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The parents of Weierstrass semigroups and non-Weierstrass semigroups [1]

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Abstract

We consider the map p between the sets of numerical semigroups sending a numerical semigroup to the one whose genus is decreased by 1. We prove that the semigrop $p(H)$, which is called the *parent* of H, of a Weierstrass (resp. non-Weierstrass) numerical semigroup H is Weierstrass(resp. non-Weierstrass) in some cases.

1 Notations and terminologies

Let $\mathbb{N}_0$ be the additive monoid of non-negative integers. A submonoid H of $\mathbb{N}_0$ is called a *numerical semigroup* if the complement $\mathbb{N}_0\backslash H$ is finite. The cardinality of $\mathbb{N}_0\backslash H$ is called the *genus* of H, denoted by $g(H)$. For a numerical semigroup H we set

$$m(H)=\min\{h\in H\mid h>0\},$$

which is called the *multiplicity* of H. In this case, the semigroup H is called an *m-semigroup* where we set $m=m(H)$. For any i with $1\leqq i\leqq m-1$ we set

$$s_i=\min\{h\in H\mid h\equiv i \bmod m\}.$$

The set $S(H)=\{m,s_1,\ldots,s_{m-1}\}$ is called the *standard basis* for H. We set

$$s_{max}=\max\{s_i\mid i=1,\ldots,m-1\}.$$

For a numerical semigroup H we set

$$c(H)=\min\{c\in\mathbb{N}_0\mid c+\mathbb{N}_0\subseteqq H\},$$

which is called the *conductor* of H. We note that $c(H)-1\notin H$. We set $p(H)=H\cup\{c(H)-1\}$, which is a numerical semigroup of genus $g(H)-1$. The numerical semigroup $p(H)$ is called the *parent* of H.

A *curve* means a complete non-singular irreducible algebraic curve over an algebraically closed field k of characteritic 0. For a pointed curve (C,P) we set

$$H(P)=\{n\in\mathbb{N}_0\mid\exists f\in k(C)\text{ such that }(f)_\infty=nP\},$$

where $k(C)$ is the field of rational functions on C and $(f)_\infty$ denotes the polar divisor of f. A numerical semigroup H is said to be *Weierstrass* if there exists a pointed curve (C,P) with $H(P)=H$.

[1] This paper is an extended abstract and the details will appear elsewhere.

2 The parents of non-Weierstrass semigroups

Let H be a numerical semigroup. For any integer $m \geqq 2$ we set

$$L_m(H) = \{l_1 + \cdots + l_m \mid l_i \in \mathbb{N}_0 \backslash H, \text{ all } i\}.$$

A numerical semigroup H is said to be *Buchweitz* if there exists an integer m such that $\sharp L_m(H) \geqq (2m-1)(g(H)-1)+1$. Buchweitz [1] showed that every Buchweitz semigroup H is non-Weierstrass. We showed the following in Lemma 4.2 of [5]:

Remark 2.1 *Let H be a primitive n-semigroup, i.e., $2n > \max\{l \mid l \notin H\} = c(H)-1$, with $g(H) \geqq n+5$. Let $\overline{H}$ be a primitive $2n$-semigroup with*

$$\mathbb{N}_0 \backslash \overline{H} = \{1, \ldots, 2n-1\} \cup \{2\ell_n, 2\ell_{n+1}, \ldots, 2\ell_{g(H)}\} \cup \{4n-3, 4n-1\}$$

where

$$\mathbb{N}_0 \backslash H = \{1, \ldots, n-1, \ell_n < \ldots < \ell_{g(H)}\}.$$

Assume that $\sharp L_2(H) \geqq 3g(H)-2$. Then we have

$$\sharp L_2(\overline{H}) \geqq 3g(\overline{H}) - 2 \text{ and } \sharp L_2(p(\overline{H})) \geqq 3g(p(\overline{H})) - 2.$$

In Example 4.2 in [5] we give the following example:

Example 2. 1 Let t and n be integers with $t \geqq 5$ and $n \geqq 4t+1$. Let H be a primitive n-semigroup whose complement $\mathbb{N}_0 \backslash H$ is

$$\{1, \ldots, n-1\} \cup \{2n-2t-1, 2n-2t-1+2\cdot 1, \ldots, 2n-2t-1+2\cdot(t-2)\} \cup \{2n-2, 2n-1\}.$$

Then H satisfies $\sharp L_2(H) = 3g(H) - 2$. For example, if we set $t = 5$ and $n = 21$, we have

$$\mathbb{N}_0 \backslash H = \{1, \ldots 20\} \cup \{31, 33, 35, 37, 40, 41\}.$$

Example 2. 2 Let H be as in the above example with $t = 5$ and $n = 21$. Let $\overline{H}$ be as in Remark 2.1. In fact, we have

$$\overline{H} = \{1 \longrightarrow 41\} \cup \{62, 66, 70, 74, 80, 82\} \cup \{81, 83\}$$

and

$$p(\overline{H}) = \{1 \longrightarrow 41\} \cup \{62, 66, 70, 74, 80, 82\} \cup \{81\}.$$

Then the semigroups $\overline{H}$ and $p(\overline{H})$ are Buchweitz, hence non-Weierstrass.

Let $\tilde{H}$ be a non-Weierstrass numerical semigroup. We consider the sequence

$$\tilde{H} \longrightarrow p(\tilde{H}) \longrightarrow p^2(\tilde{H}) \longrightarrow \cdots \longrightarrow p^{g(\tilde{H})-8}(\tilde{H}).$$

Since $g(p^{g(\tilde{H})-8}(\tilde{H}))=8$, $p^{g(\tilde{H})-8}(\tilde{H})$ is Weierstrass (see [8]). Hence, there exists i with $0 \leqq i \leqq g(\tilde{H})-7$ such that $p^i(\tilde{H}) = H$ is non-Weierstrass and $p^{i+1}(\tilde{H}) = p(H)$ is Weierstrass. In fact, we have the following example with $i=0$:

Example 2. 3 The numerical semigroup $H = \langle 8, 12, 8\ell+2, 8\ell+6, n, n+4\rangle$ with $\ell \geqq 2$ and odd $n \geqq 16\ell+19$ is non-Weierstrass (see [6]). The parent $p(H) = H + (n+8\ell-2)\mathbb{N}_0$ is Weierstrass (See [7]).

3 The parents of Weierstrass semigroups

Problem 3. 1 Let H be a numerical semigroup. When are the numerical semigroups H and $p(H)$ Weierstrass?

Let $\mathbb{N}_0 \backslash H = \{l_1, \ldots, l_{g(H)}\}$. We set $w(H) = \sum_{i=1}^{g(H)} (l_i - i)$, which is called the *weight* of H. Then it is well-known that $0 \leqq w(H) \leqq \dfrac{(g(H)-1)g(H)}{2}$.

Proposition 3.1 *If* $w(H) = \dfrac{(g(H)-1)g(H)}{2}$, *then* H *and* $p(H)$ *are Weierstrass. In fact, we have* $H = \langle 2, 2g(H)+1\rangle$ *and* $p(H) = \langle 2, 2(g(H)-1)+1\rangle$, *which are Weierstrass.*

We have the following:

Remark 3.2 0) *If* $w(H) \leqq \dfrac{g(H)}{2}$, *then* H *is primitive* (see [2]).
i) *If* H *is primitive and* $w(H) \leqq g(H)-2$, *then* H *is Weierstrass* (see [2]).
ii) *If* H *is primitive and* $w(H) = g(H)-1$, *then* H *is Weierstrass* (see [3]).

Moreover, we see the following:

Lemma 3.3 i) *If* $0 < w(H) \leqq g-1$, *then we have* $w(p(H)) \leqq w(H)-1$.
ii) *If* $w(H) \geqq g$, *then we have* $w(p(H)) \leqq w(H)-2$.

By Lemma 3.3 and Remark 3.2 we get the following:

Proposition 3.4 i) *If* $w(H) \leqq \dfrac{g(H)}{2}$, *then* H *and* $p(H)$ *are Weierstrass.*
ii) *If* $w(H) \leqq g(H)-1$ *and* H *is primitive, then* H *and* $p(H)$ *are Weierstrass.*
iii) *If* $w(H) = g(H)$ *and* H *is primitive, then* $p(H)$ *is Weierstrass,*

We note the following:

Remark 3.5 *We have* $g(H)+1 \leqq c(H) \leqq 2g(H)$.

If $c(H) = g(H) + 1$, then we obtain

$$H = \langle g(H) + 1 \to 2g(H) + 1 \rangle \text{ and } p(H) = \langle g(H) \to 2g(H) - 1 \rangle,$$

which are Weiersrass. Hence, we get the following:

Proposition 3.6 *If $c(H) = g(H) + 1$, then H and $p(H)$ are Weierstrass.*

Moreover, we can prove the following:

Theorem 3.7 *If we have $c(H) = g(H) + 2$, then H and $p(H)$ are Weierstrass.*

Proof. Since $c(H) = g(H) + 2$, we have $\mathbb{N}_0 \backslash H \subset \{1 \longrightarrow g(H) + 1\}$. Assume that $2m(H) \leqq g(H) + 1$. Since we have $m(H), 2m(H) \notin \mathbb{N}_0 \backslash H$, we get

$$\mathbb{N}_0 \backslash H \subsetneqq \{1 \longrightarrow g(H) + 1\} \backslash \{m(H), 2m(H)\}$$

which is a contradiction. Hence, we get $2m(H) > g(H) + 1$, i.e., H is primitive. We may assume that $g(H) \geqq 3$. Hence, we have some $i \geqq 3$ such that $i \in H$. In this case, we obtain

$$\mathbb{N}_0 \backslash H = \{1, \dots i - 1, i + 1, \dots g(H) + 1\}.$$

We have $w(H) = g(H) + 1 - i \leqq g(H) - 2$. By Remark 3.2 i), H is Weierstrass. Moreover, we have

$$\mathbb{N}_0 \backslash p(H) = \{1, \dots i - 1, i + 1, \dots g(H)\}.$$

By the same method as in the above we can show that $p(H)$ is Weierstrass. □

Problem 3. 2 Let H be a Weierstrass numerical semigroup. Then is the numerical semigroup $p(H)$ also Weierstrass?

Using the standard method constructing a double covering we can show the following theorem:

Theorem 3.8 *Let $c(H) = 2g(H)$, i.e., H is symmetric. If $g(H) \geqq 6g(d_2(H)) + 4$ and H is Weierstrass, then $p(H)$ is also Weierstrass.*

We set

$$d_2(H) = \{\frac{h}{2} \mid h \in H \text{ which is even}\},$$

which is also a numerical semigroup. If $\pi : C \longrightarrow C'$ is a double covering with a ramification point P, then we have $H(\pi(P)) = d_2(H(P))$. We set

$$n(H) = \min\{h \in H \mid h \text{ is odd}\}.$$

Remark 3.9 *Assume that $g(H) \geqq 6g(d_2(H)) + 4$.*
i) *We have*

$$g' + \frac{n-1}{2} \leqq g(H) \leqq 2g' + \frac{n-1}{2}$$

where we set $g' = g(d_2(H))$ and $n = n(H)$ (see [4]).
ii) *If H is Weierstrass, then so is $d_2(H)$* (see [9]).

Theorem 3.10 *Let* $g(H) \geqq 6g(d_2(H)) + 4$. *Assume that* $g(H) = 2g(d_2(H)) + \frac{n-1}{2}$ *where we set* $n = n(H)$. *In this case,* $H = 2d_2(H) + n\mathbb{N}_0$. *If* H *is Weierstrass, then so is* $p(H)$.

Proof. We have $p(H) = 2d_2(H) + n\mathbb{N}_0 + (n + 2(s_{max} - m))\mathbb{N}_0$. Since $d_2(p(H)) = d_2(H)$ is Weierstrass by Remark 3.9 ii), $p(H)$ is Weierstrass (see Proposiiton 2.4 in [6]). □

By a similar method to the proof of Proposiiton 2.4 in [6] we can prove the following:

Theorem 3.11 *Let* $g(H) \geqq 6g(d_2(H)) + 4$. *Assume that* $H \not\ni n + 2(s_{max} - m)$ *where we set* $n = n(H)$. *If* H *is Weierstrass, then so is* $p(H)$.

Moreover, we get the following:

Theorem 3.12 *We set* $\mathbb{N}_0 \backslash d_2(H) = \{l_1 < \cdots < l_{g'}\}$ *where* $g' = g(d_2(H))$. *Let* $H_i = 2d_2(H) + \langle n, n + 2l_{g'}, n + 2l_{g'-1}, \ldots, n + 2l_{g'-i}\rangle$ *where we set* $n = n(H)$. *Assume that* $g(H) \geqq 6g(d_2(H)) + 4$. *If* $H = 2d_2(H) + n\mathbb{N}_0$ *is Weierstrass, then so is* H_i *for any* i *with* $0 \leqq i \leqq g' - 1$.

Using Theorems 3.11 and 3.12 we get the following:

Corollary 3.13 *Let* $g(H) \geqq 6g(d_2(H)) + 4$. *Assume that* $g(H) = 2g(d_2(H)) + \frac{n-1}{2} - 1$. *If* H *is Weierstrass, then so is* $p(H)$.

Proof. By the assumption $H = 2d_2(H) + \langle n, n + 2(s_i - m)\rangle$ for some i with $s_i + s_j \notin S(d_2(H))$, all j (see [6]). If $s_i \neq s_{max}$, then by Theorem 3.11 we get the result. If $s_i = s_{max}$, then by Theorem 3.12 we get the result. □

By Proposition 2..4 in [4] we have the following:

Remark 3.14 *Let* $n \geqq 4g(d_2(H)) + 1$ *where we set* $n = n(H)$. *Assume that* $g(H) = g(d_2(H)) + \frac{n-1}{2}$. *In this case,* $H = 2d_2(H) + \langle n, n + 2, \ldots, n + 2(m(d_2(H)) - 1)\rangle$. *If* $d_2(H)$ *is Weierstrass, then so is* H.

By Remarks 3.14 and 3.9 ii) we get the following:

Proposition 3.15 *Let* $g(H) \geqq 6g(d_2(H)) + 4$. *Assume that* $g(H) = g(d_2(H)) + \frac{n-1}{2} + 1$ *where we set* $n = n(H)$. *If* H *is Weierstrass, then so is* $p(H)$.

Proposition 3.16 *Let* $g(H) \geqq 6g(d_2(H)) + 4$. *Assume that* $g(H) = g(d_2(H)) + \frac{n-1}{2}$ *where we set* $n = n(H)$. *If* H *is Weierstrass, then so is* $p(H)$.

Proof. We have $n(p(H)) = n - 1$. Hence, by Remarks 3.14 and 3.9 ii) we get the result. □

References

[1] R.O. Buchweitz, *On Zariski's criterion for equisingularity and non-smoothable monomial curves,* Preprint 113, University of Hannover, 1980.

[2] D. Eisenbud and J. Harris, *Existence, decomposition, and limits of certain Weierstrass points,* Invent. Math. **87** (1987) 495-515.

[3] J. Komeda, *On primitive Schubert indices of genus g and weight $g-1$,* J. Math. Soc. Japan **43** (1991) 437-445.

[4] J. Komeda, *On Weierstrass semigroups of double coverings of genus three curves*, Semigroup Forum **83** (2011) 479-488.

[5] J. Komeda, *The fractional map by two and the parent map of numerical semigroups*, 数理解析研究所講究録 **1809** (2012) 198-204.

[6] J. Komeda, *Double coverings of curves and non-Weierstrass semigroup*, Communications in Algebra **41** (2013) 312-324.

[7] J. Komeda, *Boundaries between non-Weierstrass semigroups and Weierstrass semigroups,* In preparation.

[8] J. Komeda and A. Ohbuchi, *Existence of the non-primitive Weierstrass gap sequences on curves of genus* 8, Bull. Braz. Math. Soc. **39** (2008) 109-121.

[9] F. Torres, *Weierstrass points and double coverings of curves with application: Symmetric numerical semigroups which cannot be realized as Weierstrass semigroups,* Manuscripta Math. **83** (1994) 39-58.