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Galois Connections arising in Clone Theory

Hajime Machida † and Jovanka Pantović ‡
† International Christian University, Tokyo, Japan
‡ University of Novi Sad, Serbia

Abstract

Galois connections appear in various areas in mathematics and computer science. In this article a brief review is presented on Galois connections arising in clone theory.

Keywords: Galois connection; clone; co-clone; hyperclone

1 Basic Notions

For a set S let $\mathcal{P}(S)$ denote the power set of S. For non-empty sets A and B let φ and ψ be mappings from $\mathcal{P}(A)$ to $\mathcal{P}(B)$ and $\mathcal{P}(B)$ to $\mathcal{P}(A)$, respectively:

$$\varphi : \mathcal{P}(A) \longrightarrow \mathcal{P}(B), \qquad \psi : \mathcal{P}(B) \longrightarrow \mathcal{P}(A)$$

A pair (φ, ψ) of mappings is a *Galois connection* between A and B if φ and ψ satisfy the following conditions for any $X_1, X_2, X \in \mathcal{P}(A)$ and $Y_1, Y_2, Y \in \mathcal{P}(B)$.

(1) $X_1 \subseteq X_2 \implies \varphi(X_1) \supseteq \varphi(X_2)$
$Y_1 \subseteq Y_2 \implies \psi(Y_1) \supseteq \psi(Y_2)$

(2) $X \subseteq (\psi \circ \varphi)(X), \qquad Y \subseteq (\varphi \circ \psi)(Y)$

(3) $(\varphi \circ \psi \circ \varphi)(X) = \varphi(X), \qquad (\psi \circ \varphi \circ \psi)(Y) = \psi(Y)$

Here, notice that these three conditions are not independent in a sense that Condition (3) follows from Conditions (1) and (2). In fact, the first inclusion in Condition (2) and the first implication in Condition (1), by letting $X_1 = X$ and $X_2 = (\psi \circ \varphi)(X)$, gives us $\varphi(X) \supseteq (\varphi \circ \psi \circ \varphi)(X)$ and the second inclusion in Condition (2), applied to $Y = \varphi(X)$, yields $\varphi(X) \supseteq (\varphi \circ \psi \circ \varphi)(X)$, resulting in the equality $(\varphi \circ \psi \circ \varphi)(X) = \varphi(X)$. The second equality in Condition (3) is obtained analogously.

Note: For a non-empty set A let η be a mapping from $\mathcal{P}(A)$ into itself, i.e., $\eta : \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$. The mapping η is a *closure operator* on A if it satisfies the following three conditions for any X_1, X_2, $X \in \mathcal{P}(A)$.

(1) $X_1 \subseteq X_2 \implies \eta(X_1) \subseteq \eta(X_2)$

(2) $X \subseteq \eta(X)$

(3) $\eta\,(\eta(X)) = \eta(X)$

It is well-known that a Galois connection induces closure operators. Namely, for a Galois connection (φ, ψ) between A and B, the compositions

$$\psi \circ \varphi : \mathcal{P}(A) \longrightarrow \mathcal{P}(A) \qquad \text{and} \qquad \varphi \circ \psi : \mathcal{P}(B) \longrightarrow \mathcal{P}(B)$$

are closure operators on A and B, respectively.

For a Galois connection (φ, ψ) between A and B, a subset $X \in \mathcal{P}(A)$, or $Y \in \mathcal{P}(B)$, is a *Galois closed set* if it satisfies $X = (\psi \circ \varphi)(X)$, or $Y = (\varphi \circ \psi)(Y)$, respectively.

For a non-empty set A and $n > 0$, the set of n-variable functions on A, i.e., maps from A^n into A, is denoted by $\mathcal{O}_A^{(n)}$. The set of all functions on A is denoted by $\mathcal{O}_A$, i.e., $\mathcal{O}_A = \bigcup_{n=1}^{\infty} \mathcal{O}_A^{(n)}$. For $1 \leq i \leq n$ the *projection* e_i^n on A is a function in $\mathcal{O}_A^{(n)}$ which always takes the value of the i-th variable. Denote by $\mathcal{J}_A$ the set of projections on A. A subset C of $\mathcal{O}_A$ is a *clone* on A if C contains $\mathcal{J}_A$ and is closed under (functional) composition.

For a non-empty set A and $m > 0$, A^m $(= A \times \cdots \times A)$ is the direct product of m copies of A. A subset ρ of A^m is called an m-ary *relation* on A, i.e., ρ is a relation on A if $\rho \subseteq A^m$ for some $m > 0$. Let $\mathcal{R}_A^{(m)}$ denote the set of all m-ary relations on A and $\mathcal{R}_A$ denote the set of all finitary relations on A, i.e., $\mathcal{R}_A = \bigcup_{m=1}^{\infty} \mathcal{R}_A^{(m)}$.

2 Galois Connections

2.1 Clones and Relations

For a function $f \in \mathcal{O}_A^{(n)}$ and a relation $\rho \in \mathcal{R}_A^{(m)}$, we say that f *preserves* ρ, or ρ is an *invariant relation* of f, if

$$\begin{pmatrix} f(a_{11}, a_{12}, \ldots, a_{1n}) \\ f(a_{21}, a_{22}, \ldots, a_{2n}) \\ \cdots \\ f(a_{m1}, a_{m2}, \ldots, a_{mn}) \end{pmatrix}$$

belongs to ρ whenever ${}^t(a_{11}\ a_{21}\ \cdots\ a_{m1})$, ${}^t(a_{12}\ a_{22}\ \cdots\ a_{m2})$, $\ldots$, ${}^t(a_{1n}\ a_{2n}\ \cdots\ a_{mn})$ all belong to ρ. For $\rho \in \mathcal{R}_A$, the set of functions in $\mathcal{O}_A$ which preserve ρ is called the *polymorph* of ρ and denoted by $\operatorname{Pol}\rho$.

Define mappings

$$\operatorname{Pol} : \mathcal{P}(\mathcal{R}_A) \longrightarrow \mathcal{P}(\mathcal{O}_A) \qquad \text{and} \qquad \operatorname{Inv} : \mathcal{P}(\mathcal{O}_A) \longrightarrow \mathcal{P}(\mathcal{R}_A)$$

by

$$\operatorname{Pol}(R) = \{ f \in \mathcal{O}_A \mid (\forall \rho \in R)\ f \text{ preserves } \rho \}$$

and

$$\operatorname{Inv}(F) = \{ \rho \in \mathcal{R}_A \mid (\forall f \in F)\ f \text{ preserves } \rho \}$$

for all $R \in \mathcal{P}(\mathcal{R}_A)$ and $F \in \mathcal{P}(\mathcal{O}_A)$. To rephrase, $\operatorname{Pol}(R) = \bigcap_{\rho \in R} \operatorname{Pol}\rho$.

Clearly, the pair (Pol, Inv) is a Galois connection between $\mathcal{R}_A$ and $\mathcal{O}_A$. This is the best known, and most typical, Galois connection in clone theory.

We define the following operations on $\mathcal{R}_A$. (Here, by operations we mean set-theoretical operations.) The operations ζ, τ and pr are unary operations and the operations $\cap$ and $\times$ are binary operations.

(1) For $\rho \in \mathcal{R}_A^{(1)}$ and $\rho = \emptyset$, $\zeta\rho = \tau\rho = \rho$ and $pr\rho = \emptyset$.
For $\rho \in \mathcal{R}_A^{(m)}$ where $m \geq 2$,

$$\begin{aligned} \zeta\rho &= \{(a_1, a_2, \ldots, a_m) \in A^m \mid (a_m, a_1, \ldots, a_{m-1}) \in \rho\}, \\ \tau\rho &= \{(a_1, a_2, a_3, \ldots, a_m) \in A^m \mid (a_2, a_1, a_3, \ldots, a_m) \in \rho\}, \\ pr\rho &= \{(a_2, \ldots, a_m) \in A^{m-1} \mid (\exists a_1 \in A)(a_1, a_2, \ldots, a_m) \in \rho\}. \end{aligned}$$

(2) For $\rho_1, \rho_2 \in \mathcal{R}_A^{(m)}$ where $m > 0$,

$$\rho_1 \cap \rho_2 = \{(a_1, \ldots, a_m) \in A^m \mid (a_1, \ldots, a_m) \in \rho_1 \text{ and } (a_1, \ldots, a_m) \in \rho_2\}$$

(3) For $\rho_1 \in \mathcal{R}_A^{(m)}$ and $\rho_2 \in \mathcal{R}_A^{(m')}$ where m, $m' > 0$,

$$\begin{aligned} \rho_1 \times \rho_2 = \{&(a_1, \ldots, a_m, b_1, \ldots, b_{m'}) \in A^{m+m'} \mid (a_1, \ldots, a_m) \in \rho_1 \text{ and} \\ &(b_1, \ldots, b'_m) \in \rho_2\}. \end{aligned}$$

Moreover, we define the diagonal relation Δ_A of arity 2 by

$$\Delta_A = \{ (a, a) \mid a \in A \}.$$

A subset R of $\mathcal{R}_A$ is a *co-clone* (or, *relational clone*) on A if R contains Δ_A and is closed under all of the operations ζ, τ, pr, $\cap$ and $\times$.

It is easy to see that $\text{Pol}(R)$ is a clone for any R in $\mathcal{P}(\mathcal{R}_A)$ and $\text{Inv}(F)$ is a co-clone for any F in $\mathcal{P}(\mathcal{O}_A)$. The following remarkable result was established independently by several authors (e.g., [1]).

Theorem 2.1 *Let A be a finite set with $|A| > 1$.*

(1) *For any $R \in \mathcal{P}(\mathcal{R}_A)$, if R is a co-clone then* $\text{Inv}(\text{Pol}(R)) = R$.

(2) *For any $F \in \mathcal{P}(\mathcal{O}_A)$, if F is a clone then* $\text{Pol}(\text{Inv}(F)) = F$.

In other words, clones and co-clones are Galois closed sets of the Galois connection (Pol, Inv).

2.2 Centralizers and Monoids

For functions $f \in \mathcal{O}_A^{(n)}$ and $g \in \mathcal{O}_A^{(m)}$ we say that f *commutes* with g, or f and g *commute*, if the following holds for every $m \times n$ matrix M over A with rows $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_m$ and columns $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n$.

$$f(g({}^t\boldsymbol{c}_1), \ldots, g({}^t\boldsymbol{c}_n)) = g(f(\boldsymbol{r}_1), \ldots, f(\boldsymbol{r}_m))$$

We write $f \perp g$ when f commutes with g. The relation $\perp$ is a symmetric relation on $\mathcal{O}_A$.

As a special case, let $m = 1$ and $n \geq 1$. Then, for $f \in \mathcal{O}_k^{(n)}$ and $g \in \mathcal{O}_k^{(1)}$, f commutes with g if

$$f(g(x_1), \ldots, g(x_n)) = g(f(x_1, \ldots, x_n))$$

holds for all $(x_1, \ldots, x_n) \in A^n$. Let $\mathcal{A} = (A; F)$ be an algebra. By definition, $g \in \mathcal{O}_A^{(1)}$ is an *endomorphism* of $\mathcal{A}$ if and only if $f \perp g$ holds for every $f \in F$. Denote by $\text{End}\,(\mathcal{A})$ the set of endomorphisms of $\mathcal{A}$, i.e., $\mathrm{End}(\mathcal{A}) = \{g \in \mathcal{O}_A^{(1)} \mid f \perp g \text{ for } \forall f \in F\}$.

For $F \subseteq \mathcal{O}_A$ the *centralizer* F^* of F is defined by

$$F^* = \{ g \in \mathcal{O}_A \mid g \perp f \text{ for all } f \in F \}.$$

For any subset $F \subseteq \mathcal{O}_A$ the centralizer F^* is easily verified to be a clone. When $F = \{f\}$ we often write f^* instead of F^*. We also write F^{**} for $(F^*)^*$.

Two types of Galois connections can be defined with respect to the centralizers. First, let φ and ψ be the same mapping $\varphi\ (= \psi) : \mathcal{P}(\mathcal{O}_A) \longrightarrow \mathcal{P}(\mathcal{O}_A)$ defined by

$$\varphi(F)\ (= \psi(F)) = F^*$$

for all $F \in \mathcal{P}(\mathcal{O}_A)$. Then, clearly, the pair (φ, ψ) is a Galois connection between $\mathcal{O}_A$ and itself. Hence the map $F \longmapsto F^{**}$ is a closure operator on $\mathcal{O}_A$.

The second type of a Galois connection relates the centralizers to the monoids. As is well-known, a non-empty subset M of $\mathcal{O}_A^{(1)}$ is a (transformation) *monoid* on A if it is closed under composition and contains the identity *id*. The whole set $\mathcal{O}_A^{(1)}$ is the largest monoid on A and the singleton $\{id\}$ is the smallest monoid on A. Denote by $\mathcal{M}_A$ the set of monoids on A. For a clone C on A the unary part $C^{(1)}$ of C, i.e., $C^{(1)} = C \cap \mathcal{O}_A^{(1)}$, is a monoid. In particular, for any centralizer F^* the unary part of F^*, i.e., $F^* \cap \mathcal{O}_A^{(1)}$, is a monoid.

Let us define the mappings φ and ψ

$$\varphi : \mathcal{P}(\mathcal{O}_A^{(1)}) \longrightarrow \mathcal{P}(\mathcal{O}_A) \quad \text{and} \quad \psi : \mathcal{P}(\mathcal{O}_A) \longrightarrow \mathcal{P}(\mathcal{O}_A^{(1)})$$

by

$$\varphi(M) = M^* \qquad \text{for all } M \in \mathcal{P}(\mathcal{O}_A^{(1)})$$

and

$$\psi(F) = F^* \cap \mathcal{O}_A^{(1)} \qquad \text{for all } F \in \mathcal{P}(\mathcal{O}_A).$$

Then the pair (φ, ψ) of mappings is a Galois connection between $\mathcal{O}_A^{(1)}$ and $\mathcal{O}_A$. Moreover, notice that $\varphi(M)$ is always a clone on A and $\psi(F)$ is always a monoid on A.

For $M \subseteq \mathcal{O}_A^{(1)}$, M is a *centralizing monoid* if M satisfies the equation

$$M = M^{**} \cap \mathcal{O}_A^{(1)}.$$

In other words, a monoid M on A is a centralizing monoid if M satisfies $(\psi \circ \varphi)(M) = M$, that is, a centralizing monoid M is a Galois closed set of a Galois connection (φ, ψ).

Lemma 2.2 *For $M \subseteq \mathcal{O}_A^{(1)}$ the following conditions are equivalent.*

(1) *M is a centralizing monoid.*

(2) *For some subset $F \subseteq \mathcal{O}_A$, $M = F^* \cap \mathcal{O}_A^{(1)}$*

(3) *For some algebra $\mathcal{A} = (A;\ F)$, $M = \mathrm{End}(\mathcal{A})$*

Note that Lemma 2.2 (2) asserts that a centralizing monoid is the unary part of some centralizer.

Concerning the images of monoids under φ, we have the following theorem ([10]). Let A be a finite set with $|A| > 2$. For a monoid M on A, define properties I and II in the following way:

I (Partial separation property)
For all $a, b, c, d \in A$, if $\{a, b\} \neq \{c, d\}$ and $c \neq d$ then M contains $f\ (= f_{cd}^{ab})$ which satisfies

$$f(a) = f(b) \quad \text{and} \quad f(c) \neq f(d).$$

II (Fixed-point-free property)
For every $i \in A$, M contains g_i which satisfies $g_i(i) \neq i$.

Then we obtain a sufficient condition for $\varphi(M)$ to be the least clone.

Theorem 2.3 ([10]) *For a monoid M on A, if M satisfies Properties I and II then $\varphi(M)\ (= M^*)$ is $\mathcal{J}_A$, i.e., the clone of projections.*

2.3 Hyperclones and Relations

For a set A let $\mathcal{P}_A^*$ denote the set of non-empty subsets of A, i.e., $\mathcal{P}_A^* = \mathcal{P}(A) \setminus \{\emptyset\}$. An n-ary *hyperoperation* f on A is a mapping from A^n to P_A^*. For $n \geq 1$ let $\mathcal{H}_A^{(n)}$ be the set of n-ary hyperoperations on A, and $\mathcal{H}_A$ be the set of all hyperoperations on A, i.e., $\mathcal{H}_A = \bigcup_{n\geq 1} \mathcal{H}_A^{(n)}$. For $1 \leq i \leq n$, an i-th n-ary *(hyper-) projection* $\widehat{e}_i^n$ on A is the n-ary hyperoperation defined by $\widehat{e}_i^n(x_1, \ldots, x_i, \ldots, x_n) = \{x_i\}$ for all $(x_1, \ldots, x_n) \in A^n$. For $f \in \mathcal{H}_A^{(n)}$ and $g_1, \ldots, g_n \in \mathcal{H}_A^{(m)}$ where $m, n > 0$, the composition $f(g_1, \ldots, g_n)$ of f and $g_1, \ldots, g_n$ is defined in a natural way by

$$f(g_1, \ldots, g_n)(x_1, \ldots, x_m) \ = \ \bigcup \{\, f(y_1, \ldots, y_n) \mid y_i \in g_i(x_1, \ldots, x_m) \text{ for } 1 \leq \forall i \leq n \,\}.$$

A *hyperclone* on A is a set of hyperoperations on A which is closed under composition and contains all (hyper-) projections.

Galois connections between $\mathcal{H}_A$ and $\mathcal{R}_A$ have been studied in three different ways. Here we denote them by (*dPol*, *dInv*), (*mPol*, *mInv*) and (*hPol*, *hInv*). The first one, (*dPol*, *dInv*), is independently due to F. Börner ([3]) and B. A. Romov ([16]), the second one, (*mPol*, *mInv*), due to T. Drescher and R. Pöschel ([8]) and the third one, (*hPol*, *hInv*), due to I. G. Rosenberg ([17]) and H. Machida and J. Pantović ([9]).

Let $m > 0$ be an integer. Recall that $\mathcal{R}_A^{(m)}$ denotes the set of all m-ary relations on A. Let us denote by $\mathcal{P}^*(\mathcal{R}_A^{(m)})$ the set of non-empty subsets of $\mathcal{R}_A^{(m)}$, i.e., $\mathcal{P}^*(\mathcal{R}_A^{(m)}) = \mathcal{P}^*_{\mathcal{R}_A^{(m)}}$ in the above notation.

Let $\rho \in \mathcal{R}_A^{(m)}$ be an $m\,(> 0)$-ary relation on A. We define ρ_d, ρ_m and ρ_h in $\mathcal{P}^*(\mathcal{R}_A^{(m)})$ as follows:

$$\begin{aligned}
\rho_d &= \{\, (A_1, \ldots, A_m) \mid \ A_1 \times \cdots \times A_m \subseteq \rho \,\} \\
\rho_m &= \{\, (A_1, \ldots, A_m) \mid \ \forall \ell \in \{1, \ldots, m\} \ \forall a \in A_\ell \\
&\qquad\qquad (A_1 \times \cdots \times A_{\ell-1} \times \{a\} \times A_{\ell+1} \times \cdots \times A_m) \cap \rho \neq \emptyset \,\} \\
\rho_h &= \{\, (A_1 \ldots, A_m) \mid \ (A_1 \times \cdots \times A_m) \cap \rho \neq \emptyset \,\}
\end{aligned}$$

Let x be either of d, m or h. For a hyperoperation $f \in H_A^{(n)}$ and a relation $\rho \in \mathcal{R}_A^{(m)}$, f is said to *x-preserve* ρ if

$$\begin{pmatrix} f(a_{11}, a_{12}, \ldots, a_{1n}) \\ f(a_{21}, a_{22}, \ldots, a_{2n}) \\ \cdots \\ f(a_{m1}, a_{m2}, \ldots, a_{mn}) \end{pmatrix}$$

belongs to ρ_x whenever ${}^t(a_{11}\, a_{21}\, \ldots\, a_{m1})$, ${}^t(a_{12}\, a_{22}\, \ldots\, a_{m2})$, $\ldots$, ${}^t(a_{1n}\, a_{2n}\, \ldots\, a_{mn})$ all belong to ρ. Let $x Pol\, \rho$ denote the set of all hyperoperations on A that x-preserve ρ.

Define mappings

$$x\mathrm{Pol} : \mathcal{P}(\mathcal{R}_A) \longrightarrow \mathcal{P}(\mathcal{H}_A) \qquad \text{and} \qquad x\mathrm{Inv} : \mathcal{P}(\mathcal{H}_A) \longrightarrow \mathcal{P}(\mathcal{R}_A)$$

by

$$x\mathrm{Pol}(R) \;=\; \{\, f \in \mathcal{H}_A \mid (\forall\, \rho \in R)\; f\; x\text{−preserves } \rho \,\}$$

and

$$x\mathrm{Inv}(F) \;=\; \{\, \rho \in \mathcal{R}_A \mid (\forall\, f \in F)\; f\; x\text{−preserves } \rho \,\}$$

for all $R \in \mathcal{P}(\mathcal{R}_A)$ and $F \in \mathcal{P}(\mathcal{H}_A)$. Equivalently, $x\mathrm{Pol}(R) = \bigcap_{\rho \in R} x\mathrm{Pol}\, \rho$.

It is easy to see that, for each x in $\{d, m, h\}$, the pair $(x\mathrm{Pol}, x\mathrm{Inv})$ is a Galois connection between $\mathcal{R}_A$ and $\mathcal{H}_A$. However, it should be remarked that for any x in $\{d, m, h\}$, the invariant set $x Inv\, F$ is, in general, not a co-clone on A ([6]).

Concerning the inclusion relations among $dPol\, \rho$, $mPol\, \rho$ and $hPol\, \rho$ we have the following results ([6]). The first part is an immediate consequence of the inclusion $\rho_d \subseteq \rho_m \subseteq \rho_h$.

Theorem 2.4 (a) *For any relation $\rho \in \mathcal{R}_A$, it holds that $dPol\rho \subseteq mPol\rho \subseteq hPol\rho$.*

(b) *There exists $\rho_1, \rho_2 \in \mathcal{R}_A$ which satisfy $dPol\rho_1 \subset mPol\, \rho_1$ and $mPol\, \rho_2 \subset hPol\rho_2$.*

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