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ALGEBRAIC INDEPENDENCE OF VALUES OF EXPONENTIAL TYPE POWER SERIES

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In this article we announce our results in [1] without proof.

1 Exponential type power series with periodic coefficients

Let $q \geq 2$ be an integer and let $\xi = e^{2\pi i/q}$. We consider the power series

$$e_r(z) = e_{q,r}(z) = \sum_{\substack{n=0 \\ n \equiv r \pmod q}}^{\infty} \frac{z^n}{n!} \qquad (r = 0, 1, \ldots, q-1). \tag{1}$$

Trivially the relation

$$e_0(z) + e_1(z) + \cdots + e_{q-1}(z) = e^z$$

holds. Using the formula

$$\frac{1}{q}\sum_{k=0}^{q-1} \xi^{k(n-r)} = \begin{cases} 1 & \text{if } n \equiv r \pmod q, \\ 0 & \text{otherwise}, \end{cases}$$

we have

$$\begin{aligned} e_r(z) &= \frac{1}{q}\sum_{k=0}^{q-1} \xi^{-kr} \sum_{n=0}^{\infty} \frac{\xi^{kn} z^n}{n!} \\ &= \frac{1}{q}\left(e^z + \xi^{-r} e^{\xi z} + \xi^{-2r} e^{\xi^2 z} + \cdots + \xi^{-(q-1)r} e^{\xi^{q-1} z}\right), \end{aligned}$$

or

$$\begin{pmatrix} e_0(z) \\ e_1(z) \\ e_2(z) \\ \vdots \\ e_{q-1}(z) \end{pmatrix} = C \begin{pmatrix} e^z \\ e^{\xi z} \\ e^{\xi^2 z} \\ \vdots \\ e^{\xi^{q-1} z} \end{pmatrix}, \tag{2}$$

where

$$C = \frac{1}{q}\begin{pmatrix} c_{1,1} & c_{1,2} & \dots & c_{1,q} \\ c_{2,1} & c_{2,2} & \dots & c_{2,q} \\ \vdots & \vdots & & \vdots \\ c_{q,1} & c_{q,2} & \dots & c_{q,q} \end{pmatrix} = \frac{1}{q}\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \xi^{-1} & \xi^{-2} & \dots & \xi^{-(q-1)} \\ 1 & \xi^{-2} & \xi^{-2\cdot 2} & \dots & \xi^{-(q-1)\cdot 2} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & \xi^{-(q-1)} & \xi^{-2(q-1)} & \dots & \xi^{-(q-1)(q-1)} \end{pmatrix}. \quad (3)$$

Since ξ is a root of the q-th cyclotomic polynomial, $1, \xi, \xi^2, \dots, \xi^{\varphi(q)-1}$ are linearly independent over $\mathbb{Q}$ and every ξ^k can be written as a linear combination of these $\varphi(q)$ numbers over $\mathbb{Z}$. If α is a nonzero algebraic number, $e^{\alpha}, e^{\xi\alpha}, \dots, e^{\xi^{\varphi(q)-1}\alpha}$ are algebraically independent over $\mathbb{Q}$ by the Lindemann-Weierstrass theorem and in view of (2) each of the numbers $e_0(\alpha)$, $e_1(\alpha), \dots,$ $e_{q-1}(\alpha)$ is transcendental.

Theorem 1. *Let $q \geq 3$ be an integer. If α is a nonzero algebraic number, then among q numbers*

$$e_0(\alpha),\ e_1(\alpha), \dots,\ e_{q-1}(\alpha)$$

any $\varphi(q)$ are algebraically independent over $\mathbb{Q}$. Moreover, any $\varphi(q) + 1$ of the q functions $e_0(z)$, $e_1(z), \dots,$ $e_{q-1}(z)$ are algebraically dependent over $\mathbb{Q}$.

Corollary 1. *Let $q \geq 3$ be an integer and let α be a nonzero algebraic number. Then any $\varphi(q)$ of the numbers*

$$\sum_{n=0}^{\infty} \frac{\alpha^n}{(qn+r)!} \qquad (r = 0, 1, \dots, q-1)$$

are algebraically independent over $\mathbb{Q}$.

Example 1. *In the case of $q = 2$, we have*

$$e_{2,0}^2(z) - e_{2,1}^2(z) = \cosh^2(z) - \sinh^2(z) = 1\,,$$

and for $q = 3$

$$e_0^3(z) + e_1^3(z) + e_2^3(z) - 3e_0(z)e_1(z)e_2(z) \ = \ 1\,.$$

2 Series involving fractional parts of polynomials

For any real number α we denote by $[\alpha]$ and $\{\alpha\}$ the integer and the fractional parts of α respectively.

Theorem 2. *Let $f(x) \in \mathbb{Q}[x]$, α be a nonzero algebraic number, and*

$$S = \sum_{n=0}^{\infty} \frac{\{f(n)\}}{n!}\alpha^n \neq 0\,.$$

Then S is a transcendental number.

Corollary 2. *Let $f(x) \in \mathbb{Q}[x]$, α be a nonzero algebraic number, and*

$$S = \sum_{n=0}^{\infty} \frac{[f(n)]}{n!} \alpha^n \neq 0 .$$

Then S is a transcendental number.

In the case of linear polynomials, we obtain the following results.

Theorem 3. *Let q and a are coprime integers with $q \geq 3$ and $0 < a < q$. Let*

$$f_b(z) = \sum_{n=0}^{\infty} \left\{ \frac{an+b}{q} \right\} \frac{z^n}{n!} \qquad (b = 0, 1, \ldots, q-1).$$

If α is a nonzero algebraic number, then among q numbers $f_0(\alpha), \ldots, f_{q-1}(\alpha)$ any $\varphi(q)$ are algebraically independent over $\mathbb{Q}$. Moreover, any $\varphi(q) + 1$ of the functions $f_0(z), \ldots, f_{q-1}(z)$ are algebraically dependent over $\mathbb{Q}$.

3 Series involving Fibonacci numbers

In this section we set $\rho := (1 + \sqrt{5})/2$. Let

$$F_n = \frac{1}{\sqrt{5}} \left(\rho^n - \left(-\frac{1}{\rho} \right)^n \right), \qquad L_n = \rho^n + \left(-\frac{1}{\rho} \right)^n \tag{4}$$

denote the *Fibonacci numbers* and the *Lucas numbers*, respectively.

Theorem 4. *Let $f_s(\alpha)$ and $g_s(\alpha)$ be power series defined by*

$$f_s(z) = \sum_{n=0}^{\infty} F_n^s \frac{z^n}{n!}, \qquad g_s(z) = \sum_{n=0}^{\infty} L_n^s \frac{z^n}{n!}.$$

If α is a nonzero algebraic number, then all the numbers in the set $\{f_s(\alpha) \mid s \in \mathbb{N}\} \cup \{g_s(\alpha) \mid s \in \mathbb{N}\}$ are distinct and any two are algebraically independent over $\mathbb{Q}$. Moreover, any three functions in the set $\{f_s(z) \mid s \in \mathbb{N}\} \cup \{g_s(z) \mid s \in \mathbb{N}\}$ are algebraically dependent over $\mathbb{Q}$.

Theorem 5. *Let $f_{a,b}(z)$ and $g_{a,b}(z)$ be power series defined by*

$$f_{a,b}(z) = \sum_{n=0}^{\infty} F_{an+b} \frac{z^n}{n!}, \qquad g_{a,b}(z) = \sum_{n=0}^{\infty} L_{an+b} \frac{z^n}{n!}.$$

If α is a nonzero algebraic number, then any two numbers in the set $\{f_{a,b}(\alpha) \mid a \in \mathbb{N},\ b \in \mathbb{N}_0\}$ are algebraically independent over $\mathbb{Q}$. Moreover, any three functions in the set $\{f_{a,b}(z) \mid a \in \mathbb{N},\ b \in \mathbb{N}_0\}$ are algebraically dependent over $\mathbb{Q}$. The same statements hold also for the power series $g_{a,b}(z)$.

References

[1] C. Elsner, Yu. V. Nesterenko, and I. Shiokawa, *Algebraic independence of values of exponential type power series*, submitted.