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Triple systems and structurable algebras *

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Abstract

We will show that we can always construct a $(-\varepsilon, \delta)$ Freudenthal-Kantor triple system from a given (ε, δ) Freudenthal-Kantor triple system. Moreover, we will prove that we can construct a structurable algebra from any $(-1,1)$ Freudenthal-Kantor triple system with a distinguished element e satisfying $eex = x$ for the triple product.

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1. Introduction and summary of Main Results

Let V be a vector space over a field F with a triple product denoted by jaxtaposition, $xyz \in V$ for $x, y, z \in V$. Then, the triple systems (V, xyz) is called a (ε, δ) Freudenthal-Kantor triple system ([17]) (which is hereafter abreviated as (ε, δ)FKTS), provided that

$$K, L; V \otimes V \to \text{End } V \tag{1.1}$$

defined by

$$K(x, y)z = xzy - \delta yzx, \tag{1.2}$$

$$L(x, y)z = xyz \tag{1.3}$$

satisfy

$$[L(u, v), L(x, y)] = L(uvx, y) + \varepsilon L(x, vuy) \tag{1.4}$$

and

$$K(K(u, v)x, y) = L(y, x)K(u, v) - \varepsilon K(u, v)L(x, y). \tag{1.5}$$

Here, ε and δ are constants with values 1 or -1. A ternary system (V, xy, xyz) is called structurable, ([2]), where V is a vector space with both binary product xy and ternary product xyz, satisfying the following conditions;

*This paper is a survey note and the detail is described in other article. And this is talked by a workshop in RIMS (Kyoto University) 2013,Feb.

1. The triple system (V, xyz) is a $(-1,1)$FKTS.
2. The binary algebra (V, xy) is a unital involutive algebra with involution map $: x \longrightarrow \bar{x}$, i.e.,

$$\bar{\bar{x}} = x \quad \text{and} \quad \overline{xy} = \bar{y}\bar{x}.$$

3. The triple product xyz is expressed as

$$xyz = (z\bar{y})x - (z\bar{x})y + (x\bar{y})z \tag{1.6}$$

in terms of the binary products.

We may then call the binary system (V, xy) to be a structurable (or Allison) algebra. It posseses many interesting properties. First, it satisfies the triality relation ([3],[16]). Second, its assoicated Lie algebra is a BC_1-graded Lie algebra of type B_1([5]) and is invariant under the symmetric group S_4 of degree 4, as are noted in ([8] and [9]). We may also note that structurable algebras can be formulated purely as binary algebras without any reference on the triple product ([3] and [16]). In ([14]), we have alternatively shown that it can be derived from any $(-1,1)$FKTS satisfying some additional conditions as in the following theorem:

Theorem 1.1

Let (V, xyz) be a $(-1,1)$FKTS over the field F of characteristic $\neq 2, \neq 3$, possessing a distinguished element $e \in V$ satisfying

$$eex = x \tag{1.7}$$

and

$$exe + 2xee = 3eex \tag{1.8}$$

for any $x \in V$. We then introduce a linear mapping $x \longrightarrow \bar{x}$ and a binary product xy in V by

$$\bar{x} = 2x - xee \tag{1.9}$$

$$xy = yex - \bar{x}\bar{y}e + \bar{x}ey. \tag{1.10}$$

Then, the ternary system (V, xy, xyz) is structurable, so that (V, xy) is a sturucturable algebra with the unit element e and the involution map $x \longrightarrow \bar{x}$. Conversely, for any structurable algebra (V, xy) with the unit element e, the ternary system (V, xy, xyz) is structurable, satisfying Eq.(1.7) and (1.8), if we define the triple product by Eq.(1.6).

One of main results of this paper is that we can essentially dispense with the assumption of Eq.(1.8) as in the following theorem: to be proved in other paper

Theorem 1.2([18])

Let (V, xyz) be a $(-1,1)$FKTS over the field of characteristic $\neq 2, \neq 3$, possessing a distinguished element $e \in V$ satisfying $eex = x$. Then there exists $\sigma \in$ End V satisfying

$$(1)\ \sigma^2 = 1, \ \ \sigma e = e \tag{1.11a}$$

$$(2)\ \sigma(xyz) = (\sigma x)(\sigma y)(\sigma z). \tag{1.11b}$$

Moreover, if we introduce a new triple product $\{x, y, z\}$ *in* V *by*

$$\{x, y, z\} = x(\sigma y)z, \tag{1.12}$$

then the triple system $(V, \{x, y, z\})$ *is also a* $(-1, 1)$*FKTS, satisfying*

$$\{e, e, x\} = x, \tag{1.13a}$$

$$\{e, x, e\} + 2\{x, e, e\} = 3\{e, e, x\}. \tag{1.13b}$$

Especially, if we set

$$\bar{x} = 2x - \{x, e, e\}, \tag{1.14a}$$

$$x \cdot y = \{y, e, x\} - \{\bar{x}, \bar{y}, e\} + \{\bar{x}, e, y\}, \tag{1.14b}$$

then the ternary system $(V, x \cdot y, \{x, y, z\})$ *is structurable,*
where denoted by $Rx = xee$, $Mx = exe$, $Q = 3Id - 2R$, *and* $\sigma = MQ^{-1}$.
The explicit formula for σ *will be given in ([18]).*

A similar situation exists for $(-1, -1)$FKTS.

Theorem 1.3
Let (V, xyz) *be a* $(-1, -1)$*FKTS over the field* F *of characteristic not* 2*, possessing a distinguished element* $e \in V$ *satisfying* $eex = x$ *for any* $x \in V$*. Then, there exists* $\sigma \in \mathrm{End}\ V$ *satisfying Eqs.(1.11). Moreover, introducing the new triple product* $\{x, y, z\}$ *by Eq.(1.12), the new triple system* $(V, \{x, y, z\})$ *is also a* $(-1, -1)$*FKTS, satisfying now*

$$\{e, e, x\} = \{e, x, e\} = x. \tag{1.15}$$

Especially, $(V, \{x, y, z\})$ *is unitary* $(-1, -1)$*FKTS. (The definition of the unitary FKTS be given by* $Id \in \{K(x, y)\}_{span}$*.).*

At this point, it may be worthwhile to note that any $(1, \delta)$FKTS (i.e., $\varepsilon = +1$) cannot satisfy $eex = x$, since it contradicts Eq.(1.4) for the choice of $u = v = x = y = e$. Also, in ([18]), we will show that we can always construct a $(-\varepsilon, \delta)$FKTS from any (ε, δ)FKTS, and suggest that for $(1, 1)$FKTS the condition $eex = x$ can be replaced by $efx = -fex = x$ for a pair $e, f \in V$. We can then construct a structurable algebra from any such $(1, 1)$FKTS, and vice-versa.

Appendix (due to N.Kamiya)
Proposition A.
Under the notation of above Theorem 1.3, if we define the involution and the binary product as follows;

$$\bar{x} = \{xee\} \text{ and } x \bullet y = Ry \circ x + Rx \circ y - Ry \circ Rx,$$

where $Rx = \{xee\}$, $x \circ y = \{exy\}$*, then we have*

$$\bar{\bar{x}} = x, \ \ \overline{x \bullet y} = \bar{y} \bullet \bar{x}$$

furthermore,

$$x \circ y = 1/3\{\bar{y} \bullet (x - \bar{x}) + (2x + \bar{x}) \bullet y\}$$

Proof.
From the definition of $\bar{x}$ *and* $x \bullet y$, by straightfoward calculations, we obtain

$$\overline{x \bullet y} = R(x \circ Ry) + R(y \circ Rx) - x \circ y,$$

$$\bar{y} \bullet \bar{x} = \bar{y} \circ x + \bar{x} \circ y - R(\bar{y} \circ \bar{x}).$$

On the other hand, from

$$R(x \circ y) = x \circ Ry - Ry \circ x + Ry \circ Rx - Rx \circ y + y \circ Rx,$$

it follows that

$$R(\bar{y} \circ \bar{x}) = \bar{y} \circ x - x \circ \bar{y} + x \circ y - y \circ \bar{x} + \bar{x} \circ y.$$

$$R(x \circ \bar{y}) = x \circ y - y \circ x + y \circ \bar{x} - \bar{x} \circ \bar{y} + \bar{y} \circ \bar{x}$$

$$R(y \circ Rx) = y \circ x - x \circ y + x \circ \bar{y} - \bar{y} \circ \bar{x} + \bar{x} \circ \bar{y}.$$

Therefore, combining these relations, we get

$$\overline{x \bullet y} = \bar{y} \bullet \bar{x} \quad and \quad \bar{\bar{x}} = x.$$

From

$$x \bullet y = Ry \circ x + Rx \circ y - Ry \circ Rx$$

it follows that for

$$U_0 = \{x | Rx = -x\}, \; U_2 = \{x | Rx = x\},$$

$$if \; x \in U_0, \; y \in U_0, \; then \; x \bullet y = -2y \circ x - x \circ y,$$

$$if \; x \in U_0, \; y \in U_2, \; then \; x \bullet y = 2y \circ x - x \circ y,$$

$$if \; x \in U_2, \; then \; x \bullet y = x \circ y.$$

Thus we have the relation of the $x \bullet y$ and $x \circ y$. This completes the proof.

Corollary ([2011 Glasgow J.Math.vol 53,p727-739]) *For balanced (-1,-1)Freudenthal-Kantor triple system, if we set*

$$\bar{x} = R(x) = 2 < x, e > e - x, \; x \circ y = \{exy\}$$

then we have

$$\bar{\bar{x}} = x, \; \overline{x \circ y} = \bar{y} \circ \bar{x}$$

In particular, also we get

$$\bar{\bar{x}} = x, \; \overline{x \bullet y} = \bar{y} \bullet \bar{x}$$

that is, this algebra is a quadratic algebra.

Following([2006 Lec.Notes in Pure and App.Math. vol 246,Chapter 16, CRC]), we recall the definition of the quadratic triple system defined by

$$\{xxy\} = \{yxx\} =< x|x > y, \; < x|y >=< y|x >\in \Phi.$$

Remark. we note that by means of $x \circ y = \{xey\}$, this product has the structure of a quadratic algebra with respect to $\circ$.

Proposition B. *Let A be a nonassociative algebra with involution and unit element e such that*

$$e \bullet x = x, \ x \bullet e = x, \ \bar{\bar{x}} = x, \ \overline{x \bullet y} = \bar{y} \bullet \bar{x}.$$

If $x \bullet \bar{x} \in \Phi e$ and define

$$\{xyz\} = (x \bullet \bar{y}) \bullet z - (z \bullet \bar{y}) \bullet x + (z \bullet \bar{x}) \bullet y$$

then $\{xyz\}^{\tilde{}}$ is a quadratic triple system w.r.t. new triple product $\{xyz\}^{\tilde{}} = \{yxz\}$.

Proof.

From $\{xxy\} = \{yxx\} =< x, x > y$, it is clear, where $x \bullet \bar{x}$ is denoted by $< x|x > e$.

Remark

For the above Prop.B, the triple product$\{xyz\}$ is closedly related to an anti-structurable algebras ([2010 Bull. Aust. Math. vol.81,132-155.]).

Example (a)

Let U be a set of matrix n × n and $O(n, \Phi) = \{x \in U| \ x^t x = Id\}$, where ${}^t x$ is the transpose matrix of x. We define a triple product as follows;

$$\{xyz\} = (x^t y)z - (z^t y)x + (z^t x)y, \ x, y, z \in U$$

then this triple product has a structure of (-1,-1)Freudenthal-Kantor triple system.

Furthermore, if we set

$$\{xyz\}^{\tilde{}} = \{yxz\},$$

this triple system is a quadratic triple system. On the other hand, we have

$$\{uux\} = \{uxu\} = x, \ \{xuu\} = u^t xu, \ \ \forall u \in O(n, \Phi)$$

that is, u is a left unit element of U.

If we define a binary product by $x \circ y = uxy, \ u \in O(n, \Phi)$ and an involution by $\bar{x} = Rx = \{xuu\}$ then we have

$$x \circ y = u^t xy - y^t xu + y^t ux, \ \bar{x} = u^t xu, \ \bar{\bar{x}} = x.$$

However,

$$\overline{x \circ y} \neq \bar{y} \circ \bar{x} \text{ and } \overline{x \bullet y} = \bar{y} \bullet \bar{x}.$$

Indeed, we have

$$\{xxy\}^{\tilde{}} =< x|x > y, \ \ \{yxx\}^{\tilde{}} =< x|x > y$$

$x^t x =< x|x > e, \ e = Id_n$, e is the identity matrix element. Thus, it is clear that $\{xyz\}^{\tilde{}}$ is a quadratic triple system.

On the other hand, from the relations;

$$x \circ y = u^t xy - y^t xu + y^t ux,$$

$$x \bullet y = Ry \circ x + Rx \circ y - Ry \circ Rx,$$

$$\bar{x} = Rx = \{xuu\} = u^t xu$$

by $u^t u = Id_n$ we get

$$y \circ x = u^t yx - x^t yu + x^t uy,$$

$$Ry \circ x = y^t ux - x^t uy + x^t yu,$$
$$Rx \circ y = x^t uy - y^t ux + y^t xu,$$
$$Ry \circ Rx = y^t xu - u^t xy + y^t ux.$$

Furthermore we get the relations;

$$x \bullet y = x^t yu + u^t xy - y^t ux,$$
$$\overline{x \bullet y} = y^t xu + u^t yx - u^t xu^t yu,$$
$$y \bullet x = y^t xu + u^t yx - x^t uy,$$
$$\bar{y} \bullet \bar{x} = u^t yx + y^t xu - u^t xu^t yu.$$

Therefore we obtain

$$\overline{x \bullet y} = \bar{y} \bullet \bar{x}, \ \bar{\bar{x}} = x.$$

This case has

$$\overline{x \circ y} \neq \bar{y} \circ \bar{x}, \ \bar{\bar{x}} = x.$$

Example (b).
Let U be a set of $Mat(1, n, \Phi)$. We define the triple product as follows;

$$\{xyz\} =< x|y > z- < y|z > x+ < z|x > y, \quad < x|y >=< y|x >\in \Phi.$$

Then this product has a structure of balanced (-1,-1)Freudenthal-Kantor triple system. Thus, as same as Example (a), the triple product $\{xyz\}^{\tilde{}} = \{yxz\}$ is a quadratic triple system.

In deed, it is clear that

$$\{xxy\}^{\tilde{}} = \{xyx\}^{\tilde{}} =< x|x > y.$$

On the other hand, for $\delta = 1$, we give an example, of left unit element from the structurable algebra as follows.

Let $x, y, z \in Mat(n, n; \Phi)$,

$$\{xyz\} = x^t yz + z^t yx - z^t xy,$$

where ${}^t x$ is the transpose matrix of x, and chose $e \in O(n, \Phi)$.

Then we have

$$\{eex\} = x, \quad e^t e = Id_n$$

We now define

$$\bar{x} := e^t xe \ \ and\ x \cdot y := \{\bar{x}ey\} - \{\bar{x}\sigma(\bar{y})e\} + \{yex\}$$

where $\sigma := M(5 - 2R)/3, \ Mx := \{exe\}, \ Rx := \{xee\}$.

Then we have

$Mx = 2e^t xe - x, \ Rx = 2x - e^t xe, \ (5 - 2R)x = 2x + e^t xe, \ M(5 - 2R)x = 3x$, and so $\sigma x = x$.

From these relations, it follows that

$$\{\bar{x}ey\} = \bar{x}^t ey + y^t e\bar{x} - y^t \bar{x}e$$

$$\{\bar{x}\sigma(\bar{y})e\} = \bar{x}^t\bar{y}e + e^t\bar{y}\bar{x} - e^t\bar{x}\bar{y}$$
$$\{yex\} = y^tex + x^tey - x^tye.$$

By means of $\bar{x} = e^txe$, from straightfoward calculations, we get

$$x \cdot y = x^tey \quad (as\ usual\ product)$$
$$\overline{x \cdot y} = e^tye^txe$$
$$\bar{y} \cdot \bar{x} = e^tye^tee^txe = e^tye^txe \quad (by\ e^tee = e\ ).$$

Hence we obtain

$$\overline{x \cdot y} = \bar{y} \cdot \bar{x} \ and\ \bar{\bar{x}} = x.$$

Coccluding Remark

For left unit $(-1, \delta)$ Freudenthal-Kantor triple systems, it seems that there is a relationship as follows;

$$\delta = 1 \quad ---- \ structurable\ algebra,$$
$$\delta = -1 \quad ------ \ weakly\ quadratic\ algebra.$$

In our subjects;

N.Jacobson, H.Freudenthal, I.L.Kantor, E.Zelmanov, S.Okubo, are important persons, historically in my opinion.

· In view point of generalization of numbers;

$$\mathbf{R} -- > \mathbf{C} -- > \mathbf{H} -- > \mathbf{O} -- > \mathbf{H}_3(\mathbf{O})$$
$$-- > M(H_3(\mathbf{O})) -- > E_8 \quad (Lie\ algebra\ of\ dim\ 248)$$
$$\mathbf{O} -- > Im\mathbf{O} -- > G_2, \ \mathbf{H} \otimes \mathbf{O} -- > E_7, \mathbf{O} \oplus \mathbf{O} -- > F(4).$$

· In view point of construction of Lie algebras and superalgebras:

By means of triple systems, we can construct to all simple algebras and superalgebras without concept of root systems, and also the tangent spaces of some symmetric space or bounded domain correspond to certain triple systems.

(i.e., symmetric space $\leftrightarrow$ Lie triple system etc.).

· In view point of physics application:

(Clasical)Yang-Baxter equations, Dynamical Yang-Baxter equations, M theory(hermitian triple systems), String theory.

The example of hermitian triple system is as follows;

$$\{xyz\} = x^t\bar{y}z - z^t\bar{y}x + z^tx\bar{y}, \quad x, y, z \in Mat(m, n; C)$$

where $^-$ is the conjugation of $\mathbf{C}$.

This triple system is a (-1,-1)-Freudenthal-Kantor triple system with

$$\{\alpha x, \beta y, \gamma z\} = \alpha\overline{\beta}\gamma\{xyz\}, \quad for\ all\ \ \alpha,\ \beta,\ \gamma \in \mathbf{C}$$

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