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States on residuated lattices

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Abstract

We define states on non-commutative bounded residuated lattices and consider their property. We show that, for a non-commutative bounded residuated lattice X, if s is a state then $X/\ker(s)$ is an MV-algebra.
Keywords: residuated lattice, state, state-morphism

1 Introduction

Since the notion of *state* was firstly defined on MV-algebras by Kôpka and Chovanec in [11], theory of states on algebras is applied to other algebras and now it is a hot research filed. For example, property of states on pseudo-MV algebras is considered in [3], on pseudo-BL algebras in [7], on non-commutative residuated Rℓ-monoids in [5, 6]. In [7], it is proved that the notion of (Bosbach) state is the same as the notion of Riečan for good bounded non-commutative Rℓ-monoids. On the other hand, it is proved in [1] that there is a Riečan state which is not Bosbach state on a certain (non-commutative) residuated lattice.

The algebras above all except [1] have the condition of divisibility (div): $x \wedge y = x \odot (x \to y)$ (or $x \odot (x \rightsquigarrow y) = x \wedge y = (x \to y) \odot x$ in non-commutative case), from which the algebras are distributive lattices. On the other hand there are few research about states on algebras without (div) so far ([1]). In [10], states and state-morphisms on commutative residuated lattices are defined and investigated their property. We here generalize the results to the cases of non-commutative cases. That is, we define states and state-morphism on non-commutative residuated lattices and consider their property. We show that, for a non-commutative residuated lattice X, if s is a state then $X/\ker(s)$ is an MV-algebra.

2 Residuated lattices and states

We recall a definition of non-commutative bounded residuated lattices. An algebraic structure $(X, \wedge, \vee, \odot, \to, \mathbf{0}, \mathbf{1})$ is called a non-commutative bounded residuated lattice (simply called *residuated lattice*, RL) if

(1) $(X, \wedge, \vee, \mathbf{0}, \mathbf{1})$ is a bounded lattice;

(2) $(X, \odot, \mathbf{1})$ is a monoid with unit element $\mathbf{1}$;

(3) For all $x, y, z \in X$, $x \odot y \leq z$ if and only if $x \leq y \to z$ if and only if $y \leq x \rightsquigarrow z$.

For all $x \in X$, by x^- and $x^\sim$, we mean $x^- = x \to 0$ and $x^\sim = x \rightsquigarrow 0$, respectively.

The following results are easy to prove ([9, 12]).

Proposition 1. *For all $x, y, z \in X$, we have*

(1) $\mathbf{0}^- = \mathbf{1} = \mathbf{0}^\sim, \mathbf{1}^- = \mathbf{0} = \mathbf{1}^\sim$

(2) $x \odot x^\sim = \mathbf{0} = x^- \odot x$

(3) $x \odot (x \rightsquigarrow y) \leq y, \ \ (x \to y) \odot x \leq y$

(4) $(x \vee y)^- = x^- \wedge y^-, \ \ (x \vee y)^\sim = x^\sim \wedge y^\sim$

(5) $x \leq y \iff x \to y = \mathbf{1} \iff x \rightsquigarrow y = \mathbf{1}$

(6) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$

(7) $x \leq y \implies x \odot z \leq y \odot z, \ z \odot x \leq z \odot y$

(8) $x \leq y \implies z \to x \leq z \to y, \ y \to z \leq x \to z$

(9) $x \leq y \implies z \rightsquigarrow x \leq z \rightsquigarrow y, \ y \rightsquigarrow z \leq x \rightsquigarrow z$

(10) $x \to y \leq (y \to z) \rightsquigarrow (x \to z)$

(11) $x \rightsquigarrow y \leq (y \rightsquigarrow z) \to (x \rightsquigarrow z)$

(12) $x \to y \leq (z \to x) \to (z \to y)$

(13) $x \rightsquigarrow y \leq (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y)$

Some well-known algebras, MTL-algebras, BL-algebras, MV-algebras, Heyting algebras and so on, are considered as algebraic semantics for so-called fuzzy logics, monoidal t-norm logic, Basic logic, many valued logic, intuitionistic logic and so on, respectively.

The algebras above can be generalized to non-commutative cases. For example, any non-commutative residuated lattice satisfying the divisibility condition

$$\text{(div)} \quad x \odot (x \rightsquigarrow y) = x \wedge y = (x \to y) \odot x$$

is called a *pseudo $R\ell$-monoid* ([5, 6]). Any such algebra which support is a non-commutative residuated lattice has an attached name *pseudo.* For example pseudo MV-algebra (or GMV-algebra) is a residuated lattice with satisfying the conditions (div) : $x \odot (x \rightsquigarrow y) = x \wedge y = (x \to y) \odot x$

(dn) : $x^{-\sim} = x = x^{\sim -}$

(p-lin) : $(x \to y) \vee (y \to x) = 1 = (x \rightsquigarrow y) \vee (y \rightsquigarrow x)$

These algebras can be defined as axiomatic extensions of residuated lattices as follows:

$$\begin{aligned} \text{pseudo MTL} &= \text{RL} + \{\text{p} - \text{lin}\} \\ \text{pseudo BL} &= \text{RL} + \{\text{div}\} + \{\text{p} - \text{lin}\} \\ &= \text{pseudo MTL} + \{\text{div}\} \\ \text{pseudo MV} &= \text{pseudo BL} + \{\text{dn}\} \end{aligned}$$

To treat the state theory of such algebras uniformly, we define states on residuated lattices according to [5, 6] and investigate their property. Let X be a residuated lattice. A map $s : X \to [0,1]$ is called a *state* on X if it satisfies

(S1) $s(x) + s(x \to y) = s(y) + s(y \to x)$

(S2) $s(x) + s(x \rightsquigarrow y) = s(y) + s(y \rightsquigarrow x)$

(S3) $s(\mathbf{0}) = 0$ and $s(\mathbf{1}) = 1$

The condition (S1) above has another equivalent notion.

Proposition 2. *For a map $s : X \to [0,1]$ with meeting (S3) above, the following conditions are equivalent:*

(S1) $s(x) + s(x \to y) = s(y) + s(y \to x)$ *for all* $x, y \in X$

(S1)' $1 + s(x \wedge y) = s(x \vee y) + s(d_1(x,y))$ *for all* $x, y \in X$,
where $d_1(x,y) = (x \to y) \wedge (y \to x)$

(S1)" $1 + s(x \wedge y) = s(x) + s(x \to y)$ *for all* $x, y \in X$

Similarly we have

Proposition 3. *For a map $s : X \to [0,1]$ with meeting (S3) above, the following conditions are equivalent:*

(S2) $s(x) + s(x \rightsquigarrow y) = s(y) + s(y \rightsquigarrow x)$ *for all* $x, y \in X$

(S2)' $1 + s(x \wedge y) = s(x \vee y) + s(d_2(x,y))$ *for all* $x, y \in X$,
where $d_2(x,y) = (x \rightsquigarrow y) \wedge (y \rightsquigarrow x)$

(S2)" $1 + s(x \wedge y) = s(x) + s(x \rightsquigarrow y)$ *for all* $x, y \in X$

The following results are proved in [5, 6] under the condition that the support algebras are Rℓ-monoids. We can show them without the divisibility condition (div).

Proposition 4. *Let s be a state on a residuated lattice X. Then for any $x, y \in X$ we have,* (S4) $s(x \to y) = s(x \rightsquigarrow y)$

(S5) $s(d_1(x,y)) = s(d_2(x,y))$

(S6) $s(x^-) = 1 - s(x) = s(x^\sim)$

(S7) $s(x^{--}) = s(x^{-\sim}) = s(x^{\sim -}) = s(x^{\sim\sim}) = s(x)$

(S8) $x \le y \Rightarrow 1 + s(x) = s(y) + s(y \to x) = s(y) + s(y \rightsquigarrow x)$

(S9) $x \le y \Rightarrow s(x) \le s(y)$

(S10) $s(x \odot y) = 1 - s(x \to y^-) = 1 - s(y \rightsquigarrow x^\sim)$

(S11) $s(x) + s(y) = s(x \odot y) + s(y^- \to x) = s(y \odot x) + s(y^\sim \rightsquigarrow x)$

(S12) $s(x^- \to y^-) = 1 + s(x) - s(x \vee y) = s(y \to x) = s(x^\sim \rightsquigarrow y^\sim)$

(S13) $s(x^- \vee y^-) = 1 - s(x) - s(y) + s(x \vee y) = s(x^\sim \vee y^\sim)$

(S14) $s(x^{-\sim} \vee y^{-\sim}) = s(x^{\sim -} \vee y^{\sim -}) = s(x \vee y)$

(S15) $s(x) + s(y) = s(x \wedge y) + s(x \vee y)$

(S16) $s(d_1(x,y)) = s(d_1(x^{-\sim}, y^{-\sim}))$

(S17) $s(d_1(x^-, y^-)) = s(d_1(x^\sim, y^\sim)) = s(d_1(x,y))$

Proof. We only show the cases of (S11) and (S15), because other cases can be proved similarly as in [5, 6].

(S11) $s(x) + s(y) = s(x \odot y) + s(y^- \to x) = s(y \odot x) + s(y^\sim \rightsquigarrow x)$: It follows from $(S1)$ and $(S10)$ that $s(y^-) + s(y^- \to x) = s(x) + s(x \to y^-) = s(x) + 1 - s(x \odot y)$ and hence that $1 - s(y) + s(y^- \to x) = s(x) + 1 - s(x \odot y)$, that is, $s(x) + s(y) = s(x \odot y) + s(y^- \to x)$. Similarly we get $s(x) + s(y) = s(y \odot x) + s(y^\sim \rightsquigarrow x)$

(S15) $s(x) + s(y) = s(x \wedge y) + s(x \vee y)$: From $x \le x \vee y$, we have $1 + s(x) = s(x \vee y) + s(x \vee y \to x) = s(x \vee y) + s(y \to x)$. This implies that $1 + s(x) + s(y) = s(x \vee y) + s(y) + s(y \to x) = s(x \vee y) + 1 + s(x \wedge y)$ and hence that $s(x) + s(y) = s(x \wedge y) + s(x \vee y)$. We note that the condition can be proved without divisibility nor pre-linearity condition $(a \to b) \vee (b \to a) = \mathbf{1}$.

□

We note that especially (S15) and (S16) above are proved in several papers ([4, 5, 6]) under the condition of divisibility. But our proof says that the condition is not necessary to prove them.

It follows from the results above that the next important property of states on residuated lattices can be proved.

Lemma 1. *Let s be a state on a residuated lattice X. Then for all $x, y \in X$, we have*

(S18) $1 + s(d_1(x,y)) = s(x \to y) + s(y \to x)$ *and*
$1 + s(d_2(x,y)) = s(x \rightsquigarrow y) + s(y \rightsquigarrow x)$

(S19) $s((x \to y) \vee (y \to x)) = 1$ *and* $s((x \rightsquigarrow y) \vee (y \rightsquigarrow x)) = 1$

(S20) $s(d_1(x,y)) = s(d_1(x \to y, y \to x))$ *and* $s(d_2(x,y)) = s(d_2(x \rightsquigarrow y, y \rightsquigarrow x))$

Proof. (S19) : It follows from (S18) and (S15) that

$$\begin{aligned}1+s(d_1(x,y)) &= s(x\to y)+s(y\to x)\\ &= s((x\to y)\wedge(y\to x))+s((x\to y)\vee(y\to x))\\ &= s(d_1(x,y))+s((x\to y)\vee(y\to x))\end{aligned}$$

and thus $s((x\to y)\vee(y\to x))=1$.

□

3 Filter

We define filters of residuated lattices. Let X be a residuated lattice. A non-empty subset $F\subseteq X$ is called a *filter* of X if

(F1) If $x,y\in F$ then $x\odot y\in F$;

(F2) If $x\in F$ and $x\leq y$ then $y\in F$.

It is easy to prove that, for a non-empty subset F of X, F is a filter if and only if it satisfies the condition

(DS) If $x\in F$ and $x\to y\in F$ then $y\in F$, or equivalently,

(DS)' If $x\in F$ and $x\rightsquigarrow y\in F$ then $y\in F$.

A filter F is called *normal* when $x\to y\in F$ if and only if $x\rightsquigarrow y\in F$ for all $x,y\in X$.

For every normal filter F, we define a relation $\equiv_F$ on X as follows:

$$x\equiv_F y \Longleftrightarrow x\to y, y\to x\in F \text{ or equivalently } x\rightsquigarrow y, y\rightsquigarrow x\in F$$

We see that if F is a normal filter then $\equiv_F$ is a congruence. In this case, we consider a quotient structure $X/F=\{x/F\,|\,x\in X\}$ by the congruence $\equiv_F$ and we consistently define operations on it, for $x/F,y/F\in X/F$

$$\begin{aligned}x/F\wedge y/F &= (x\wedge y)/F\\ x/F\vee y/F &= (x\vee y)/F\\ x/F\to y/F &= (x\to y)/F\\ x/F\rightsquigarrow y/F &= (x\rightsquigarrow y)/F\\ x/F\odot y/F &= (x\odot y)/F\\ \mathbf{0} &= 0/F\\ \mathbf{1} &= 1/F.\end{aligned}$$

Since the class of all residuated lattices is a variety, we see that the quotient structure $X/F=(X/F,\wedge,\vee,\odot,\to,\rightsquigarrow,\mathbf{0},\mathbf{1})$ is also a residuated lattice.

A proper filter P(i.e., $P\neq X$) is called *prime* if it satisfies $x\in P$ or $y\in P$ provided $x\vee y\in P$ for all $x,y\in X$. A filter H is called *maximal* if there is no proper filter containing H properly. It is easy to prove that, for a filter F, F is a maximal filter if and only if there exists $n\geq 1$ such that $(x^n)^-\in F$ for $x\notin F$ if and only if there exists $n\geq 1$ such that $(x^n)^\sim\in F$ for $x\notin F$.

Lemma 2. *If H is a normal maximal filter of a residuated lattice X, then it is also a prime filter.*

Proof. Let H be a normal maximal filter of a residuated lattice X. If there are some $a, b \in X$ such that $a \vee b \in H$ but $a \notin H$ and $b \notin H$, then we have $(a^n)^-, (b^n)^- \in H$ for some $n \geq 1$ and thus $(a^n)^- \wedge (b^n)^- = (a^n \vee b^n)^- \in H$. On the other hand, since $a \vee b \in H$, we also have $H \ni (a \vee b)^{2n} \leq a^n \vee b^n$ and thus $a^n \vee b^n \in H$. But this is a contradiction. This means that if H is a normal maximal filter then it is a prime filter. □

For a state s on X, we define

$$\ker(s) = \{x \in X \mid s(x) = 1\},$$

the *kernel* of s. Since $\ker(s)$ is a proper normal filter of X, we can consider the quotient residuated lattice $X/\ker(s)$.

Lemma 3. *If X is a residuated lattice and s is a state on X, then the following conditions are equivalent:*

(i) $x/\ker(s) = y/\ker(s)$

(ii) $s(x) = s(y) = s(x \wedge y)$

(iii) $s(x \wedge y) = s(x \vee y)$

Proof. (i) $\Rightarrow$ (ii):Suppose $x/\ker(s) = y/\ker(s)$. We have $x \rightarrow y, y \rightarrow x \in \ker(s)$ and thus $s(x \rightarrow y) = s(y \rightarrow x) = 1$. Since $s(x) + s(x \rightarrow y) = s(y) + s(y \rightarrow x) = 1 + s(x \wedge y)$, we get $s(x) = s(y) = s(x \wedge y)$.

(ii) $\Rightarrow$ (iii):We assume that $s(x) = s(y) = s(x \wedge y)$. Since $s(x) + s(y) = s(x \wedge y) + s(x \vee y)$, we have $s(x) = s(x \vee y)$ and thus $s(x \wedge y) = s(x \vee y)$.

(iii) $\Rightarrow$ (i):Assume $s(x \wedge y) = s(x \vee y)$. Since $x \wedge y \leq x, y \leq x \vee y$, it follows from assumption that $s(x \wedge y) = s(x) = s(y) = s(x \vee y)$. The fact that $s(x) + s(x \rightarrow y) = 1 + s(x \wedge y) = s(y) + s(y \rightarrow x)$ yields to $s(x \rightarrow y) = 1 = s(y \rightarrow x)$. This means that $x \rightarrow y, y \rightarrow x \in \ker(s)$ and thus $x/\ker(s) = y/\ker(s)$. □

We note that $s(x) = s(y)$ if and only if $x/\ker(s) = y/\ker(s)$ for all x and y with $x \leq y$.

Lemma 4. *If s is a state on X, then $s(x \wedge y) = s(x \odot (x \rightsquigarrow y)) = s((x \rightarrow y) \odot x)$.*

Proof. Since $s(x \rightarrow y) + s(x) = s((x \rightarrow y) \odot x) + s(x^- \rightarrow (x \rightarrow y)) = s((x \rightarrow y) \odot x) + s(1) = s((x \rightarrow y) \odot x) + 1$ by (S11), it follows from (S1)" that we have $s((x \rightarrow y) \odot x) = s(x \rightarrow y) + s(x) - 1 = s(x) + s(x \rightarrow y) - 1 = s(x \wedge y) + 1 - 1 = s(x \wedge y)$. The other case $s(x \odot (x \rightsquigarrow y)) = s(x \wedge y)$ can be proved similarly. □

If s is a state on X, we denote by $\hat{X} = \{\hat{x} := x/\ker(s) \mid x \in X\}$ the corresponding quotient residuated lattice. Let $\hat{s}$ be the map on $\hat{X}$ defined by $\hat{s}(\hat{x}) = s(x)$ $(x \in X)$.

The lemma above means that $\hat{X}$ satisfies the divisibility condition.

Theorem 1. *Let X be a residuated lattice and s be a state on X, then we have*

(i) $\hat{s}$ is a state on $\hat{X}$.

(ii) $\hat{X} = X/\ker(s)$ is an MV-algebra.

Proof. We only show the case of (ii), because (i) is proved easily (c.f. [5]).

It follows from the above and (S19) that the residuated lattice $\hat{X}$ satisfies the divisibility condition and pre-linearity. Moreover, it follows from $s(x^{-\sim}) = s(x) = s(x^{\sim -})$ and $x \leq x^{-\sim}, x^{\sim -}$ that $s(x) = s(x \wedge x^{-\sim}) = s(x \vee x^{-\sim}) = s(x^{-\sim})$. This means that $(\hat{x})^{-\sim} = (x/\ker(s))^{-\sim} = x^{-\sim}/\ker(s) = x/\ker(s) = \hat{x}$. Similarly, $(\hat{x})^{\sim -} = \hat{x}$. Thus, $\hat{X}$ is a pseudo MV-algebra. On the other hand, it was proved in [3] that for all pseudo MV-algebra A, if there exists a state t on it then the quotient set $A/\ker(t)$ is an MV-algebra. Since $\hat{s}$ is a state on $\hat{X}$ and $\ker(\hat{s}) = \{1/\ker(s)\}$, it follows from $(X/\ker(s))/\ker(\hat{s}) \cong X/\ker(s)$ that $X/\ker(s)$ is an MV-algebra. □

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