic representations in the discrete spectrum. We also give a sketch of the main ideas thereof, as exemplified in Arthur's endoscopic classification for classical groups, and then discuss its relation with the Hiraga-Saito theory for the group SL(N) and its inner forms. This is based on a talk given in the RIMS workshop "Automorphic Representations and Related Topics", Kyōto 2013.

1 Multiplicities in the discrete spectrum

Let F be a number field and $\mathbb{A} := \mathbb{A}_F$ its ring of adèles. Fix an algebraic closure $\bar{F}$ of F. We define $\Gamma_F := \mathrm{Gal}(\bar{F}/F)$ and denote its Weil group by W_F. The Weil-Deligne group of F is denoted by W'_F.

For a connected reductive F-group G, one of the main concerns of the theory of L^2-automorphic forms is to study the right regular representation of $G(\mathbb{A})$ on

$$L^2(G(F)\backslash G(\mathbb{A})^1) = L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A})^1) \oplus (\text{continuous spectrum})$$

where $G(\mathbb{A})^1$ is the kernel of the Harish-Chandra homomorphism $H_G : G(\mathbb{A}) \to \mathfrak{a}_G$.

It is known that the discrete part $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A})^1)$ decomposes into $\bigoplus_\pi m(\pi)\pi$ with multiplicities $m(\pi) < \infty$ for all $\pi = \bigotimes'_v \pi_v$. Our main goal is the study of $m(\pi)$. In this article, we adopt the usual convention that the archimedean components of π are viewed as Harish-Chandra modules. Assume hereafter:

1) the existence of the *automorphic Langlands group* $L_F \to W_F$ (we shall write $L'_F := W_F \times \mathrm{SU}(2)$);

2) G is quasisplit.

The first assumption is of course too extravagant; we use it only to streamline the exposition. In particular, we can then talk about the A-parameters $\psi : L'_F \to {}^L G := \hat{G} \rtimes W_F$. The $\hat{G}$-conjugacy classes of A-parameters are expected to parametrize packets of automorphic representations of $G(\mathbb{A})$.

The internal structure of A-packets are expected to be controlled by the groups

$$\begin{aligned} S_\psi &:= \left\{ \hat{g} \in \hat{G} : \hat{g}\psi\hat{g}^{-1} = a \cdot \psi,\ a \in \ker{}^1(W_F, Z_{\hat{G}}) \right\}, \\ S_{\psi,\mathrm{ad}} &:= S_\psi / Z_{\hat{G}}, \\ \mathcal{S}_\psi &:= \pi_0(S_{\psi,\mathrm{ad}}, 1). \end{aligned}$$

The idea is that elements in $\mathcal{S}_\psi$ gives rise to endoscopic data of G by which ψ factors through.

The group $S_\psi \times L'_F$ acts on $\hat{\mathfrak{g}} := \mathrm{Lie}\,(\hat{G})$, which gives a representation

$$\tau_\psi = \bigoplus_\alpha (\lambda_\alpha \boxtimes \mu_\alpha \boxtimes \nu_\alpha) \quad \text{(decomposition into irreducibles)}$$

where the exterior tensor products are taken with respect to the product $S_\psi \times L_F \times \mathrm{SU}(2)$. The relevance of these objects are explained as follows.

i) We define a sign character $\varepsilon_\psi : \mathcal{S}_\psi \to \{\pm 1\}$ by setting

$$\varepsilon_\psi(x) := \prod_\alpha \det(\lambda_\alpha(s))$$

where $s \in S_\psi$ projects to $x \in \mathcal{S}_\psi$, and the index α ranges over those with μ_α symplectic and $\varepsilon(\frac{1}{2}, \mu_\alpha) = 1$.

ii) It is expected that to ψ is associated an A-packet Π_ψ of representations of $G(\mathbb{A})$, together with a map

$$\begin{aligned} \mathcal{S}_\psi \times \Pi_\psi &\to \mathbb{C}^\times, \\ (x, \pi) &\mapsto \langle x, \pi \rangle. \end{aligned}$$

iii) Set

$$m_\psi(\pi) := \frac{1}{|\mathcal{S}_\psi|} \sum_{x \in \mathcal{S}_\psi} \varepsilon_\psi(x) \langle x, \pi \rangle.$$

Now we can state Arthur's conjecture on the multiplicities [1].

Conjecture 1.1. For every admissible irreducible representation π of $G(\mathbb{A})$, we have

$$m(\pi) = \sum_{\psi} m_{\psi}(\pi)$$

where ψ ranges over the $\hat{G}$-conjugacy classes of A-parameters.

Remark 1.2. We note that in many cases (eg. the classical groups), this formula is expected to come from a decomposition into direct sums:

$$L^2_{\text{disc}}(G(F)\backslash G(A)^1) = \bigoplus_{\psi} L^2_{\psi}.$$

Consequently, every π in the discrete L^2 spectrum should belong to at most one A-packet, say that corresponding to ψ, and we expect $m(\pi) = m(\psi)$.

2 Known cases

Arthur's conjectures are largely inspired by his study of the trace formula: see [3] for an excellent introduction. Here are a few known cases.

A. For the quasisplit groups $\mathrm{SO}(2n+1)$, $\mathrm{Sp}(2n)$, this is proved in [5], by using the selfdual irreducible cuspidal automorphic representations of $\mathrm{GL}(n)$ as a substitute for the A-parameters. In particular, there is no need to assume the existence of L_F. This is done by realizing these classical groups as elliptic endoscopic groups for the twisted space $\widetilde{\mathrm{GL}}(n)$.

B. For the quasisplit groups $\mathrm{SO}(2n)$, a coarse version "up to outer automorphisms" is proved in [5], in which one can only identify the $\mathrm{O}(2n)$-orbits of ψ.

C. The case of $U(3)$ is proved earlier by Rogawski [12].

D. Arthur's machine is adopted to the quasisplit unitary groups $U(n)$ by Chung Pang Mok [11]. There is no ambiguity of outer automorphisms.

E. For the group $\mathrm{SL}(N)$, Hiraga and Saito [8] have obtained the multiplicity formula for the generic spectrum by using the representations of $\mathrm{GL}(N)$ as substitutes of the A-parameters as before. They also obtained coarser results for the inner forms of $\mathrm{SL}(N)$.

As regards the classical groups SO, Sp and U, it would be interesting to consider the non-quasisplit cases as well, as alluded in [5, Chapter 9]. Some modifications of the definitions of S-groups are needed. The same remark certainly applies to $\mathrm{SL}(N)$ and its inner forms.

We will return to these issues later.

3 Arthur's approach

Grosso modo, Arthur's approach in [5, Chapter 4] can be summarized by the triad

multiplicity formula

global intertwining relation ———— stable multiplicity formula

in which any two terms imply the third one. The so-called stable multiplicity formula is a stable variant of our objective, the multiplicity formula. It pertains only to quasisplit groups. Note that in the Endoscopic Classification for classical G, these three properties are proved altogether in a long interlocking argument.

4 Stable multiplicity formula

Let S be a union of connected components of a reductive $\mathbb{C}$-group; these components generate a group $\langle S\rangle$, whose neutral component is denoted by S°. Fix a maximal torus T in S° and set

$$\begin{aligned} W^\circ &:= W(S^\circ, T), \\ W &:= W(S, T) = N_S(T)/T. \end{aligned}$$

As usual, put $\mathfrak{a}_T := \mathrm{Hom}(X^*(T), \mathbb{R})$ and set

$$W_{\mathrm{reg}} := \{w \in W : \det(w - 1|\mathfrak{a}_T) \neq 0\}.$$

Fix a Borel subgroup of S° containing T. For each $w \in W$, set

$$\varepsilon(w) := (-1)^{\#\{\alpha\in\Sigma(S^\circ,T):\alpha>0, w\alpha<0\}}$$

where $\Sigma(S^\circ, T)$ is the set of roots of (S°, T). We also write $\varepsilon^G(w)$ to emphasize the ambient group G. The first goal is to "stabilize" the expression

$$i(S) := \frac{1}{|W^\circ|} \sum_{w\in W_{\mathrm{reg}}} \varepsilon(w)|\det(w-1)|^{-1}.$$

Theorem 4.1. *There exist unique constants $\sigma(S_1)$ for each connected reductive $\mathbb{C}$-group S_1, such that*

i) $\sigma(S_1) = \sigma(S_1/Z_1)/|Z_1|$ for every central subgroup Z_1, this means in particular that $\sigma(S_1) = 0$ if S_1 is not semisimple;

ii) for S as above, we have

$$i(S) = \sum_{\substack{s \in S/conj \\ \#Z(S_s^\circ) < \infty}} |\pi_0(S_s, 1)|^{-1} \sigma(S_s^\circ)$$

where $S_s := Z_S(s)$.

Assume hereafter in this section that G is quasisplit. By assuming the local Langlands correspondence and the endoscopic character relations, to each A-parameter ψ for G we may attach a stable distribution $f \mapsto f(\psi)$ on $G(\mathbb{A})$. It satisfies $f(\psi) = \prod_v f_v(\psi_v)$ if $f = \prod_v f_v \in C_c^\infty(G(\mathbb{A})^1)$ and ψ_v is the local A-parameter deduced from ψ.

On the other hand, recall the stable trace formula for G, written as

$$I_{\mathrm{disc}}^G(f) = \sum_{G'} \iota(G, G') S_{\mathrm{disc}}^{G'}(f')$$

(cf. [2, §7]), where

- I_{disc}^G is the discrete part of Arthur's *invariant trace formula* for G;
- G' ranges over the elliptic endoscopic data of G, identified somehow abusively with the associated endoscopic group;
- $\iota(G, G')$ are explicit positive constants;
- $f' \in C_c^\infty(G'(\mathbb{A}))$ is a Langlands-Shelstad transfer of f;
- $S_{\mathrm{disc}}^{G'}$ is the discrete part of the stabilized trace formula for G', which is a stable distribution on $G'(\mathbb{A})$.

Remark 4.2. In Arthur's works, he has to introduce a parameter $t > 0$ and consider the distributions $I_{\mathrm{disc},t}^G$, etc., to ensure absolute convergence. We deliberately omit this technical complication.

Now one can state the conjectural stable multiplicity formula.

Conjecture 4.3. For each $f \in C_c^\infty(G(\mathbb{A})^1)$, we have

$$S_{\mathrm{disc}}^G(f) = \sum_{\psi} |\mathcal{S}_\psi|^{-1} \sigma((S_{\psi,\mathrm{ad}})^\circ) \varepsilon^G(\psi) f(\psi).$$

As mentioned above, this is essentially proved for the groups SO, Sp and U.

5 Intertwining relations

We consider only the local intertwining relation, as the global version [5, Corollary 4.2.1] is simply the product of its local avatars. Our main reference is [4].

Let F be a local field of characteristic zero, G be a connected reductive F-group and M a Levi subgroup of G. To each A-parameter $\psi : W'_F \times \mathrm{SU}(2) \to {}^{\mathrm{L}}G$, we may define the groups S_ψ, $\mathcal{S}_\psi$. Moreover, if ψ factors through ${}^{\mathrm{L}}M \hookrightarrow {}^{\mathrm{L}}G$, say via $\psi_M : W'_F \times \mathrm{SU}(2) \to {}^{\mathrm{L}}M$, then by assuming the local Langlands correspondence, we may form the A-packet Π_{ψ_M} of M.

Let $\sigma \in \Pi_{\psi_M}$ and $w \in N_G(M)(F)$ such that $w\pi := \pi \circ \mathrm{Ad}(w^{-1})$ is isomorphic to π. Fix such an isomorphism $\pi(w) : w\pi \xrightarrow{\sim} \pi$. The variety Mw becomes a M-bitorsor under multiplication by M, as w normalizes M. That is, Mw is a twisted space in the sense of Labesse [10]. The assignment $mwm' \mapsto \pi(m)\pi(w)\pi(m')$ gives rise to an irreducible representation of the twisted space Mw (see *loc. cit.*) Denote it by π_w.

Assume that ψ_M is invariant under the Weyl element associated to w. Then ψ_M can be plugged into the formalism of twisted endoscopy [9] for Mw. Define $\Pi^w_{\psi_M} \subset \Pi_{\psi_M}$ to be the w-fixed elements in Π_{ψ_M}.

Let $(M', s, \ldots)$ be an elliptic endoscopic datum of the twisted space Mw by which ψ_M factors through via $\psi' : W'_F \times \mathrm{SU}(2) \to {}^{\mathrm{L}}M'$. Consider a "lifting" of the elliptic endoscopic datum to G, upon replacing s by $s' \in sZ^{\Gamma_F}_{\widehat{Mw}}/Z^{\Gamma_F}_{\hat{G}}$:

$$\begin{array}{ccc} G' \text{ - - - } (G, \text{inner twist by } \mathrm{Ad}(w)) & = & G \text{ untwisted} \\ \text{Levi} \uparrow \qquad\qquad \uparrow \text{Levi} & & \\ M' \text{ - - - - - - - - - - } Mw & & \end{array}$$

where the dashed line means connection via elliptic endoscopic datum. We also assume that an L-embedding ${}^{\mathrm{L}}G' \hookrightarrow {}^{\mathrm{L}}G$ is chosen.

Conjecture 5.1. Given a lifting as above, there exists a canonical map

$$\begin{array}{ll} \Delta : & \text{transfer factor for } (G', G) \\ \quad\downarrow & \\ \Delta_w : & \text{twisted transfer factor for } (M', Mw), \end{array}$$

and there exist explicit constants $c(\psi_{M,w})$ depending on the choice of an additive character $\theta_F : F \to \mathbb{C}^\times$, which should satisfy a global product formula, such that

$$f'(\psi') \to c(\psi_{M,w}) \sum_{\pi \in \Pi^w_{\psi_M}} \Delta_w(\psi'_w, \pi_w) \mathrm{tr}\,(R_P(\pi_w, \psi_M) I_P(\pi, f))$$

for all $f \in C_c^\infty(G(F))$where

- $\Delta_w(\psi'_w, \pi_w)$ is the spectral transfer factor corresponding to the geometric one Δ_w;
- $I_P(\pi)$ is the normalized parabolic induction with respect to a parabolic subgroup $P = MU$;
- $R_P(\pi_w, \psi_M)$ is the normalized intertwining operator attached to $\pi_w \in \Pi^w_{\psi_M}$ and θ_F;
- $f' \in C_c^\infty(G'(F))$ is a transfer of f.

Note that $R_P(\pi_w, \psi_M)$ and $\Delta_w(\psi'_w, \pi_w)$ depends on the choice of $\pi(w) : w\pi \xrightarrow{\sim} \pi$. But the ambiguities cancel with each other in the final expression. If G is quasisplit, we can normalize things by Whittaker models.

Remark 5.2. (a) For classical groups including the unitary groups, this conjecture can be simplified somehow and is proved in [5, 11]; note that the case of SO($2n$) is more delicate. (b) By taking $M = G$ and $w = 1$, we revert to the endoscopic character formula for A-packets:

$$f'(\psi') = \sum_{\pi \in \Pi_\pi} \Delta(\psi', \pi) f(\pi)$$

where $f(\pi) := \mathrm{tr}\pi(f)$. (c) This local intertwining relation is used to construct general A-packets, as well as the relevant character identities, from the "elliptic" ones.

6 The work of Hiraga and Saito

The inner forms of SL(N) serve as a reality check for Arthur's conjectures. Let F be a local or global field of characteristic zero. Let D be a finite-dimensional central division algebra over F. Write

$$N = \dim_F D \cdot n$$

and consider

$$G^\sharp := \mathrm{SL}(n, D) \lhd \mathrm{GL}(n, D) =: G.$$

This construction yields all the inner forms of $\mathrm{SL}(N, F) \lhd \mathrm{GL}(N, F)$. A familiar technique for the study of representations of $G^\sharp$ is to use the restriction from G to $G^\sharp$. The restriction ought be dual to the L-homomorphism ${}^{\mathrm{L}}G \twoheadrightarrow {}^{\mathrm{L}}G^\sharp$ in view of the principle of functoriality. This is systematically done in [8], which we recall below.

When F is local, for every admissible irreducible representation π of $G(F)$, we define Π_π to be the set of irreducible constituents of $\pi|_{G^\sharp(F)}$. Note that $\pi|_{G^\sharp(F)}$ is known to be semisimple of finite length. The finite sets Π_π are our candidates for the A-packets. For those π corresponding to a generic representation of $\mathrm{GL}(N)$ via Jacquet-Langlands correspondence, Hiraga and Saito (a) related the internal structure of packets in terms of the $\mathcal{S}$-groups; (b) established the endoscopic character relations conjectured by Langlands.

When F is global, Hiraga and Saito studied the restriction of cusp forms. For cuspidal representations $\pi = \bigotimes'_v \pi_v$ that are locally generic (up to Jacquet-Langlands correspondence), they derived a multiplicity formula à la Arthur, but with some undetermined constant in the non-quasisplit case. They made the assumption that $G^\sharp$ is split at every archimedean place. Thanks to [6], this hypothesis is nowadays unnecessary.

One of the technical ingredients thereof is to reduce to the *automorphic induction* from $\mathrm{GL}(N/d, E)$ to $\mathrm{GL}(n, D)$, where E/F is a cyclic extension of degree d. This reduction hinges on the seemingly folklore connection

$$\boxed{\text{Endoscopy of } G^\sharp} \longleftrightarrow \boxed{\text{Endoscopy of } G \text{ twisted by } \mathbf{a}, \text{ for various } \mathbf{a}}$$

where $\mathbf{a}$ is an element in the continuous cohomology $Z^1(W_F, Z_{\hat{G}})$ for F local (resp. $\ker^1(W_F, Z_{\hat{G}})$ for F global).The latter box is exactly the case of automorphic induction for E/F, where E/F is the cyclic extension corresponding to $\mathbf{a}$ by class field theory. The required endoscopic character identities then follow from those of automorphic induction by a "restriction" procedure for endoscopy.

It seems possible to verify Arthur's conjectures using this formalism: one may try to formulate and verify

- the local intertwining relation for $G^\sharp$ or its twisted variant for automorphic induction;
- the stable multiplicity formula for $\mathrm{SL}(N)$, which should be relatively easy.

The first obstacle is of course the extension of the local results in [8] to non-generic setting. The upshot is the character relation for automorphic induction of the *Speh representations*. Professor Hiraga has an unpublished proof for this using Zelevinsky involution (private communication). Granting this, it would be relatively easy to verify Arthur's conjectures for $G^\sharp = \mathrm{SL}(N)$ such as the stable multiplicity formula.

For the non-quasisplit case, it may help us to see the necessary modifications for Arthur's conjectures in the non-quasisplit setting, such as the use of modified S-groups, etc. For example, in the study of local intertwining relations, some phenomena unseen for classical groups might appear for the inner forms of $\mathrm{SL}(N)$, cf. [7].

All these are obviously some immature thoughts. We hope to address the relevant issues in some future papers.

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