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Schubert Eisenstein Series for $GL(3)$

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1 Schubert Eisenstein series

We try to explain the concept of Schubert Eisenstein series and possible arithemetic implication. This is joint work with D. Bump and a survey talk at RIMS. The original work can be found in [1]

1.1 Schubert Eisenstein series

Let us start with a general set up.

Let G be a split reductive algebraic group over a global field F. Let $\hat{T}$ be the maximal torus of the group $\hat{G}$ with opposite root data, so that $\hat{G}(\mathbb{C})$ is the connected Langlands L-group. Let $\nu\in\hat{T}(\mathbb{C})$. Then ν parametrizes a character χ_{ν} of $T(\mathbb{A})/T(F)$, where $\mathbb{A}$ is the adele ring of F. Extending χ_{ν} to the Borel subgroup $B(\mathbb{A})$, let f_{ν} be an element of the corresponding induced representation, so that

$$f_{\nu}(bg)=(\delta^{1/2}\chi_{\nu})(b)\,f(g), \qquad b\in B(\mathbb{A}). \tag{1}$$

The flag variety $X=B\backslash G$ is a projective variety. We recall its decomposition into Schubert cells. We have the Bruhat decomposition $G=\bigcup BwB$, a disjoint union over $w\in W$, and let Y_{w} be the image of BwB in X. The Schubert cell X_{w} is the Zariski closure of Y_{w}. It equals

$$\bigcup_{\substack{u\in W\\ u\leqslant w}} Y_{u},$$

where $u\leqslant w$ is the Bruhat order. Let G_{w} be the subset of G that is the union of BuB for $u\leqslant w$. It is not a subgroup. Let $X_{w}(F)$ be the set of $\gamma\in B_{F}\backslash G_{F}$ belonging to X_{w}. Thus $X_{w}(F)=B_{F}\backslash G_{w}(F)$. We may now define the *Schubert Eisenstein series*

$$E_{w}(g,f,\chi)=\sum_{\gamma\in X_{w}(F)}f(\gamma g).$$

1.2 Bott-Samelsom varieties

Let us recall the Bott-Samelson varieties and their relationship with Schubert varieties. We will denote by α_i and s_i the simple roots and corresponding simple reflections. Let $w \in W$ and let $\mathfrak{w} = (s_{i_1}, s_{i_2}, \cdots, s_{i_k})$ be a reduced decomposition of w into a product of simple reflections: $w = s_{i_1} \cdots s_{i_k}$. Let P_j be the minimal parabolic subgroup generated by B and s_j. We define a left action of B^k on $P_{i_1} \times \cdots P_{i_k}$ by

$$(b_1, \cdots, b_k) \cdot (p_{i_1}, \cdots, p_{i_k}) = (b_1 p_{i_1} b_2^{-1}, b_2 p_{i_2} b_3^{-1}, \cdots, b_k p_{i_k}). \tag{2}$$

The quotient $B^k \backslash (P_{i_1} \times \cdots \times P_{i_k})$ is the *Bott-Samelson variety* $Z_{\mathfrak{w}}$. There is a morphism $\mathrm{BS}_{\mathfrak{w}} : Z_{\mathfrak{w}} \longrightarrow X_w$ induced by the multiplication map that sends

$$(p_{i_1}, \cdots, p_{i_k}) \longmapsto p_{i_1} \cdots p_{i_k}.$$

This map is a surjective birational morphism.

It is known that Bott-Samelson varieties are always nonsingular, so this gives a resolution of the singularities of the Schubert variety X_w. The map $\mathrm{BS}_{\mathfrak{w}} : Z_{\mathfrak{w}} \longrightarrow X_w$ may not be an isomorphism. In special cases where it is an isomorphism, every element of X_w has a unique representation as a product $i_{\alpha_1}(\gamma_1) \cdots \iota_{\alpha_k}(\gamma_k)$, where if α is a root (in this case a simple root) ι_α is the Chevalley embedding of SL(2) into G corresponding to α, so the image of ι_{α_i} lies in the Levi subgroup of P_i. Beyond these special cases where $\mathrm{BS}_{\mathfrak{w}}$ is an isomorphism, in every case each element of X_w has such a factorization, and if the element is in general position, it is unique, since $\mathrm{BS}_{\mathfrak{w}}$ is birational. Let us call this a *Bott-Samelson factorization*. This means that we may write

$$E_{s_1 \cdots s_k}(g, \nu) = \sum_{\gamma_k \in B_{\mathrm{SL}_2}(F) \backslash \mathrm{SL}_2(F)} E_{s_1 \cdots s_{k-1}}(\iota_{\alpha_k}(\gamma_k) g, \nu), \tag{3}$$

building up the Schubert Eisenstein series by repeated SL_2 summations. If $\mathrm{BS}_{\mathfrak{w}}$: $Z_{\mathfrak{w}} \longrightarrow X_w$ is not an isomorphism, a modification of this method should be applicable.

1.3 GL₃ Schubert Eisenstein series, Explicit Computation

Let us be more precise : let $G = GL_3$ and let

$$\zeta^*(s) = |D_F|^{\frac{s}{2}} \prod_v \zeta_v(s), \qquad \zeta_v(s) = \begin{cases} (1 - q_v^{-s})^{-1} & \text{if } v \text{ is nonarchimedean,} \\ \Gamma_v(s) & \text{if } v \text{ is archimedean} \end{cases}$$

where we recall that D_F is the discriminant of F. For simplicity we will assume that the character χ is unramified at every place. Find $\nu_1, \nu_2 \in \mathbb{C}$ such that

$$(\delta^{1/2}\chi)\begin{pmatrix} y_1 & & \\ & y_2 & \\ & & y_3 \end{pmatrix} = |y_1|^{2\nu_1+\nu_2}|y_2|^{\nu_2-\nu_1}|y_3|^{-\nu_1-2\nu_2}.$$

We will denote this character χ_{ν_1,ν_2}. Also, take $f = f^\circ$ where

$$f^\circ(g) = f^\circ_{\nu_1,\nu_2}(g) = \prod_v f^\circ_v(g_v).$$

Thus if $k \in K$

$$f^\circ_{\nu_1,\nu_2}\left(\begin{pmatrix} y_1 & * & * \\ & y_2 & * \\ & & y_3 \end{pmatrix} k\right) = |y_1|^{2\nu_1+\nu_2}|y_2|^{\nu_2-\nu_1}|y_3|^{-\nu_1-2\nu_2}.$$

For each $w \in W$ normalize the Schubert Eisenstein series and denote

$$E^*_w(g;\nu_1,\nu_2) = \zeta^*(3\nu_1)\zeta^*(3\nu_2)\zeta^*(3\nu_1+3\nu_2-1)E_w(g;\nu_1,\nu_2)$$

and

$$\hat{E}^*_{s_1s_2}(\nu_1,\nu_2) = E^*_{s_1s_2}(\nu_1,\nu_2) - E^*_{s_2}(\nu_1,\nu_2) - E^*_{s_2}(\nu_2, 1-\nu_1+\nu_2). \tag{4}$$

Similarly

$$\hat{E}^*_{s_2s_1}(\nu_1,\nu_2) = E^*_{s_2s_1}(\nu_1,\nu_2) - E^*_{s_1}(\nu_1,\nu_2) - E^*_{s_1}(1-\nu_1+\nu_2, \nu_1). \tag{5}$$

Theorem 1 *[1] $E^*_{s_1s_2}(g;\nu_1,\nu_2)$ has meromorphic continuation to all ν_1,ν_2. It has a functional equation*

$$E^*_{s_1s_2}(g;\nu_1,\nu_2) = E^*_{s_1s_2}\left(g; \frac{2}{3}-\nu_1, \nu_1+\nu_2-\frac{1}{3}\right).$$

Moreover $\hat{E}^{**}_{s_1s_2}(g;\nu_1,\nu_2)$ is an entire function.

The similar result holds for $\hat{E}^*_{s_2s_1}(\nu_1,\nu_2)$.

1.4 Kronecker Limit Formula

Bump and Goldfeld proved the following result. If $K/\mathbb{Q}$ is a cubic field, and $\mathfrak{a}$ is an ideal class of K one may associate with $\mathfrak{a}$ a compact torus of GL_3, and if $L_{\mathfrak{a}}$ is the period of $\kappa(g)$ over this torus, then the Taylor expansion of the L-function $L(s,\mathfrak{a})$ has the form $\rho s^{-1}+L_{\mathfrak{a}}+\cdots$. Therefore if θ is a character of the ideal class group then $L(s,\theta)=\sum\theta(\mathfrak{a})L_{\mathfrak{a}}$. The proof involves showing that the torus period of the Eisenstein series equals a Rankin-Selberg integral of a Hilbert modular Eisenstein series.

Considering Taylor expansions of E_w for various w at $\nu_1=\nu_2=0$ we get

Theorem 2 *[1] We have*

$$\kappa(g)=\frac{\rho}{3}\zeta^*(2)\left[\hat{E}^{**}_{s_2s_1}(g;0,0)+E^{**}_{s_1}(g;1,0)\right]+c_0.$$

Furthermore

$$\kappa(g)=\frac{\rho}{3}\zeta^*(2)\left[\hat{E}^{**}_{s_1s_2}(g;1,0)+\phi_{s_2}(g)\right]+c_0'.$$

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References

[1] Daniel Bump and YoungJu Choie, Schubert Eisenstein Series for $GL(3)$, Preprint (2011)