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MONSTROUS LIE ALGEBRAS

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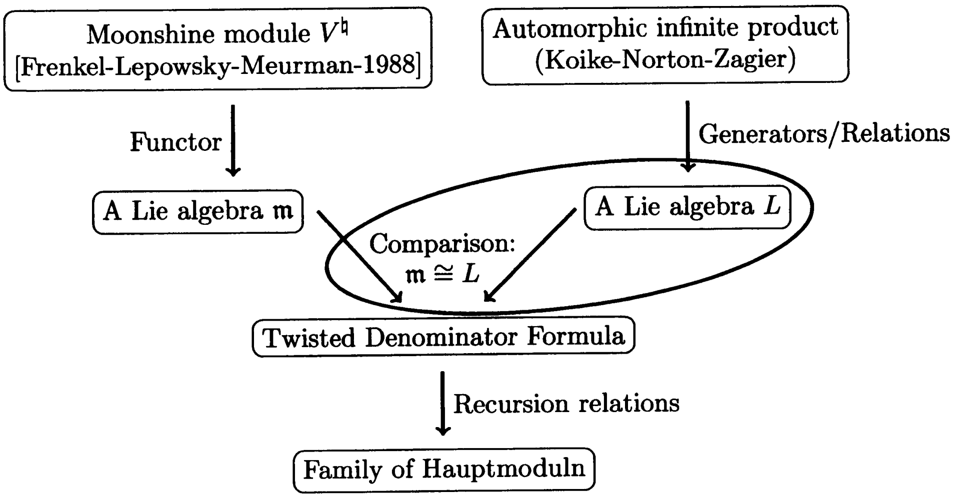
ABSTRACT. We describe a generalization of Borcherds's two constructions of the Monster Lie algebra by producing, for each element of the monster simple group, a pair of infinite dimensional Lie algebras. We outline a proof of the fact that most cases of Norton's Generalized Moonshine Conjecture for twisted modules can be reduced to the existence of an isomorphism between the Lie algebras in a pair.

CONTENTS

1. INTRODUCTION

In this paper, we outline some recent work concerning the construction of Lie algebras related to Monstrous Moonshine and Generalized Moonshine. Monstrous Moonshine is the study of how the largest sporadic finite simple group, called the monster $\mathbb{M}$, is related to modular functions, i.e., certain holomorphic functions on the complex upper half-plane $\mathfrak{H}$. Conway and Norton, in formulating the Monstrous Moonshine conjecture [Conway-Norton-1979], produced a list of modular functions indexed by conjugacy classes in $\mathbb{M}$, and asserted that there exists an infinite dimensional graded representation $\bigoplus V_n$ of the monster, such that the graded character $\sum_n Tr(g|V_{n+1})q^n$ of each element g, viewed as a power series (called the McKay-Thompson series), is equal to the Fourier expansion of the corresponding modular function. The candidate functions T_g are Hauptmoduln, that is, they are invariant under an infinite discrete group $\Gamma_g \subset SL_2(\mathbb{R})$, such that T_g generates the function field of the quotient space $\mathfrak{H}/\Gamma_g$. The quotient space is then necessarily a sphere with finitely many punctures, i.e., genus zero. A candidate representation called $V^\natural$ was constructed by Frenkel, Lepowsky, and Meurman in [Frenkel-Lepowsky-Meurman-1988], and the Monstrous Moonshine conjecture for $V^\natural$ was proved by Borcherds in [Borcherds-1992].

Here is a diagram outlining Borcherds's proof of the Conway-Norton conjecture:



The blocks in this diagram are mathematical objects that had to be constructed, and the arrows are roughly ways to pass from one object to the construction of another. As you can see, the proof involves the construction of two Lie algebras. The Lie algebra $\mathfrak{m}$ is distinguished by the fact that it has a natural, faithful action of the Monster simple group, and the Lie algebra L is distinguished by the property that its homology (in particular, its simple roots) are encoded by the normalized modular function $J(\tau) = q^{-1} + 196884q + 21493760q^2 + \cdots$. By establishing an isomorphism between these Lie algebras and using *transport de structure*, one obtains a Lie algebra with both a faithful action of the monster and explicitly described homology. This combination of information is essential to establishing the recursion relations that prove the Hauptmodul property of the McKay-Thompson series.

The Lie algebra L and the comparison step (indicated in the ellipse) are rarely noticed, but they are very useful. Indeed, the main theorem of [Cummins-Gannon-1997] implies that any group action on L satisfying a certain compatibility condition on root spaces will yield characters that are Hauptmoduln. That is, we don't strictly need an isomorphism with $\mathfrak{m}$ to get a modularity result for characters. However, it is in general quite difficult to construct an interesting group action on an infinite dimensional Lie algebra by diagram automorphisms. That is why $V^\natural$ and $\mathfrak{m}$ are necessary in a practical sense.

Because the Lie algebras $\mathfrak{m}$ and L are isomorphic, it is reasonable to say that Borcherds gave two constructions of a single Lie algebra, known as the Monster Lie Algebra. Indeed, this is the description in the Wikipedia "Monster Lie algebra" article [Wikipedia]. However, in the review literature one often finds an additional simplification: because the part in the ellipse is buried in the proof of [Borcherds-1992] Theorem 7.2, this part of Borcherds's argument is outlined as a single Lie algebra construction, followed by a computation of the simple roots. While such a description outlines a mathematically correct proof, I have found it to be somewhat inconvenient to generalize.

1.1. **Generalized moonshine.** There is a good reason why we want to generalize Borcherds's Lie algebra constructions: Norton proposed a generalization of the Conway-Norton conjecture [Norton-1987], based on numerical evidence computed by Queen [Queen-1981] and himself. One of the principal claims is that certain characters of a graded representation of a group form Hauptmoduln, much like the original conjecture. This claim is the target of my study. If we view the outline of Borcherds's proof as a blueprint for a Hauptmodul-making machine, it is reasonable to try to construct analogues of $\mathfrak{m}$ and L that suit Norton's conjecture. In particular, for each element g in the monster $\mathbb{M}$, we want:

(1) a Lie algebra $\mathfrak{m}_g$ with a projective action of the centralizer $C_{\mathbb{M}}(g)$.

(2) a Borcherds-Kac-Moody Lie algebra L_g with a good description of simple roots.

Then, we want to show $\mathfrak{m}_g \cong L_g$. If we can show that $\mathfrak{m}_g$ is Borcherds-Kac-Moody, then the isomorphism will follow from an equality of root multiplicities. By a suitable generalization of the methods in [Cummins-Gannon-1997], proved in [Carnahan-2010], such an isomorphism can then be used to show that certain characters are Hauptmoduln.

Hoehn [Höehn-2003] already proved that this method works for the case when g lies in conjugacy class 2A. Since Hoehn's argument used some explicit information about the Baby monster and its distinguished representation that we don't have in the general case, we need to replace his explicit calculations with structural theorems.

Here is a diagram of my program for a proof:

$$\begin{tikzcd}
\boxed{\text{A generalized vertex algebra } V_g} \arrow[d, "\text{Functor}"'] & & \boxed{\text{Automorphic infinite product}} \arrow[d, "\text{Generators/Relations}"'] \\
\boxed{\text{A Lie algebra } \mathfrak{m}_g} \arrow[dr] \arrow[rr, phantom, "\text{Comparison: } \mathfrak{m}_g \cong L_g"] & & \boxed{\text{A Lie algebra } L_g} \arrow[dl] \\
& \boxed{\text{Twisted Denominator Formula}} \arrow[d, "\text{Hecke operators}"] & \\
& \boxed{\text{Family of Hauptmoduln}} &
\end{tikzcd}$$

Much like the special case of Borcherds's theorem, the size of any given piece of this diagram is not representative of its difficulty. Frenkel, Lepowsky, and Meurman's construction of the vertex operator algebra $V^\natural$ was a very substantial undertaking, and the top left corner of this diagram also requires much technical work. In contrast, the construction of L_g from an infinite product formula only requires a page or two of proof.

2. Construction of $\mathfrak{m}_g$

The construction of $\mathfrak{m}_g$ can be broken into two steps:

(1) Construct a generalized vertex algebra of central charge 24 from irreducible twisted modules of $V^\natural$ and intertwining operators between them.
(2) Apply a functor **Quant** to a conformal vertex algebra of central charge 26, to get a Lie algebra.

The second step is well-established, dating back to the early 1970s, but I feel that its significance is still poorly appreciated in mathematics.

2.1. The functor Quant. Recall that Borcherds constructed $\mathfrak{m}$ from $V^\natural$ using a functor. The monster action is then automatically transferred by functoriality. We will use the same functor to construct the Lie algebras $\mathfrak{m}_g$.

Let V be a Virasoro representation with central charge $c \in \mathbb{C}$, and a Virasoro-invariant bilinear form $\langle -, - \rangle$. That is, in addition to the vector space structure on V, we are given operators $\{L_n\}_{n \in \mathbb{Z}}$, satisfying $[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}c \cdot Id_V$, and a symmetric bilinear form such that L_n and L_{-n} are adjoint for all integers n.

Then, the quotient $V/rad\langle -, - \rangle$ by the singluar subspace has a natural bilinear form induced by $\langle -, - \rangle$. If we restrict to the "primary weight 1" subspace $P^1V \subset V$, whose vectors satisfy

$L_0 v = v$ and $L_n v = 0$ for all $n > 0$, then we may construct the quotient $\mathbf{Quant}(V) := P^1V/(P^1V \cap rad\langle -,-\rangle)$. This yields our functor **Quant**:

$$\boxed{\begin{array}{c}Vir\text{-modules with}\\ Vir\text{-invariant forms}\end{array}} \xrightarrow{\quad\mathbf{Quant}\quad} \boxed{\begin{array}{c}\text{Vector spaces with}\\ \text{bilinear forms}\end{array}}$$

There is more than one valid choice for defining the morphisms in these categories, but for the purposes of our construction, we may restrict to Virasoro-module isomorphisms that preserve the form.

The functor **Quant** has two compelling properties that we need:

(1) First, **Quant** is compatible with certain products, in the sense that it refines to a functor:

$$\boxed{\begin{array}{c}\text{Conformal vertex algebras}\\ \text{with invariant forms}\end{array}} \xrightarrow{\quad\mathbf{Quant}\quad} \boxed{\begin{array}{c}\text{Lie algebras with}\\ \text{invariant bilinear forms}\end{array}}$$

The Lie bracket is given by $[u, v] = u_0 v$, i.e., the z^{-1}-coefficient of the vertex algebra product $Y(u, z)v$.

(2) Second, at critical dimension, **Quant** satisfies an oscillator-cancellation property: if $V \cong W \otimes \pi_\lambda^{1,1}$ for W unitarizable of central charge 24, then

$$\mathbf{Quant}(V) \cong \begin{cases} W_1 \oplus W_0^{\oplus 2} & \lambda = 0 \\ W_{1-\lambda^2} & \lambda \neq 0 \end{cases}.$$

Here, $\pi_\lambda^{1,1}$ is the irreducible Heisenberg module with central charge 2 attached to the vector $\lambda \in \mathbb{R}^{1,1}$, and W_α denotes the subspace of W on which L_0 acts by $\alpha \in \mathbb{R}$. This oscillator-cancellation property was conjectured by Lovelace in 1971 [Lovelace-1971] and proved (under some unnecessary hypotheses) by Goddard and Thorn in 1972 as part of the no-ghost theorem [Goddard-Thorn-1972]. At critical dimension, **Quant** is naturally isomorphic to the cohomology functor H^1_{BRST} (see [D'Hoker-1997] for a nice explanation), so we have an alternative conceptual foundation.

Oscillator cancellation will allow us to identify root spaces of Borcherds-Kac-Moody Lie algebras as homogeneous spaces in generalized vertex algebras.

To get a Lie algebra with projective $C_{\mathbf{M}}(g)$ action from **Quant**, we therefore need:

(1) a conformal vertex algebra V with projective $C_{\mathbf{M}}(g)$ action by conformal vertex algebra automorphisms.
(2) a decomposition of V as a sum of tensor products $\bigoplus_\lambda (V_\lambda \otimes \pi_\lambda^{1,1})$, with each V_λ a unitarizable Virasoro representation of central charge 24.

If the set $\{\lambda\}$ is a submonoid of $\mathbb{R}^{1,1}$, then the Heisenberg modules $\pi_\lambda^{1,1}$ naturally assemble into a generalized vertex algebra (more precisely, an abelian intertwining algebra in the sense of Dong-Lepowsky [Dong-Lepowsky-1993]). In fact, to form our conformal vertex algebra, it is necessary and sufficient to put a generalized vertex algebra structure on $\bigoplus_\lambda V_\lambda$, with opposite braiding. To this end, we need to figure out a way to assemble generalized vertex algebras out of parts.

2.2. **Making generalized vertex algebras.** Shortly after Norton formulated the Generalized Moonshine Conjecture, Dixon, Ginsparg, and Harvey gave a physical interpretation in terms of twisted sectors of a conformal field theory with monster symmetry [Dixon-Ginsparg-Harvey1988]. In mathematical language, the candidate representation of $C_{\mathbf{M}}(g)$ is an irreducible g-twisted $V^\natural$-module. We will refer to these irreducible twisted modules by the notation $V^\natural(g)$ - it is unambiguous, because they are known to be unique up to isomorphism by [Dong-Li-Mason-2000].

Following the physical interpretation, we expect our generalized vertex algebra to be built from the twisted $V^\natural$-modules. In particular, the pieces V_λ mentioned earlier are eigenspaces in twisted modules $V^\natural(g^i)$ under a lifted action of g. Because the action of $C_{\mathbb{M}}(g)$ is projective, this action is a priori only defined up to a scalar. However, the decomposition into summands parametrized by a $\mu_{|g|}$-torsor of eigenvalues is canonical.

To assemble a generalized vertex algebra from these pieces, we need to define a multiplication operation, i.e., a set of compatible intertwining operators $V^\natural(g^i) \otimes V^\natural(g^j) \to V^\natural(g^{i+j})((z^{1/|g|}))$.

We split this problem into two parts:

(1) Show that for each i,j the vector space $I_{i,j}^{i+j}$ of intertwining operators is one dimensional, and that composition induces appropriate isomorphisms between tensor products of these vector spaces.
(2) Show that a good choice $\{m^{i,j}(z) \in I_{i,j}^{i+j}\}_{i,j=0}^{|g|-1}$ can be drawn from these spaces.

The first part is sketched in my doctoral dissertation [Carnahan-2007], and a full treatment is in preparation. The one-dimensionality follows from a calculation using conformal blocks, combining the theorems of [Nagatomo-Tsuchiya-2005] and [Frenkel-Szczesny-2004].

The second part, choosing elements from one-dimensional spaces of multiplication maps, is a homological algebra problem that is solved in [Carnahan≥2013]. It is similar to the construction of a group ring from one-dimensional subspaces, but with an additional complication coming from monodromy. By some power series manipulations, we obtain the following:

Theorem 1. *Given an abelian group A, a vertex algebra $V = M^0$, a family $\{M^i\}_{i\in A}$ of V-modules, and one-dimensional spaces $I_{i,j}^{i+j}$ of intertwining operators whose composition maps are compatible with associativity and skew-symmetry, any choice of nonzero intertwining operators $\{m^{i,j}(z) \in I_{i,j}^{i+j}\}_{i,j=0}^{|g|-1}$ defines an abelian intertwining algebra structure on $\bigoplus_\lambda V_\lambda$ for a 4-cocycle on the Eilenberg-MacLane complex $K(A,2)$ with coefficients in $\mathbb{C}^\times$ (see* [Eilenberg-Mac Lane-1954] *section 26). Furthermore, changing the choice of elements $m^{i,j}(z)$ will change the cocycle by a coboundary, so the cohomology class in $H^4(K(A,2),\mathbb{C}^\times)$ is invariant. The cohomology class of the 4-cocycle determines the braiding, and is determined, up to an "evenness" ambiguity, by the L_0-spectrum.*

In particular, there always exists a lattice Λ of signature $(1,1)$, such that the fractional part of the L_0-spectrum of twisted $V^\natural$ modules is cancelled by that of Heisenberg modules in $\bigoplus_{\lambda\in\Lambda}(V_\lambda \otimes \pi_\lambda^{1,1})$. The tensor product then has trivial braiding.

This yields a conformal vertex algebra structure on $\bigoplus_{\lambda\in\Lambda}(V_\lambda \otimes \pi_\lambda^{1,1})$, and we obtain the Lie algebra $\mathfrak{m}_g := \mathbf{Quant}(\bigoplus_{\lambda\in\Lambda}(V_\lambda \otimes \pi_\lambda^{1,1}))$. A second homological manipulation shows that our conformal vertex algebra has a projective $C_{\mathbb{M}}(g)$-action by conformal vertex algebra automorphisms, so $\mathfrak{m}_g$ has a projective action by Lie algebra automorphisms. It is not particularly difficult to check that $\mathfrak{m}_g$ is a Borcherds-Kac-Moody Lie algebra, and oscillator cancellation identifies its root spaces (as projective $C_{\mathbb{M}}(g)$-modules) with certain L_0-eigenspaces in g-eigenspaces V_λ of twisted modules.

3. Construction of L_g

We now outline how we construct Borcherds-Kac-Moody Lie algebras using automorphic infinite products (known as Borcherds products) for $O(2,2)$. In Borcherds's proof of the Conway-Norton Monstrous Moonshine conjecture, the necessary infinite product identity was the Koike-Norton-Zagier identity:

$$J(w) - J(z) = p^{-1} \prod_{m>0,n\in\mathbb{Z}} (1-p^m q^n)^{c(mn)},$$

where $p = e^{2\pi i w}$ and $q = e^{2\pi i z}$ are Fourier series variables, and the c in the exponent is defined by $J(z) = \sum_{n\geq -1} c(n)q^n = q^{-1} + 196884q + 21493760q^2 + \cdots$. The infinite product on the right converges when w and z have sufficiently large imaginary part, and the identity holds on this domain.

Borcherds used this infinite product identity to construct a Borcherds-Kac-Moody Lie algebra L. The Lie algebra L is then distinguished up to isomorphism by the property that its Weyl-Kac-Borcherds denominator formula is given by the Koike-Norton-Zagier identity.

To generalize this to the construction of Lie algebras L_g, we need analogues of the Koike-Norton-Zagier formula, and in particular a way to identify the exponents in an infinite product expansion of an automorphic function.

3.1. **From products to Lie algebras.** We wish to construct Lie algebras from infinite products, using generators and relations. This is not hard to do in general: Given any non-negative formal power series $f(q) \in q^{-1} + \mathbb{Z}_{\geq 0}[[q]]$, one has a product expansion:

$$f(p) - f(q) = p^{-1} \prod_{m>0, n\in\mathbb{Z}} (1 - p^m q^n)^{c(m,n)}$$

where the exponents $c(m,n)$ are non-negative integers. This product expansion is the Weyl-Kac-Borcherds denominator formula for a generalized Kac-Moody Lie algebra. For each m and n, the integer $c(m,n)$ gives the multiplicity of the degree (m,n) root space. The Cartan matrix, while infinitely large in general, is straightforward to describe in terms of the power series f.

The hard part is finding the right power series f and the right infinite product in the first place. Recall that for generating L, we used the product expansion of $J(w) - J(z)$. For generalized moonshine, it is natural to replace $J(z)$ with the McKay-Thompson series $T_g(z)$. Indeed, in 1992, Borcherds showed that the product expansions of $T_g(w) - T_g(z)$ at the cusp (∞, ∞) of $\mathfrak{H} \times \mathfrak{H}$ have exponents that are linear combinations of coeffiicients of Hauptmoduln. However, when $T_g(z)$ has negative coefficients, the products cannot describe Lie algebras, because the root multiplicities become negative. The standard solution to the problem of negative multiplicities is to count them as a contribution of odd roots of a Lie superalgebra, but as far as we can tell, these superalgebras are not helpful for Generalized Moonshine.

When I started this project, Borcherds suggested I compute some expansions of $T_g(w) - T_g(z)$ at different cusps in $\mathfrak{H} \times \mathfrak{H}$ to get an idea of how they behave. Some experimentation suggested that one always gets non-negative exponents by expanding $T_g(w) - T_g(z)$ at the $(\infty, 0)$ cusp in $\mathfrak{H} \times \mathfrak{H}$, i.e., forming a product from the Fourier expansion of $T_g(w) - T_g(-1/z)$. This is consistent with the construction of L, since $J(z) = J(-1/z)$. Furthermore, I later realized that it is a natural choice, because Norton's conjecture leads to the prediction that the twisted module $V^\natural(g)$ has character given by $T_g(-1/z)$. That is, if we want a Lie algebra related to a twisted module, it is a good sign when the simple root multiplicities are given by the character of the twisted module. We end up with the following dichotomy:

(1) The Fricke case: $T_g(z)$ is invariant under a Fricke involution $z \mapsto 1/Nz$ for some N, and has non-negative integer coefficients. Then the expansion of $T_g(w) - T_g(-1/z)$ is essentially the same as the expansion of $T_g(w) - T_g(z)$ computed by Borcherds in [Borcherds-1992], and yields a Lie algebra with one real simple root.
(2) The non-Fricke case: $T_g(z)$ is not invariant under any Fricke involutions. By the Hauptmodul property, this means $T_g(z)$ is regular at the 0 cusp. The product expansion is substantially different from what Borcherds computed, and the corresponding Lie algebra has no real simple roots.

3.2. **Identification of root multiplicities.** It remains to identify the root multiplicities in the Lie algebras described by the infinite product expansions of $T_g(w) - T_g(-1/z)$. There are a few reasons why we should expect the multiplicities $c(m,n)$ to be coefficients of modular functions:

First, that was the case for the Lie algebra L. Second, by looking to the construction of $\mathfrak{m}_g$, we know that the root multiplicities should be related to subspaces of twisted $V^\natural$-modules, and the dimensions of such subspaces should come from modular forms by general modularity considerations in conformal field theory, and specifically a compatibility asserted in Norton's conjecture. Third, Borcherds developed a theory of automorphic infinite products that describe infinite dimensional Lie algebras with root multiplicities given by coefficients of modular forms, and $T_g(w) - T_g(z)$ is an example of an automorphic function on $\mathfrak{H} \times \mathfrak{H}$. With this evidence in hand, the question of what modular function provides the root multiplicities of L_g becomes a natural one. In [Carnahan-2012], we establish that the exponents $c(m,n)$ are the coefficients of a vector-valued modular function F built from McKay-Thompson series.

Our vector-valued function F is constructed by a two-step process due to Borcherds (derived from [Borcherds-1998] Lemma 2.6). We begin with a set of functions $\{f^{(m)}\}_{m|N}$, with $f^{(m)}$ invariant under $\Gamma(N/m)$, and define $f^{(k)} = f^{(k,N)}$ for all integers k. For our purposes, we will set $f^{(m)}$ to be the McKay-Thompson series T_{g^m} for our chosen element g in the Monster.

(1) First step: We construct an $N \times N$ matrix of functions:

$$\hat{F}_{i,j}(\tau) = f^{((i,j))}\left(\frac{*\tau + *}{\frac{i}{(i,j)}\tau + \frac{j}{(i,j)}}\right).$$

The asterisks in the numerator do not need to be specified, because of the $\Gamma(N/m)$-invariance of $f^{((i,j))}$. The components of $\hat{F}$ then form a complete list of images of all $f^{(m)}$ under $SL_2(\mathbb{Z})$ transformations.

(2) Second, we apply a discrete Fourier transform on the rows of the matrix to get the vector-valued modular function:

$$F_{i,k} = \frac{1}{N}\sum_{j=0}^{N-1} e^{-2\pi ijk}\hat{F}_{i,j}$$

The vector-valued function F transforms according to Weil's representation, and one can check that this is equivalent to the modular invariance properties of the components of $\hat{F}$ by a straightforward use of the discrete Fourier transform.

The construction of F is best motivated by passing to conjectural interpretations of the components of F and $\hat{F}$ in moonshine. Recall that we constructed a generalized vertex algebra structure on $V_g = \bigoplus_{i=1}^N V^\natural(g^i)$, the sum of twisted modules along a cyclic group. If we assume a refined version of Generalized Moonshine for cyclic subgroups of $\mathbb{M}$ (proved to hold up to a constant ambiguity in [Dong-Li-Mason-2000]), there is a linear (not just projective) action of some cyclic extension H of $\langle g \rangle$ on V_g, where we write g for a distinguished generator of H, such that the graded trace of g^j on the g^i-twisted module is given by the expansion of $\hat{F}_{i,j}(\tau)$. That is, $\hat{F}$ conjecturally describes the traces of elements of H on twisted modules whose twisting ranges among elements of H. For example, $\hat{F}_{0,0}$ is the graded trace of 1 on the untwisted module $V^\natural$, so it is equal to J. We can call $\hat{F}$ the vector-valued character of H acting on V_g.

The discrete Fourier transform takes traces to eigenvalue multiplicities. More precisely, in this interpretation, the q^k coefficient of $F_{i,j}$ gives the dimension of the subspace of $V^\natural(g^i)$ on which g acts by $e^{2\pi ij/N}$ and L_0 acts by $k+1$. We can call F the vector-valued dimension of V_g under the H-character decomposition. While this interpretation of $\hat{F}$ and F in terms of traces and multiplicities is conjectural, and in fact equivalent to the assertion that $L_g \cong \mathfrak{m}_g$, we can still use the functions themselves to produce automorphic products.

3.2.1. *Hecke operators.* In [Carnahan-2012], the root multiplicities in the Lie algebra L_g are identified with coefficients of F in two ways. Both methods use the "finite order Hecke-monic" property of the McKay-Thompson series, which is equivalent to being completely replicable and finite level. Here, the name Hecke shows up because we employ Hecke operators for elliptic curves equipped with torsors (see e.g., [Ganter-2009]). Given a finite group G, one has a moduli space of elliptic curves with G-torsors, which can be written as an analytic quotient of a disjoint union of finitely many complex upper half-planes $\coprod_{(g,h):\mathbf{Z}\times\mathbf{Z}\to G}\mathfrak{H}$. The quotient map is given by the fact that any G-torsor is determined up to isomorphism by its monodromy along a basis of H_1, given by a conjugacy class of a pair of commuting elements of G, and points on the upper half-plane parametrize elliptic curves equipped with an oriented basis of H_1 (a more precise description of the moduli problem can be found in [Carnahan-2012] section 3.1.1).

Given a function f on this space, we can define a Hecke operator by setting:

$$(T_n f)(P \to E) = \frac{1}{n} \sum_{n\text{-isogenies } \pi:E'\to E} f(\pi^* P \to E')$$

By lifting along the analytic quotient map, we can write f as $f(g,h,\tau)$ for a commuting pair $g,h \in G$, and the Hecke operator is:

$$T_n f(g,h,\tau) = \frac{1}{n} \sum_{ad=n, 0\le b<d} f\left(g^d, g^{-b}h^a, \frac{a\tau+b}{d}\right)$$

Given a function f on $\mathcal{M}^G_{ell}$, written as $f(g,h,\tau)$ for commuting elements $g,h \in G$, we define "Hecke-monic" to be the property that for each fixed g,h, the function $n(T_n f)(g,h,\tau)$ is a monic polynomial in $f(g,h,\tau)$. We use the Hecke-monic property of the McKay-Thompson series to identify $\log e^{2\pi i w}(T_g(w) - T_g(z))$ with $\sum_{n=1}^\infty e^{2\pi i n w} T_n T_g(z)$ in an analytic neighborhood of the cusp (∞,∞) in $\mathfrak{H}\times\mathfrak{H}$. The proof is in fact quite similar to the proof of the Koike-Norton-Zagier identity for the J function.

3.2.2. *First identification.* The first method for identifying roots of L_g is a straightforward analytic continuation: The logarithm of the infinite product whose exponents are drawn from coefficients of F becomes a sum of $e^{2\pi i n w}T_n T_g(z)$ expanded at the cusp $(\infty, 0)$, and converges in a neighborhood U of that cusp. The region U overlaps with the domain of convergence of the sum of $e^{2\pi i n w}T_n T_g(z)$ expanded at the cusp (∞,∞). This yields the identification with $T_g(w) - T_g(-1/z)$.

3.2.3. *Second identification.* The second method for identifying roots of L_g is the Borcherds-Harvey-Moore regularized theta-lift [Borcherds-1998]. Following a discovery by Harvey and Moore [Harvey-Moore-1996] with string-theoretic motivation, Borcherds found a general method for describing the expansions of certain automorphic functions at all cusps by infinite products. In the case of interest to us, the machine takes in a vector-valued modular function F, and produces a function Ψ on $\mathfrak{H}\times\mathfrak{H}$, invariant under $\Gamma_0(N)\times\Gamma_0(N)$ modulo a possible correction term. This function has the following key properties:

(1) The zeroes and poles of Ψ lie on quadratic divisors in $\mathfrak{H}\times\mathfrak{H}$, and their multiplicity is determined by the coefficients of poles of F. In particular, we need the poles of F to have integer coefficients to make Ψ a single-valued function.
(2) The weight of Ψ as a modular form on $Y_0(N)\times Y_0(N)$ is given by the constant term of $F_{0,0}$. In our case, this term is zero (because the weight 1 subspace of $V^\natural$ is zero dimensional), so we have an invariant function.
(3) At each cusp, Ψ admits an infinite product expansion, and the exponents are given by the coefficients of regular terms in F.

To show that the Borcherds-Harvey-Moore lift Ψ is equal to $T_g(w) - T_g(-1/z)$, we need only compare product expansions at the cusp (∞,∞), where $\log e^{2\pi i w}\Phi$ simplifies to $\sum_{n=1}^\infty e^{2\pi i n w}T_n T_g(z)$. Then the product expansion at $(\infty, 0)$ is what we wanted.

In summary, we have $T_g(w) - T_g(-1/z) = p^{-1}\prod_{m>0,n\in\frac{1}{N}\mathbb{Z}}(1-p^m q^n)^{c(m,n)}$ with $c(m,n)$ equal to the q^{mn} coefficient of the vector-valued function $F_{m,Nn}$ for all $m \in \mathbb{Z}, n \in \frac{1}{N}\mathbb{Z}$. In fact, we obtain such an identification if we replace T_g with any completely replicable function of finite level, but McKay-Thompson series yield non-negative exponents (hence honest Lie algebras) in a particularly natural way. With this identification of exponents, we identify the root multiplicities $c(m,n)$ of L_g with coefficients of F.

4. Comparison

We now have a Lie algebra L_g with automorphic denominator formula and known simple roots, and a Lie algebra $\mathfrak{m}_g$ with a projective action of $C_{\mathbb{M}}(g)$. Both are Borcherds-Kac-Moody algebras.

As in Borcherds's theorem, we need an isomorphism $L_g \cong \mathfrak{m}_g$ to get a Lie algebra with both a finite group action, and known simple roots. Because Borcherds-Kac-Moody algebras have a denominator formula, the Cartan matrix is determined by the root multiplicities. In particular, to prove the existence of an isomorphism, it suffices to show that the two Lie algebras have the same root multiplicities.

To compare root multiplicities, we must compare the multiplicity of each eigenspace of g and L_0 on twisted modules $V^\natural(g^i)$ with the coefficients of the vector-valued forms $F_{i,k}$. Using the discrete Fourier transform, this is equivalent to matching the traces of g^j on L_0-eigenspaces of $V^\natural(g^i)$ with the coefficients of the matrix function $\hat{F}_{i,j}$. The problem lies with the definition of g: the canonical action of $C_{\mathbb{M}}(g)$ on $V^\natural(g^i)$ is projective, so the eigenspaces have a cyclic shift ambiguity, i.e., we need a way to eliminate the constant ambiguity in the trace of each g^j. This problem is partially resolved by the fact that there is a preferred lift of g on $V^\natural(g)$, given by $e^{2\pi i L_0}$. While there exists a unique extension to a preferred lift of g as a linear automorphism of the generalized vertex algebra $\bigoplus_i V^\natural(g^i)$, I do not know enough about this lift of g to precisely determine the root multiplicities when g has composite order. However, when g has prime order, the root-of-unity ambiguities do not appear.

Theorem 2. *If g has prime order, or lies in conjugacy class 4B, then $L_g \cong \mathfrak{m}_g$.*

The extra case 4B is proved by appealing to the Borcherds-Kac-Moody structure of $\mathfrak{m}_g$ to resolve a sign ambiguity in the action of g on the g^2-twisted module. Specifically, if the isomorphism did not exist, then $\mathfrak{m}_g$ would have collinear real simple roots. Unfortunately, this trick only seems to work once.

5. Hauptmoduln

It remains to use the isomorphism $L_g \cong \mathfrak{m}_g$ to show that characters of twisted modules are Hauptmoduln.

The denominator formula of L_g is a manifestation of the isomorphism $\bigwedge^* E_g \cong H_* E_g$ of virtual vector spaces, where E_g is the positive subalgebra of $\mathfrak{m}_g$. The projective action of $C_{\mathbb{M}}(g)$ on $\mathfrak{m}_g$ promotes the isomorphism $\bigwedge^* E_g \cong H_* E_g$ to an equivariant map. The twisted denominator formula then implies any character of a centralizing element is weakly Hecke-monic.

More precisely, for any commuting $g, h \in \mathbb{M}$, we choose a fixed lift of h together with the preferred lift of g to linear automorphisms of the generalized vertex algebra V_g, and construct orbifold partition functions:

$$Z(g^k, g^\ell h^m, \tau) = \sum_{s\in\frac{1}{N}\mathbb{Z}} \sum_{r\in\frac{1}{N}\mathbb{Z}/\mathbb{Z}, s\in kr+\mathbb{Z}} Tr(g^\ell h^m | V^\natural(g^k)^r_{1+s}) e^{2\pi i s\tau}.$$

Here, $V^\natural(g^k)^r_{1+s}$ is the subspace of the irreducible g^k-twisted $V^\natural$-module on which L_0 acts by $1+s$, and the preferred lift of g acts by $e^{2\pi i r}$. We define equivariant Hecke operators:

$$T_nZ(g,h,\tau) = \frac{1}{n}\sum_{ad=n,0\le b<d} Z\left(g^d, g^{-b}h^a, \frac{a\tau+b}{d}\right)$$

We say that Z is weakly Hecke-monic at (g,h) if for any $n \geq 1$, there is a degree n monic polynomial $x \mapsto P_n^{g,h}(x)$ such that $nT_nZ(g,h,\tau) = P_n^{g,h}(Z(g,h,\tau))$.

By following the analysis of Cummins and Gannon, we find that:

Theorem 3. *([Carnahan-2010]) Any weakly Hecke-monic function Z with a pole at infinity, and expansion coefficients that are algebraic integers, either has the form $aq^{-1}+b+cq$, or is a Hauptmodul.*

The degenerate case is eliminated using the modularity work in [Dong-Li-Mason-2000]. Unfortunately, we do not have a corresponding result for weakly Hecke-monic functions that are regular at infinity, so for now we are limited to considering the case where the twisting element is Fricke. We conclude that for the 141 Fricke classes, out of 194 total classes, an isomorphism of Monstrous Lie algebras implies the Hauptmodul part of Norton's Generalized Moonshine Conjecture for twisted modules:

Theorem 4. *([Carnahan≥2013]) For any Fricke g for which $L_g \cong \mathfrak{m}_g$, and any $h \in C_{\mathbb{M}}(g)$, the function $Z(g,h,\tau)$ is a Hauptmodul.*

In particular, we obtain a full proof for g in 17 classes:

Corollary 5. *If $g \in \mathbb{M}$ lies in a conjugacy class of type 4B or 3C or pA for a prime p, and $h \in C_{\mathbb{M}}(g)$, then $Z(g,h,\tau)$ is a Hauptmodul.*

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