数理解析研究所講究録
第 1872 巻 2014 年 72-82

Interaction between lights and shadows for the quasithin groups

Yasuhiko Tanaka
Oita University

In this short article, we introduce 'shadows' emerging in the classification of the quasithin simple groups. As is probably known, the classification of the finite simple groups is divided into two major subclasses: even groups and odd groups. However, a subsequent study in depth shows that another even/odd partition may be better than the classical one. We will pay attention to a new type of partition, which causes in return an obstruction called shadows.

1 What is a shadow?

Let $\mathcal{F}$ be a set (of the isomorphism classes) of known finite simple groups. A finite group G is said to be similar to $\mathcal{F}$ if the structure of G is close to that of a member of $\mathcal{F}$. We do not give a precise definition of the word 'close,' though.

Let G be a finite group. Then one of the following holds:

(1) G is a member of $\mathcal{F}$;

(2) G is not a member of $\mathcal{F}$, but similar to $\mathcal{F}$;

(3) G is not similar to $\mathcal{F}$.

A group arising in the second case is called a *shadow* from $\mathcal{F}$, while a group arising in the first case is called a *light* from $\mathcal{F}$.

We have in our mind a revision program of the classification of the finite simple groups. In an actual classification process, we often fix a set $\mathcal{S}$ (of the isomorphism classes) of the finite simple groups satisfying certain conditions. We hope that a finite group is a light from $\mathcal{S}$ under those conditions. However, we often encounter shadows which force us extra hard work to eliminate them.

Historically speaking, the whole classification was divided into several subclasses in a certain point of view. The two of the most important partitions were done by chracteristic and size: the partition into even/odd simple groups, and the partition into small/large simple groups. We accept the partitions for now, and we focus attention on the even small groups, which are called quasithin groups.

Quasithin simple groups were a final fort against all the efforts of the classification problem. It was long and complicated, which seems to symbolize the whole classification. Starting with dissatisfaction to the original classification, revision projects have proceeded for about thirty years. One large project by Gorenstein, Lyons, Solomon [GLS] has been in progress (even now). The project aims to construct a classification of the second generation. Apart from the GLS project, it was Aschbacher and Smith that gave an answer to the classification of quasithin simple groups, which forms a heavy two-volume book [AS] of 1200 pages. The AS work applies to a part of the GLS project. It seems quite predestined the work after thirty years holds a central position in a proof of the second generation. Now, is it satisfactory for all mathematicians, or in particular for the group theorists?

Before going to the classification of the quasithin groups themselves, let us keep in mind the special aspects of the whole classification of the finite simple groups.

First of all, the prime 2 plays a specific role in the finite group theory. For example, the following theorems are of fundamental importance. They are used everywhere in the classification of the finite simple groups.

Theorem 1 (Feit-Thompson) *A group of odd order is solvable.*

Theorem 2 (Bender-Suzuki) *A simple group with strongly embedded subgroups is isomorphic to one of the Lie type groups of characteristic* 2 *of rank* 1*:* $PSL_2(q)$, $Sz(q)$, $PSU_3(q)$ *(*q*: powers of* 2, ≥ 4*).*

Of course, the formal substitution of an odd prime for the prime 2 gives false statements. Why is the prime 2 so different from odd primes? Among others, the following aspects are considered to be quite essential:

(1) $2 = 1 + 1$,

(2) the even prime,

(3) the minimum prime.

We will give a short comment for each of them. For (1), we raise the Burnside theorem: a group with a fixed point free automorphism of order 2 is abelian. The statement is not true for odd primes although the Thompson theorem says such a group must be nilpotent. For (2), we raise a basic property of permutations: a cycle of prime length p is an odd permutation if and only if $p = 2$. So, if a group has a subgroup M of index $2k$ with odd k and an element of order 2 is not conjugate any element of M, then the group has a normal subgroup of index 2. For (3), we raise the Burnside theorem: if a group G has a cyclic Sylow p-subgroup for the minimum prime divisor p of $|G|$, then G has a normal p complement. The statement is not true for nonminimum primes.

The next thing to note is that the classification proceeds by an induction of the order of groups. It was correct for the existing proof, and it will be so even if a completely new approach would be developed in the future. Because the finite groups do not have 'structure' except group multiplication, we can rely only on the finiteness of the groups.

In order to proceed by an induction, we need a notion of 'known' groups. Let $\mathcal{K}$ be the set of (the isomorphism classes of) known finite simple groups. The set $\mathcal{K}$ consists of the following simple groups: the cyclic groups of prime order, the alternating groups of degree five or more, the Lie type groups defined over finite fields, and the twenty-six sporadic simple groups.

Now we reach the definition of $\mathcal{K}$-groups. A *$\mathcal{K}$-group* is a finte group all of whose simple sections are members of $\mathcal{K}$, where a section is defined to be a factor group of a subgroup. Note that Lie type groups have a recursive structure. The factor group of a parabolic subgroup by the maximal solvable normal subgroup should be another Lie type group of the same kind. Since the most of the finite simple groups should become groups of Lie type, we will have a closer look at various sections in a simple group.

It is important to capture subgroups of a simple group which corresponds to parabolic subgroups of a Lie type group. In order to do so, we consider how to abstract the parabolic subgroups.

By definition, a *local subgroup* of a finite group is the normalizer of a nonidentity solvable subgroup. In particular, for a prime number p, a *p-local subgroup* is the normalizer of a nonidentity p-subgroup. It is easily seen that a p-local subgroup is a generalization of a parabolic subgroup of a Lie type group of characteristic p. To identify a finite simple group with a known simple group, we have to analyze the structure and embeddings of local subgroups. This means that we believe local properties determine the whole structure of finite simple groups.

By the special properties of the prime 2 and from the inductive treatment in the whole proof, it is quite appropriate to divide the classification into the four subclasses: the classification of the even small groups, the even large groups, the odd small groups, and the odd large groups. We express the situation as the following table called the *classification grid.* We will in fact focus on the even small groups in this article although we do not give the precise definition of the four subclasses.

	even	odd
small		
large		

2 The existing classification

Let $\mathcal{K}$ be the set of (the isomorphism classes of) the known simple groups. Let G be a finite simple group. We will say that G is in the *large* (or generic) case if the following condition holds: there exist a prime p, an element $t \in G$ of order p, a group G^* in $\mathcal{K}$, and an element $t^* \in G^*$ of order p such that both $C_G(t)$ and $C_{G^*}(t^*)$ are similar. Historically speaking, the prime 2 is preferable to odd primes for p. We will say that G is in the *small* case if G is not in the large case. The goal of the existing classification is to show that a finite simple group is a light from $\mathcal{K}$, namely, similar to a known simple group, in either case.

As stated above, the existing classification of the finite simple groups is divided into four subclasses: even small, even large, odd small, odd large.

Partition to even/odd groups is done by a generalized notion of 'characteristic.' Classically, a finite group G of even order is said to be *of characteristic* 2 *type* if $C_L(O_2(L)) \subseteq O_2(L)$ for every 2-local subgroup L of G. The above condition is in fact equivalent to somewhat weaker condition: $C_L(O_2(L)) \subseteq O_2(L)$ for every maximal 2-local subgroup L of G. The typical examples of the groups of characteristic 2 type are simple groups of Lie type defined over finite fields of characteristic 2. Recently, slightly different definitions are used to characterize even groups. Let G be a group of even order. We will say that G is *of even characteristic* if $C_L(O_2(L)) \subseteq O_2(L)$ for every 2-local subgroup L of G of odd index. Also, we will say that G is *of even type* if

(1) $O_{2'}(C_G(t)) = 1$,

(2) certain components are allowed in $C_G(t)$

for every involution t of G. In this article, we do not care the difference of the three definitions so much, and use the terminology 'even groups' for the groups of any one of the three types. We only note here that the range of the even groups may become wider than before.

Partition to small/large groups is done by a generalized notion of 'size.' Let G be a group of characteristic 2 type. The *2-local p-rank* $m_{2,p}(G)$ of G for an odd prime p is the maximum of the p-rank of L, where L ranges over the set of 2-local subgroups of G. The *rank* $e(G)$ of G is the maximum of $m_{2,p}(G)$, where p ranges over the set of odd primes. In this article, we use the definition of the rank not only for the groups of characteristic 2 type but also for the even groups. Considering the historical and technical reasons, we will use the terminology 'small groups' for the groups of rank 2 or less.

In the remainder of this article, we will focus our attention to the even small groups. Classically, such groups were quasithin groups. The original definition of the quasithin groups is as follows. Let G be a group of characteristic 2 type. We will say that G is *thin* if $e(G) \leq 1$, and that G is *quasithin* if $e(G) \leq 2$. As is well known, the rank is a good approximation of the Lie rank.

We will define the rank also for the even groups. Hence, by abuse of terminology, we may call an even small group a quasithin group as well. So our goal is to classify the quasithin simple groups, which should be Lie type groups of characteristic 2 and low rank, quite many sporadic groups, or a few other groups.

	even	odd
small	Quasithin groups	
large		

The partition is not satisfactory in the existing classification because the proof is still too long and complicated to understand. As stated above, the range of the even groups in the classification of the next generation is becoming wider than before. It seems, however, that we do not encounter the best partition yet. The grid is still swinging.

3 Setup

Throughout the remainder of this article, we use the following notation.

Let G be a simple quasithin even group, and let T be a Sylow 2-subgroup of G. For a set $\mathcal{X} = \mathcal{X}_G$ of subgroups of G, and a subgroup S of G, denote by $\mathcal{X}(S)$ the set of members of $\mathcal{X}$ containing S. Let $\mathcal{M} = \mathcal{M}_G$ be the set of maximal 2-local subgroups of G, and let $\mathcal{N} = \mathcal{N}_G$ be the set of maximal subgroups of G.

There are two cases in view of the number of the elements in $\mathcal{M}(T)$:

(1) Uniqueness subgroup case, or $|\mathcal{M}(T)| = 1$, i.e. there is a unique maximal 2-local subgroup containing T.

(2) Amalgam method case, or $|\mathcal{M}(T)| > 1$, i.e. there is a pair (M, N) of 2-local subgroups containing T.

A subgroup U is said to be a *uniqueness subgroup* if $|\mathcal{M}(U)| = 1$, i.e. there is a unique maximal 2-local subgroup containing U. Uniqueness subgroups play crucial roles in the analysis of the subgroup structure of simple groups. Uniqueness subgroups are important because they control the structure and embeddings of 2-local subgroups. For example, let $G = PSL_2(2^n)$, $T \in Syl_2(G)$, and $B = N_G(T)$. Then we have $\mathcal{M}(T) = \{B\}$. Thus, T is a uniqueness subgroup of G.

There is a general theory to treat the Uniqueness subgroup case. Suppose that T is a uniqueness subgroup of G, i.e. $|\mathcal{M}(T)| = 1$. Define the characteristic core as follows:

$$C(G,T) = \langle N_G(C) \mid 1 \neq C \text{ char } T \rangle.$$

The following theorem classifies the simple groups having a proper characteristic core.

Theorem 3 (Aschbacher) *If $C(G,T) \neq G$, then G is isomorphic to one of the following groups: $PSL_2(q)$, $Sz(q)$, $PSU_3(q)$ (q: powers of 2, ≥ 4), J_1.*

We will not mention any more the uniqueness subgroup case in this article.

4 Amalgam method case

Suppose that T is not a uniqueness subgroup of G, i.e. $|M(T)| > 1$. Then there exists a pair (M, N) of 2-local subgroups of G containing T with $O_2(\langle M, N \rangle) = 1$. We would like to know the structure of M and N.

The first thing we consider is to choose M and N carefully enough to find their precise structure. We only need the structure of M and/or N to appeal recognition theorems to identify G. So, it is all right if $G \neq \langle M, N \rangle$. Also, smaller M and N are better. The structures of $M/O_2(M)$ and $N/O_2(N)$ are restricted as M and N are quasithin. The structures of the chief factors of M and N are restricted by analysis of amalgams.

By amalgams, we mean lattices induced by M and N. Let $x \in M - N$ and $y \in N - M$. Consider the conjugate subgroups

$$\ldots,\ M^{yx},\ N^x,\ M,\ N,\ M^y,\ N^{xy},\ \ldots$$

of M and N. Then we have the lattice generated by those subgroups. A different choice of the elements x and y gives a different lattice. We analyze the structure of the lattices to obtain the structure of M and N.

There are too many possible structures of M and N. We need to develop a way to reduce the possibilities. So we will take a special subgroup instead of one of the maximal subgroups.

Suppose that $|\mathcal{M}(T)| > 1$. We will select an element $M \in \mathcal{M}(T)$ later. First, we will choose another subgroup to make an amalgam. Let $\mathcal{H} = \mathcal{H}_G$ be the set of subgroups H of G with $O_2(H) \neq 1$. If $H \in \mathcal{H}(T)$, then $m_p(H) \leq 2$ for each odd prime p.

We will take a pair of subgroups (H, M) instead of (M, N). The reader perhaps notice that the pair (H, M) seems to correspond to a pair of a minimal and maximal parabolic subgroup with a common Borel subgroup in a Lie type group. So we define here an abstract minimal parabolic subgroup and an abstract maximal parabolic subgroup for an arbitrary simple group.

Let G be a finite group. A subgroup P of G is said to be an abstract minimal parabolic of G if $|\mathcal{N}_P(S)| = 1$ and $1 \neq O_2(P) \neq S \subset P$ for a Sylow 2-subgroup S of G.

Suppose that P is an abstract minimal parabolic of G. Let $S \in Syl_2(P)$. Then $S \in Syl_2(G)$, and one of the following holds:

(1) If P is solvable, then $P = O_{2,p,2}(P)$, $S/O_2(P)$ acts irreducibly on $(O_{2,p}(P)/O_2(P))/\Phi(O_{2,p}(P)/O_2(P))$.

(2) If P is not solvable, then $P = O_{2,2',E,2}(P)$, $S/O_2(P)$ permutes the simple components of $P/O_{2,2'}(P)$.

The structure of the minimal parabolic subgroups is fairly restricted.

A subgroup P of G is said to be an abstract maximal parabolic of G if $|\mathcal{M}(P)| = 1$, $1 \neq O_2(P) \neq S \subset P$ for a Sylow 2-subgroup S of G, plus technical conditions. We do not care about precise information on the additional conditions. Compared with abstract minimal parabolic subgroups, there are still too many possibilities for abstract maximal parabolic subgroups even if we add a strong condition for restriction.

As stated above, we are considering the following type of amalgams.

Let $P = P_G, Q = Q_G$ be the sets of abstract minimal and maximal parabolic subgroups of G, respectively. A pair of subgroups (X, Y) is said to be an amalgam of G over T if $X \in P(T)$, $Y \in Q(T)$, $O_2(\langle X, Y \rangle) = 1$.

For example, let $G = PSL_n(q)$ (q: powers of 2), $T \in Syl_2(G)$, $B = N_G(T)$. Let P and Q be a minimal parabolic and a maximal parabolic over B, respectively, with $P \not\subseteq Q$. Put $X = \langle T^P \rangle$, $Y = \langle T^Q \rangle$. Then (X, Y) is an amalgam of G over T.

Let $\mathcal{H}(T; M) = \{H \in \mathcal{H}(T) \mid H \not\subseteq M\}$. Define $\mathcal{H}^*(T; M)$ to be the set of minimal elements of $\mathcal{H}(T; M)$, ordered by inclusion. Suppose that $H \in \mathcal{H}^*(T; M)$. Then H is an abstract minimal parabolic, and $O_2(\langle H, M \rangle) = 1$. So, H has very restricted structure, compared with M.

Now, we would like to have a way to restrict the structure of M. Suppose that there exists a uniqueness subgroup U of M, or $\mathcal{M}(U) = \{M\}$. Then $1 \neq O_2(U) \neq T \subseteq U, O_2(\langle H, U \rangle) = 1$. Here, U is an abstract maximal parabolic, so (H, U) is an amalgam of G over T.

Is it always possible to choose M so that M has a uniqueness subgroup? For an appropriate choice of M, there is a general way to construct a uniqueness subgroup of M. The uniqueness subgroup is generated by component-like subgroups explained below.

Let $H \subseteq G$. (For H, imagine a parabolic subgroup of a Lie type group for H.) Let $\mathcal{C} = \mathcal{C}_H$ be the set of $\mathcal{C}$-components, or subgroups L of H minimal subject to $1 \neq L = L' \trianglelefteq \trianglelefteq H$.

We have $H^\infty = \langle \mathcal{C}_H \rangle$. If $L_1, L_2 \in \mathcal{C}_H$ and $L_1 \neq L_2$, then $[L_1, L_2] \subseteq O_2(L_1) \cap O_2(L_2) \subseteq O_2(H)$. If $L \in \mathcal{C}_H$, then $L \trianglelefteq H$, or $|L^H| = 2$. Let $\mathcal{L}(G, T)$ be the set of subgroups L with $L \in \mathcal{C}_{\langle L, T \rangle}$, $T \in Syl_2(\langle L, T \rangle)$, and $O_2(\langle L, T \rangle) \neq 1$. Define $\mathcal{L}^*(G, T)$ be the set of maximal elements of $\mathcal{L}(G, T)$, ordered by inclusion. (Imagine maximal parabolics for $\langle L, T \rangle$.) If $L \in \mathcal{L}^*(G, T)$, then $\mathcal{M}(\langle L, T \rangle) = \{N_G(\langle L^T \rangle)\}$, and $\langle L, T \rangle$ is a uniqueness subgroup of G.

If we consider the amalgam of abstract minimal maximal parabolic subgroups, we will encounter too many possible structures of the maximal parabolic M. So we like to restrict M before analysis of amalgams. In order to do so, we must know interaction between M and other 2-locals. We hope that $|\mathcal{M}|$ must be small, which means that M should contain many 2-local subgroups. For example, let $\mathcal{M}(U) = \{M\}$, and let V be a chief factor of U. Then $O_2(C_G(V))U \subseteq N_G(V) \subseteq M$.

Let H and U be as above, or a min-max parabolic pair. The structure of G is reduced to the structure and embedding of H, U, which is again reduce to the structure of the chief factors of H (or U) as $GF(2)H$- (or $GF(2)U$-) modules. Thus the properties of $GF(2)$-representation of groups of even order are of fundamental importance. In particular, properties of FF-modules and quadratic modules, ... are repeatedly applied.

Let us consider a lattice induce by (H, M). Let $x \in H-M$ and $y \in M-H$, and make the conjugate subgroups

$$\dots,\ H^{yx},\ M^x,\ H,\ M,\ H^y,\ M^{xy},\ \dots$$

of H and M as before. Then we have the lattice generated by H and M.

Define $Q = O_2(H)$ and $V = \Omega_1(Z(Q))$. Suppose that $[V, O^2(H)] \neq 1$. If $|V : V \cap Q^y| \leq |V^y : V^y \cap Q|$, then V is an FF-module. Hence $H/C_H(V)$ is similar to a direct product of some copies of $PSL_2(q)$, and V is similar to a direct sum of the natural $PSL_2(q)$-modules. Since G is quasithin, both $H/C_H(V)$ and V are further restricted. For example, the number of direct factor (or direct summands) is forced to be 2 or less.

Simplicity of G is critical to restrict the structure of U. In the analysis of amalgams, we do not use the simplicity of the whole group. Rather, the important property of amalgams (H, U) is reduced to $O_2(H) \neq 1$, $O_2(U) \neq 1$, $O_2(\langle H, U\rangle) = 1$. Therefore, nonsimple groups having 2-local structure similar to that of simple groups happen to appear in the stage.

5 Shadows

Now we stand in the situation of classification of simple quasithin even groups. In the following, a light is an actual group in the conclusion of the classification theorem, while a shadow is a group not in the conclusion, whose local structure is close to that of a light.

We will begin with typical shadows.

First, we give a 'large rank' shadows. Let $G = PSL_4(q)$ (q: powers of 2, ≥ 4). Note that G is not quasithin because G has Cartan subgroups of rank 3. Let M be a maximal parabolic subgroup corresponding to an end node of the Dynkin diagram. Let L be the uniqueness subgroup of L, i.e. $\mathcal{M}(L) = \{M\}$. Then we have $L/O_2(L) \sim PSL_3(q)$, and, in particular, L is quasithin itself. A simple quasithin even group with a uniqueness subgroup isomorphic to L possibly appears in the stage at first although it is finally impossible. This suggests that groups of rank 3 may appear at first.

Next, we give a 'automorphic' shadows. Let L be a simple group of Lie type of characteristic 2, and let t be an automorphism of L of order 2. We will consider the following types of groups: L, $G = L\langle t\rangle$, $H = (L \times L^t)\langle t\rangle$, and so forth.

Such groups are often called groups *of characteristic* 2 *like.* We would like to regard groups of characteristic 2 like as even groups. We are sure everyone agrees with that. For example,

(1) $L = A_6 \cong Sp_4(2)'$, $G = S_6 \cong Sp_4(2)$,

(2) $L = A_5 \cong PSL_2(4)$, $G = S_5 \cong PSL_2(4)\langle f \rangle$, $H \cong PSO_4^+(4)$,

(3) $L = A_8 \cong PSL_4(2)$, $G = S_8 \cong PSL_4(2)\langle g \rangle$,

where we denote by f and g a field and graph automorphism of L, respectively.

Let L be a simple group of Lie type of characteristic 2, and let t be an automorphism of L of order 2. Put $G = L\langle t \rangle$ and $H = (L \times L^t)\langle t \rangle$. Of course, both G and H are of characteristic 2 like. In general, both $C_G(t)$ and $C_H(t)$ may have components.

Let X be a simple group with an involution whose centralizer is isomorphic to $C_G(t)$ or $C_H(t)$. In the classical definition, such a group is called of component type, and treated as an odd group. Thus, if we want to treat X as an even group, another even/odd partition is necessary, which will yield that whole classification should be restructured to avoid difficulties. That is why the even/odd partition swings in the classification.

Below is a final word for the classification of the quasithin simple groups. The GLS project seems to have taken much longer time than expected. The work of Aschbacher and Smith has made two contributions:

(1) it gives a proof which is conceptually easier to understand;

(2) it covers the counter part in the GLS project.

However, it seems not clear that more people becomes able to read and understand the whole proof. One of the reasons is that there still exist too many possibilities of amalgams considered because of shadows.

We note here the earlier work of Gomi, Hayashi, Tanaka [GH] [HT], which classifies the simple groups of characteristic 2 type all of whose 2-local subgroups are solvable. Their analysis of the simple groups began with the amalgams of abstract minimal parabolics (X, Y). In their work, precise structure of X and Y were determined not beforehand but through analysis of amalgams themselves. We hope that their method applies to decrease the possible structure of the uniqueness subgroups.

References

[GLS] D. Gorenstein, R. Lyons, and R. M. Solomon, *The classification of the finite simple groups*, AMS Surveys and Monographs **40** No. 1–6, 1994–2005.

[AS] M. Aschbacher and S. D. Smith, *The classification of quasithin groups*, AMS Surveys and Monographs **111**, **112**, 2004.

[GH] K. Gomi and M. Hayashi, A pushing-up approach to the quasithin simple finite groups with solvable 2-local subgroups, J. Algebra **146** (1992), 412–426.

[HT] M. Hayashi and Y. Tanaka, On the finite simple groups all of whose 2-local subgroups are solvable, J. Algebra **210** (1998), 365–384.

Faculty of Engineering
Oita University
Oita 870-1192
JAPAN

大分大学・工学部　田中　康彦