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Computing as Dynamics of Information: Classification of Geometric Dynamical Information Systems Based on Properties of Closure Spaces

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Abstract. *Going beyond Turing machines in the theory of computation requires some sufficiently general conceptual framework opening a wide perspective on the possible generalizations. Dynamics of information can be such a framework when information is understood as identification of a variety, and with the formalism for information in terms of closure spaces. In this context, there is an analogy between Turing machine computing and familiar straightedge and compass constructions, where the cells of the tape are replaced by the points of the plane and drawing of points and lines is replacing the change from 0 to 1. Moreover, it is possible to consider a wide range of dynamic geometric information processes (such as constructions of conics) as a form of computation. Finally, the property of character n for closure spaces provides a classification of the geometric information systems, and therefore of the types of computation, into a hierarchy indexed by natural numbers. This hierarchy has Turing machines at its lower extreme with closure spaces of character zero or one.*

Key words: *Closure space, Closure operator, Classification of closure spaces, Turing machines, Geometric information systems, Hyper-computing.*

1. INTRODUCTION

Discussions about transcending the concept of computing based on the model of Turing machine in the contexts of hyper-computing, Church-Turing Thesis, natural or naturalized computing, have a fundamental flaw of not having clearly formulated idea of what could be a more general form of computation. Typically, the attempts of generalization are based on a more or less vague concept of information, and this concept is virtually always referring to some form of a language, natural or artificial. This brings a danger that the rejection of the possibility of going beyond Turing machines is simply a result of the vicious circle: every form of computation (explicitly or implicitly understood as process based on the Turing machine model) can be performed by a Turing machine.

To avoid this type of invalid reasoning, the present author introduced into consideration a form of dynamics of information, based on his own very general concepts of information with its dualistic manifestations (selective and structural) and of information integration [1,2]. These concepts were used to develop a mathematical formalism for information and its integration within the framework of general algebra and theory of closure spaces [3, 4]. Due to the high level of generality of the concept of information, computing does not have to be restricted to the use of any language or human conventions. Since author's interests were focused on naturalization of computing, the first step in his search for generalization of computing was to separate the elements which constitute its theoretical mechanism of information processing and describe its internal dynamics from those of a pragmatic character dependent on human intervention and interpretation.

An example of this human intervention into the description of Turing machine is interpretation of the state of the tape (before and after computing) as a natural number, from which we actually get computation as the name of the process (even in times of Turing, computer was a person performing calculations, primarily on natural numbers). There is nothing in the concept of Turing machine or in the computation itself which could justify this association. The sequence of zeros and ones on the tape (or of any characters) can be interpreted as a natural number, but only by an external interpreter capable to integrate the information distributed into the cells of the tape. No part of the machine is capable to do it. This interpretation is a purely human intervention into the process. Of course, we can construct a peripheral device which can read the tape and provide any representation of the number we wish, but then this representation has to be interpreted by a human mind, or by a possible information system capable of information integration.

This should be considered an issue completely different from the problem of an observer of the physical process of computation or of its outcome. It is not a matter of observation, but of interpretation.

Another problem even more closely related to the naturalization of computing and therefore to its implementation in the physical world, is in the assumption of the goal directed action of the machine. The central idea of Turing machine is a decomposition of the process of computing into the most elementary steps carried out with the minimum means. At first sight, it seems that Turing machine with its metaphor of a tape and reading/printing head is extremely simple. However, it requires involvement of a cause-effect relationship and achieving some goals, such as typing of a character or reading the content of a cell. This requires an assumption of a one-way action, which is absent in the scientific, physical view of the world.

Directed, one-way action is possible only in cases of very complex systems. In the case of simple systems we have exclusively interactions. The view that an apple falls on Earth or Earth is revolving around Sun "because of the gravitation" is the result of our continuing pre-scientific habits of thinking. Actually, mechanical dynamics require that every process is an interaction, and every mechanical process should be invertible in time. Directed processes are possible and highly probable for very complex systems. If we want to consider computation as a process carried out in the physical reality at a very elementary level, we should consider computation as an interactive process. For this purpose, the author proposed a generalization of Turing A-machine, to an interactive S-machine (where S stands for symmetric) in which the roles of head and tape are symmetric and the head may have variable instructions modified in the time of computing [2]. A-machines are special cases of S-machines, under the very restrictive assumption that the instructions in the head never change. This assumption can be considered justified for very special cases of complex systems, such as artifacts produced by humans, but in the context of natural implementation of computing seems too restrictive.

Within this conceptual framework, the present paper is devoted to the study of information systems which can be a side in the dynamical interaction which constitutes computation. Of course, a tape of the Turing A-machine is one of fundamental examples. The question asked and answered here is in what sense it is carrying information, when the machine is considered an autonomous dynamical system and the interpretation by natural numbers of the content of a tape is an external human intervention which is not an integral part of computation. Then, the question is whether we can find other examples of information systems which can play this role. Such examples are identified in geometry, with most important instance of a straightedge and compass constructions. Finally, an attempt is made to provide a classification of such geometric systems.

2. PRELIMINARIES: CLOSURE SPACES AND INFORMATION SYSTEMS

Closure spaces are usually classified by additional properties which are added to the three axioms of a closure operator understood as a function f on the power set of a set S such that:

(1) For every subset A of S, $A \subseteq f(A)$;

(2) For all subsets A, B of S, $A \subseteq B \Rightarrow f(A) \subseteq f(B)$;

(3) For every subset A of S, $f(f(A)) = f(A)$.

Additional conditions for classifications of closure spaces are being selected from the properties of particular examples which played significant roles in the domains of application of this concept. Some of these properties are just a matter of convenience, for instance the property of normalization:

(N) $f(\varnothing) = \varnothing$, written in short as $f \in N(S)$, where N(S) means the set of all normalized closure operators on the set S, or the property well known from topology:

(T_1) $\forall a \in S: f(\{a\}) = a$, written in short $f \in T_1(S)$.

Although the concept of a closure space is already extremely general, sometimes more general concepts are considered. For this reason, to emphasize that the closure operator is transitive (its third defining property), the property of transitivity has its own symbolic:

(I) For every subset A: $f(f(A)) = f(A)$. In such case we can write $f \in I(S)$.

In the attempts to axiomatize many domains of mathematics, several properties have been distinguished as those domain-specific. For instance, the property of finite additivity:

(fA) For all subsets A, B: $f(A \cup B) = f(A) \cup f(B)$ is associated with topology, although there is no specific justification beyond tradition for this choice [5]. But of course, all classical examples of topological spaces satisfy this condition. Even less justified is the choice of "weak exchange" property:

(wE) $\forall A \subseteq S\, \forall x,y \in S, x \neq y: x \notin f(A)\ \&\ x \in f(A \cup \{y\}) \Rightarrow y \in f(A \cup \{x\})$ as the main axiom of geometry, since it characterizes all decomposable closure spaces [5]. Of course, the property is always assumed explicitly or implicitly in all types of axiomatic geometry, but there is nothing specifically "geometric" in it. Instead the author identified another property of "character n" which makes closure space geometric [5]:

(C_n) $\forall A \subseteq S: A = f(A)$ *iff* $\forall B \subseteq A: |B| \leq n \Rightarrow f(B) \subseteq A$.

This property was derived by the author from one of the equivalent forms of the "finite character" property which usually is associated with algebraic closure spaces (defined by subalgebras of an algebra):

(fC) $\forall A \subseteq S\, \forall x \in S: x \in f(A) \Rightarrow \exists B \in Fin(A): x \in f(B)$, where Fin(A) means a set of all finite subsets of A).

More extensive studies of closure spaces can be found in literature of the subject [6]. For our purpose it will be useful to recall that every transitive closure operator on a set S is associated with a unique Moore family $\Im$ of subsets of S (family having S as its member and closed with respect to arbitrary intersections). For given closure operator f, it is the family of closed subsets. For any Moore family $\Im$, the corresponding closure operator f is defined for every subset A of S by: f(A) is the least element of $\Im$ including A. Moore families are always complete lattices with respect to set inclusion.

Finally, we will consider a special case of a trivial closure operator f(A) = A where A is arbitrary subset of S, for which all subsets are closed and its Moore family of closed subsets is a Boolean lattice (Boolean algebra).

The concept of information was defined by the author as an identification of a variety, or that which makes one out of the many [7]. From this definition two main manifestations can be derived, selective (that which is distinguishing one out of many) and structural (structure which binds many into one).

The concept of information requires a variety, which here is understood as an arbitrary set S (called a carrier of information). Information system is this set S equipped with a Moore family $\Im$ of subsets of the set S. Information itself is a distinction of a subset $\Im_0$ of $\Im$, such that it is closed with respect to (pairwise) intersection and such that with each subset of S belonging to $\Im_0$, all subsets of S including it belong to $\Im_0$ (i.e. in mathematical terminology $\Im_0$ is a filter).

Moore family $\Im$ can represent a variety of structures (e.g. geometric, topological, algebraic, etc.) of the particular type which defined on the subsets of S as substructures. This corresponds to the structural manifestation of information. Filter $\Im_0$ in turn, serves identification, i.e. selection of an element within the family $\Im$, and under some conditions in the set S.

The complete lattice defined on the Moore family $\Im$ by inclusion is called a logic of information system by an extension of the association of the Boolean algebra of the power set of set S with traditional logic. Similarly, if the closure operator is defined on the Hilbert space (utilized in the formalism of quantum mechanics) by all its closed subspaces, this lattice is associated with quantum logic [8]. Certainly, in the case of information systems this generalized concept of the logic of information is going very far beyond traditional logic developed in the context of a language.

The formalism for information developed by the author allows consideration of different levels of information integration. For this purpose we can analyze the logic (complete lattice of the Moore family) of given information system. The level of information integration is expressed by the decomposability (reducibility) of the logic into direct product [4]. Every atomic Boolean algebra is totally decomposable into a direct product of simple two-element Boolean algebras. This corresponds to completely disintegrated information. Another extreme case is for quantum logics, which are totally irreducible to direct products, and therefore correspond to completely integrated information. Reducibility or irreducibility of the logics of information corresponds to a decomposition of the closure spaces into disjoint sums, but this was discussed extensively elsewhere, and will not be used in the present paper [5].

3. COMPUTATION

As it was mentioned above, computation can be considered a dynamical process of interaction between two information systems. In this context it is necessary to take into account both manifestations of information, the selective and the structural. In every information system they appear in parallel, but on different varieties related in a hierarchic relationship. We can find it in every information system, but for the objectives of the present paper the case of a Turing machine seems most appropriate as an example.

When we consider a tape in Turing machine, we can think about the selective information manifested in each of its cells. Information is related to the choice of a character in the cell. If the only characters are 0 and 1, the volume of information in each cell is 1 bit. However, we can consider the structure of 0's and 1's in all cells at the same time, which exhibits structural manifestation. Of course, the volume of information in this case is unlimited. At the former (let's call it local) level the variety consists of the set of characters that can be placed in each cell. At the latter (global) level, the variety consists of all possible configurations of 0's and 1's on the tape.

To present computation as an interaction of the two information systems (we will call them to maintain tradition "head" and "tape") we can generalize slightly the concept of an A-machine of Turing. Symmetric Turing machine (S-machine) as its prototype A-machine consists of two information systems, which in turn split into the global and local levels. Their roles are here essentially the same or symmetric. The generalization consists in possibility of a modification of the instructions at the time of their execution. Thus the tape has cells, with one cell active and involved in interaction, the head has analogous instruction list divided into instruction list positions ("ilp"), with one ilp active. Active cell and active ilp interact, and each is a subject to selection of its new content from the pre-existing variety of characters and instructions, respectively.

As a result of the interaction the active cell changes its content – character (it is not changed by a head in a one-way action, but through the interaction with its active ilp!) according to the current state of the active ilp. The active ilp changes its content too, i.e. changes instruction according to the current state of the active cell. Then the activation of the cell and ilp is changing according to both, the current state of active cell and the current state of active ilp.

There are two levels of interaction, at the level of active local elements (the active cell interacts with the active ilp), and at the global level when activation of cells and ilp's is changing. When we are talking about the change, we have an option of the void change, i.e. no change. In the special case when all changes of ilp's are void, we have an orthodox A-machine. Thus, the concept of an S-machine is a generalization of the concept of an A-machine.

The crucial aspect of information dynamics in computation is the involvement of the selective-structural duality of information. The interaction of active cell and active ilp is at the local level where the selective manifestation of information is transforming the content of these local elements. This local change of selective information is contributing to the global, structural information expressed in the state of all tape and all list of instructions in the head. However we have also transformation of the selective manifestation of information at the global level, when based on the content of both active elements there is selection of next active pair of local elements (cell and ilp). This selection can be considered only at the global level.

If we illustrate the local-global relation by vertical direction and the distinction between information systems (head and tape) in horizontal, then the process of computation (or of producing of the outcome of computation) can be interpreted as a change of structural manifestation of information at the upper level with the mechanism consisting in interaction of selective information of both lower and upper level. The structural manifestation of information at the lower level is not directly involved in this process. We have preexisting and not-changing list of characters and list of instructions, if we assume that the system has only two levels. We just make selection of the characters and selection of instructions. However, nothing prevents us from considering multilevel systems, such as living objects [1].

Now, the dynamic of computation can be understood purely in terms of interaction of information systems, not of the one-way action of the active head on the passive tape, serving only as a record of information. This dynamic can be designed through human intervention, or could be conditioned by natural forces and interactions.

In the present paper we are interested in the types of information systems involved in interaction, not in the dynamics of changes, and therefore the focus is on their properties.

4. BEYOND INFORMATION SYSTEMS OF A TURING MACHINE

Our main objective is to identify the exact characteristics of information systems of a Turing machine. Since in the generalization to a symmetric Turing S-machine the head and tape are essentially equivalent and the tape has basically the same characteristics in A-machine and in S-machine, it is easier to use as a model of information system the latter.

First, we will consider the simplest case of a Turing machine whose tape can have only multiple configurations of two symbols 0 and 1. Customary approach is to think about the global state of the tape as a natural number. However, there is nothing in Turing machine which can perform the transition from the configuration of 0's and 1's to the binary representation of the number. How would we know that the configuration is a binary representation, not just accidental (although extremely unlikely) decimal representation in which it happens that no other digit occurs? How do we know at all that it is a positional representation?

The question is what remains when we exclude human interpretation integrating the string into a unified object. The only answer that does not involve arbitrary assumptions is that the string is an element of an atomic Boolean algebra with the countable atom space. Each element is determined by the distinction of its atoms (atoms comparable with the element) by 1's. In the process of computation, currently selected element of the algebra is being changed in such a way that only one atom can be changed at a time.

Thus, computation is a travel across the Boolean algebra. Since every element of the algebra can be in principle be selected as the outcome of computation (possibly in the infinite number of steps) the logic of the information system is this Boolean algebra, and the information system is the trivial one described by the closure space with all sets closed. Of course, we have here the extreme case of totally disintegrated information.

There is a natural question regarding the possibility of having a computing machine with integrated information. For this purpose the logic of information system has to be irreducible. We could use here quantum logic and the tape physically would become a quantum mechanical system. There are of course many other information systems with completely irreducible logic. A good source of examples can be found in geometry.

Now, we can ask whether Turing machine is the only theoretical device in which algorithmic processes are used to modify the global state of the system by gradual alternation of the local states of the components. We have a very old idea of different but in some respects similar "machine" – straightedge and compass construction utilized in proving of geometric theorems.

Instead of cells we have points of the plane, which itself is playing the role of the tape. Input into this "geometric computing machine" consists of the points belonging to the "given objects". The difference is that the changes based on only few input "cells" (points) can involve an infinite number of points. Drawing of a line or circle is changing an infinite number of "white" points (0-cells) into "black" points (1-cells). As in Turing machines where the tape has unlimited number of cells, we have here an idealization allowing constructions with unlimited extension of lines and unlimited radii of circles.

How can we fit straightedge and compass constructions within the framework presented above? To maintain the high level of generality, we can limit ourselves to the closure space axioms similar to those usually used in formalization of geometry in the terms of general algebra [9]. As it was mentioned above, the author provided argumentation for using as a fundamental axiom of geometry not the weak exchange property (Steinitz property), which characterizes all direct product irreducible systems, but the n-character property [5]:

$$(C_n)\ \forall A \subseteq S: A = f(A) \text{ iff } \forall B \subseteq A: |B| \leq n \Rightarrow f(B) \subseteq A.$$

Actually, classical geometry is of 2-character, as the basic concept is of a straight line. It is a straightforward consequence of the definitions that [5]:

$$f \in C_n(S) \Rightarrow f \in C_{n+1}(S) \Rightarrow f \in fC(S),$$

and therefore every classical geometric closure space is of any n-character, where n is at least two.

It is not easy to place straightedge and compass constructions within so general concept of geometry, since we have in this framework straight lines, but not circles. However, we can talk about straightedge constructions. Thus we can consider geometric information systems defined by closure operator $\mathbf{f \in NT_1C_2I(S)}$ (or $\mathbf{f \in NT_1C_2wEI(S)}$ in the direct product irreducible or coherent case) whose closed sets consist of singleton subsets (points) and closures of two points (straight lines). Closures of any triple of non-collinear points is all set S.

If we want to consider geometries with straight lines and circles (without extending the list of axioms to be able to re-introduce metric/distance and define circles as points equidistant from a given point, which puts very strong restriction of the necessity to add an additional structure beyond that of closure space), we can change the closure space to 3-character, i.e. with the closure operator $\mathbf{f \in NT_1C_3I(S)}$ (or $\mathbf{f \in NT_1C_3wEI(S)}$).

In this case, pairs of points become closed sets, but closures of three points can be straight lines or circles. It is interesting, that in this case we have only Euclidean models, as the requirement that through every three non-collinear points there is a circle to which they belong is equivalent to the Fifth Postulate. Sets of four points which do not belong to a straight line or a circle have as their closure all set S. Within this type of geometric information systems we can have those of straightedge and compass constructions.

Within the system, there is no way to make a distinction between straight lines and circles. It is only when we are using as a model the structure of traditional metric geometry, we can decide about it. Also, the model based on lines and circles of a metric space are not the only model of such geometry. An alternative model can be found when pairs of points form closed sets and closures of triple point sets in a metric space are parabolas with distinguished unique direction of their axes.

If we want to consider a geometric information system for the machine which can draw, i.e. construct conics (including those degenerated such as straight lines and their intersecting pairs) we have to use **$f \in NT_1C_5I(S)$** (or in coherent case **$f \in NT_1C_5wEI(S)$**), as conics are determined by five points. Sets with less than five points are closed in this case.

Using the rudiments of algebraic geometry, we can state that geometric systems with all curves of degree d being closed subsets, we have to use **$f \in NT_1C_nwEI(S)$**, or (**$f \in NT_1C_nwEI(S)$**) where $n = 1/2(d+1)(d+2) - 1$. However, there are models for geometric systems for every n. For instance, when the closure operator **$f \in NT_1C_4wEI(S)$**, straight lines and curves given by the equation $y = ax^3+bx^2+cx+d$ are closures of quadruple sets of points which belong to them, while sets of five points have as their closure all S. We can see that with the property C_n as a fundamental axiom for geometry we get a hierarchical classification of geometric information systems.

There is a natural question how this classification of geometric systems by n-character is related to Turing machines. The answer comes with the following proposition relating the n-character property closure spaces for lowest values of n with binary relations:

PROPOSITION [5]:

i) $f \in IC_0(S)$ iff $\exists T \subseteq S$: $f(A) = A$ for $T \subseteq A$ and $f(A) = A \cup T$ otherwise. If $T=\emptyset$, then f is a trivial closure operator for which $f(A)=A$ for every subset A.

ii) $f \in INC_1(S)$ iff there exists a reflexive and transitive relation (quasiorder) R on S, such that $\forall A \subseteq S$: $f(A) = R^e(A) := \{y \in S: \exists x \in S: xRy\}$.

iii) $f \in INT_0C_1(S)$ iff there exists partial order R, such that $f(A)=R^e(A)$

iv) $f(A)=R^e(A)$ and R is an equivalence relation iff $f \in INC_1(S)$ and f satisfies: $\forall x,y \in S$: $x \in f(\{y\}) \Rightarrow y \in f(\{x\})$.

From the first part of the proposition and the fact that the tape of Turing machine as an information system can be described by an atomic Boolean algebra with countable atom space, or in other words by a trivial closure operator $f(A) = A$ for every subset A of S, we get that Turing machines with binary alphabet are geometric information systems of character 0.

If the alphabet consists of more than two characters, for instance k characters, we can establish correspondence between characters and appropriate sequence of 0's and 1's (their binary encoding) in such a way that each character corresponds to an equivalence class defined by an equivalence relation on the atom space of a Boolean algebra. Following the fourth part of the proposition, the closure operator becomes in this case a transitive operator of character 1. This is a natural consequence of the fact that the sequences of 0's and 1's corresponding to characters cannot be decomposed into separate parts in the process of computing.

5. CONCLUSIONS

We can conclude that when computing is understood as dynamic interaction of information systems and information is formalized in terms of closure spaces, there is an analogy between computing by Turing machines and the familiar straightedge and compass constructions.

Moreover, it is possible to consider a wide range of dynamic geometric information processes (such as constructions of conics) which can be understood as computation. Finally, the property of character n for closure spaces provides a classification of geometric information systems, and therefore of processes of computation into a hierarchy indexed by natural numbers. This hierarchy has Turing machines at its lower extreme with closure spaces of character 0 or 1.

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