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(a, b)-type balancing numbers

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Abstract

A positive n is called a balancing number if

$$1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r)$$

for some positive integer r. Several authors investigated balancing numbers and their various generalizations. The goal of this paper is to survey some interesting properties and results on balancing and generalized balancing numbers.

1 Introduction

In [3] A. Behera and G. K. Panda gave the notion of balancing number.

Definition 1 ([3]). *A positive integer n is called a balancing number if*

$$1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r)$$

for some positive integer r. This number is called the balancer corresponding to the balancing number n. The mth term of the sequence of balancing numbers is denoted by B_m.

Remark 1. *It is clear from Definition that the following statements are equivalent to each other:*

- *n is a balancing number,*

- *n^2 is a triangular number (i.e. $n^2 = 1 + 2 + \cdots + k$ for some $k \in \mathbb{N}$),*
- *$8n^2 + 1$ is a perfect square.*

It is easy to see that 6, 35, and 204 are balancing numbers with balancers 2, 14 and 84, respectively.

2 Properties of balancing numbers

2.1 Generating balancing numbers

In [3] A. Behera and G. K. Panda proved other interesting properties about balancing numbers. Let us consider the following functions:

$$F(x) = 2x\sqrt{8x^2+1} \tag{1}$$

$$G(x) = 3x + \sqrt{8x^2+1} \tag{2}$$

$$H(x) = 17x + 6\sqrt{8x^2+1} \tag{3}$$

They proved that these functions always generate balancing numbers.

Theorem 1 ([3]). *For any balancing number n, $F(n)$, $G(n)$, and $H(n)$ are also balancing numbers.*

Remark 2. *Using the theorem above we get that if n a balancing number, then $G(F(n)) = 6n\sqrt{8n^2+1} + 16n^2 + 1$ is an odd balancing number, because $F(n)$ is always even and $G(n)$ is odd when n is even.*

For generating balancing numbers they proved the following theorems.

Theorem 2 ([3]). *If n is any balancing number, then there is no balancing number k such that $n < k < 3n + \sqrt{8n^2+1}$.*

They proved that a balancing number can also be generated by two balancing numbers.

Theorem 3 ([3]). *If n and k are balancing numbers, then*

$$f(n,k) = n\sqrt{8k^2+1} + k\sqrt{8n^2+1} \tag{4}$$

is also a balancing number.

2.2 A recurrence relation and other properties

In [3] they proved that the balancing numbers fulfill the following recurrence relation

$$B_{m+1} = 6B_m - B_{m-1} \quad (m > 1)$$

where $B_0 = 1$ and $B_1 = 6$. Using this recurrence relation they get interesting relations between balancing numbers. They proved the following elementary result.

Theorem 4 ([3]). *For any $m > 1$ we have*

- $B_{m+1}B_{m-1} = (B_m + 1)(B_m - 1)$,
- $B_m = B_k B_{m-k} - B_{k-1}B_{m-k-1}$ *for any positive integer $k < m$,*
- $B_{2m} = B_m^2 - B_{m-1}^2$,
- $B_{2m+1} = B_m(B_{m+1} - B_{m-1})$.

He proved another interesting result about the greatest common divisor of balancing numbers.

Theorem 5 ([25]). *If m and k are natural numbers then*

$$\gcd(B_m, B_k) = B_{(m,k)}.$$

2.3 Fibonacci and Lucas balancing numbers

In [21] K. Liptai gave a few results about special type of balancing numbers. Let us consider the definition below:

Definition 2 ([21] and [22]). *We call a balancing number a* Fibonacci *or a* Lucas balancing number *if it is a Fibonacci or a Lucas number, too.*

Using this definition and companion polynomial of B_m K. Liptai proved that the balancing numbers are solutions of a Pell's equation.

Theorem 6 ([21]). *The terms of the second order linear recurrence $R(6, -1, 1, 6)$ are the solutions of the equation*

$$x^2 - 8y^2 = 1$$

for some integer z.

There is also a connection between Fibonacci or Lucas numbers and Pell's equation. The following theorem is due to D. E. Ferguson:

Theorem 7 ([7]). *The only solutions of the equation*

$$x^2 - 5y^2 = \pm 4$$

are $x = \pm L_m$, $y = \pm F_m$ $(n = 0, 1, 2 \ldots)$, *where* L_m *and* F_m *are the* m*th terms of the Lucas and Fibonacci sequences, respectively.*

To find all Fibonacci or Lucas balancing numbers K. Liptai proved that there are finitely many common solutions of the Pell's equations above using a method of A. Baker and H. Davenport.

The main theorem in [21] and [22] are the following:

Theorem 8 ([21] and [22]). *There is no Fibonacci or Lucas balancing number.*

Remark 3. *Using another method L. Szalay got the same result for the solutions of simultaneous Pell equations in [34]. In this method he converted simultaneous Pell's equations into a family of Thue equations which ones can be solved.*

3 Generalizations

3.1 (k, l)-numerical centers

Definition 3 ([23]). *Let* y, k *and* l *be fixed positive integers with* $y \geq 4$. *A positive integer* x $(x \leq y - 2)$ *is called a* (k, l)*-power numerical center for* y*, or a* (k, l)*-balancing number for* y *if*

$$1^k + 2^k + \cdots + (x-1)^k = (x+1)^l + \cdots + (y-1)^l.$$

Remark 4. *In [8] R. Finkelstein studied "The house problem" and introduced the notion of first-power numerical center which is consistent with notion of balancing number* B_m. *He proved that infinitely many integers* y *possess* $(1, 1)$*-power centers and there is no integer* $y > 1$ *with a* $(2, 2)$*-power numerical center. In his paper, he conjectured that if* $k > 1$ *then there is no integer* $y > 1$ *with* (k, k)*-power numerical center. Later in [32] his conjeture was confirmed for* $k = 3$. *Recently, Ingram in [17] proved Finkelstein's conjecture for* $k = 5$.

In [23] the authors proved a general result about (k,l)-balancing numbers, but they could not deal with Finkelstein's conjecture in its full generality. Their main results are the following theorems.

Theorem 9 ([23]). *For any fixed positive integer $k > 1$, there are only finitely many positive pairs of integers (y,l) such that y possesses a (k,l)-power numerical center.*

For the proof of this theorem they used a result from [30]. Thus the previous Theorem is ineffective in case $l \leq k$ in the sense that no upper bound was made for possible numerical centers except for the cases when $l = 1$ or $l = 3$.

Theorem 10 ([23]). *Let k be a fixed positive integer with $k \geq 1$ and $l \in \{1,3\}$. If $(k,l) \neq (1,1)$, then there are only finitely many (k,l)-balancing numbers, and these balancing numbers are bounded by an effectively computable constant depending only on k.*

Remark 5. *There are numerical centers, because in [23] authors gave an example in the case when $(k,l) = (2,1)$. After solving an elliptic equation by MAGMA they got three $(2,1)$-power numerical centers x, namely 5, 13 and 36.*

3.2 (a,b)-type balancing numbers

Another generalization is the following by T. Kovács, K. Liptai and P. Olajos.

Definition 4 ([20]). *Let a, b be nonnegative coprime integers. We call a positive integer $an + b$ an (a,b)-type balancing number if*

$$(a+b) + (2a+b) + \cdots + (a(n-1)+b) = (a(n+1)+b) + \cdots + (a(n+r)+b)$$

for some $r \in \mathbb{N}$. Here r is called the balancer corresponding to the balancing number. We denote the positive integer $an+b$ by $B_m^{(a,b)}$ if this number is the mth among the (a,b)-type balancing numbers.

Remark 6. *We have to mention that if we use notation $a_n = an + b$ then we get sequence balancing numbers and if $a = 1$ and $b = 0$ for (a,b)-type balancing numbers than we get balancing numbers B_m.*

Using the definition the authors get the following proposition:

Lemma 1 (Proposition 1 in [20]). *If $B_m^{(a,b)}$ is an (a,b)-type balancing number then the following equation*

$$z^2 - 8\left(B_m^{(a,b)}\right)^2 = a^2 - 4ab - 4b^2 \tag{5}$$

is valid for some $z \in \mathbb{Z}$.

3.2.1 Polynomial values among balancing numbers

Let us consider the following equation for (a, b)-type balancing numbers

$$B_m^{(a,b)} = f(x) \tag{6}$$

where $f(x)$ is a monic polynomial with integer coefficients. By the previous Lemma and the result from Brindza ([5]) they proved the following theorem:

Theorem 11 ([20]). *Let $f(x)$ be a monic polynomial with integer coefficients, of degree ≥ 2. If a is odd, then for the solutions of* (6) *we have* $\max(m, |x|) < c_0(f, a, b)$*, where $c_0(f, a, b)$ is an effectively computable constant depending only on a, b and f.*

Let us consider a special case of Theorem 11 with $f(x) = x^l$. Using one of the results from Bennett ([1]) the authors get the following theorem:

Theorem 12 ([20]). *If $a^2 - 4ab - 4b^2 = 1$, then there is no perfect power (a, b)-balancing number.*

Remark 7. *There are infinitely many integer solutions of equation $a^2 - 4ab - 4b^2 = 1$.*

The authors are interested in combinatorial numbers (see also Kovács [19]), that is binomial coefficients, power sums, alternating power sums and products of consecutive integers. For all $k, x \in \mathbb{N}$ let

$$\begin{aligned}
S_k(x) &= 1^k + 2^k + \cdots + (x-1)^k, \\
T_k(x) &= -1^k + 2^k - \cdots + (-1)^{x-1}(x-1)^k, \\
\Pi_k(x) &= x(x+1)\ldots(x+k-1).
\end{aligned}$$

We mention that the degree of $S_k(x)$, $T_k(x)$ and $\Pi_k(x)$ are $k+1$, k and k, respectively and $\binom{x}{k}$, $S_k(x)$, $T_k(x)$ are polynomials with non-integer coefficients. Moreover, in the case when $f(x) = \Pi_k(x)$ Theorem 11 is valid but parameter a is odd.

Let us consider the following equation

$$B_m^{(a,b)} = p(x), \tag{7}$$

where $p(x)$ is a polynomial with rational integer coefficients. In this case they gave effective results for the solutions of equation (7).

Theorem 13 ([20]). *Let $k \geq 2$ and $p(x)$ be one of the polynomials $\binom{x}{k}$, $\Pi_k(x)$, $S_{k-1}(x)$, $T_k(x)$. Then the solutions of equation* (6) *satisfy* $\max(m, |x|) < c_1(a, b, k)$*, where $c_1(a, b, k)$ is an effectively computable constant depending only on a, b and k.*

3.2.2 Numerical results

In [20] T. Kovács, K. Liptai and the author completely solve the above type equations for some small values of k that lead to genus 1 or genus 2 equations. In this case the equation can be written as

$$y^2 = 8f(x)^2 + 1, \tag{8}$$

where $f(x)$ is one of the following polynomials. Beside binomial coefficients $\binom{x}{k}$, we consider power sums and products of consecutive integers, as well. We have to mention that in their results, for the sake of completeness, they provide all integral (even the negative) solutions to equation (8).

Genus 1 and 2 equations They completely solve equation (8) for all parameter values k in case when they can reduce the equation to an equation of genus 1. We have to mention that a similar argument has been used to solve several combinatorial Diophantine equations of different types, for example in [9], [10], [12], [13], [18], [19], [28], [29], [33], [36], [37]. Further they also solved a particular case ($f(x) = S_5(x)$) when equation (6) can be reduced to the resolution of a genus 2 equation. To solve this equation, they used the so-called Chabauty method. We have to note that the Chabauty method has already been successfully used to solve certain combinatorial Diophantine equations, see e.g. the corresponding results in the papers [6], [11], [14], [15], [31], [35] and the references given there.

Theorem 14. *Suppose that $a^2-4ab-4b^2 = 1$. Let $f(x) \in \{\binom{x}{2}, \binom{x}{3}, \binom{x}{4}, \Pi_2(x), \Pi_3(x), \Pi_4(x), S_1(x), S_2(x), S_3(x), S_5(x)\}$. Then the solutions (m, x) of equation (6) are those contained in Table 1. For the corresponding parameter values we have $(a, b) = (1, 0)$ in all cases.*

Remark 8. *In [20] the authors considered some other related equations that led to genus 2 equations. However, because of certain technical problems, they could not solve them by the Chabauty method. They determined the "small" solutions(i.e. $|x| \leq 10000$) of equation (8) in cases*

$$f(x) \in \left\{\binom{x}{6}, \binom{x}{8}, \Pi_6(x), \Pi_8(x), S_7(x)\right\}.$$

Their conjecture is that that there is no solution for these equations.

$f(x)$	Solutions (m, x) of (6)
$\binom{x}{2}$	$(1,-3),(1,4)$
$\binom{x}{3}$	$(2,-5),(2,7)$
$\binom{x}{4}$	$(2,-4),(2,7)$
$\Pi_2(x)$	$(1,-3),(1,2)$
$\Pi_3(x)$	$(1,-3),(1,1)$
$\Pi_4(x)$	$\emptyset$
$S_1(x)$	$(1,-4),(1,3)$
$S_2(x)$	$(3,-8),(3,9),(5,-27),(5,28)$
$S_3(x)$	$\emptyset$
$S_5(x)$	$\emptyset$

Table 1:

References

[1] Bennett, M. A., Rational approximation to algebraic numbers of small height: the Diophantine equation $|ax^n - by^n| = 1$, *J. Reine Angew. Math.*, 535 (2001) 1–49.

[2] Baker, A., Wüstholz, G., Logarithmic forms and group varieties, *J. Reine Angew. Math.*, 442 (1993) 19–62.

[3] Behera, A., Panda, G. K., On the square roots of triangular numbers, *Fibonacci Quarterly*, 37 No. 2 (1999) 98–105.

[4] Bérczes, A., Liptai, K., Pink, I., On generalized balancing numbers, *Fibonacci Quarterly*, (submitted).

[5] Brindza, B., On S-integral solutions of the equation $y^m = f(x)$, *Acta Math. Hungar.* 44 (1984) 133–139.

[6] Bruin, N. , Győry, K., Hajdu, L., Tengely T., Arithmetic progressions consisting of unlike powers, *Indag. Math.* 17 (2006) 539–555.

[7] Ferguson, D. E., Letter to the editor, *Fibonacci Quarterly*, 8 (1970) 88–89.

[8] Finkelstein, R. P., The House Problem, *American Math. Monthly*, 72 (1965) 1082–1088.

[9] Hajdu, L., On a diophantine equation concerning the number of integer points in special domains II, *Publ. Math. Debrecen*, 51 (1997) 331–342.

[10] Hajdu, L., On a diophantine equation concerning the number of integer points in special domains, *Acta Math. Hungar.*, 78 (1998) 59–70.

[11] Hajdu, L., Powerful arithmetic progressions, *Indeg. Math.*, 19 (2008) 547–561.

[12] Hajdu, L., Pintér, Á., Square product of three integers in short intervals, *Math. Comp.*, 68 (1999) 1299–1301.

[13] Hajdu, L., Pintér, Á., Combinatorial diophantine equations, *Publ. Math. Debrecen*, 56 (2000) 391–403.

[14] Hajdu, L., Tengely Sz., Arithmetic progressions of squares, cubes and n-th powers, *J. Functiones es Approximatio* (submitted).

[15] Hajdu, L., Tengely, Sz., Tijdeman, R., Cubes in products of terms in arithmetic progression, *Publ. Math. Debrecen*, 74 (2009) 215–232.

[16] Hoggatt Jr., V. E., Fibonacci and Lucas numbers, *Houghton Mifflin Company IV*, (1969) 92 p.

[17] Ingram, P., On the k-th power numerical centres, *C. R. Math. Acad. Sci. R. Can.*, 27 (2005) 105–110.

[18] Kovács, T., Combinatorial diphantine equations - the genus 1 case, *Publ. Math. Debrecen*, 72 (2008) 243–255.

[19] Kovács, T., Combinatorial numbers in binary recurrences, *Period. Math. Hungar.*, 58 (2009) No. 1 83–98.

[20] Kovács, T., Liptai, K., Olajos, P., About (a,b)-type balancing numbers, *Pub. Debrecen.*

[21] Liptai, K., Fibonacci balancing numbers, *Fibonacci Quarterly*, 42 No. 4 (2004) 330–340.

[22] Liptai, K., Lucas balancing numbers, *Acta Math. Univ. Ostrav.*, 14 No. 1 (2006) 43–47.

[23] Liptai, K., Luca F., Pintér, Á., Szalay L., Generalized balancing numbers, *Indagationes Math. N. S.*, 20 (2009) 87–100.

[24] Panda, G. K., Sequence balancing and cobalancing numbers, *Fibonacci Quarterly*, 45 (2007) 265–271.

[25] Panda, G. K., Some fascinating properties of balancing numbers, *Proceedings of the Eleventh International Conference on Fibonacci Numbers and their Applications*, Cong. Numer. 194 (2009) 185–189.

[26] Panda, G. K., Ray, P. K., Cobalancing numbers and cobalancers, *Int. J. Math. Sci.*, No. 8 (2005) 1189–1200.

[27] Panda, G. K., Ray, P. K., Some links of balancing and cobalancing numbers and with Pell and associated Pell numbers, (communicated).

[28] Pintér, Á., A note on the Diophantine equation $\binom{x}{4} = \binom{y}{2}$, *Publ. Math. Debrecen*, 47 (1995) 411–415.

[29] Pintér, Á, de Weger, B. M. M., $210 = 14 \times 15 = 5 \times 6 \times 7 = \binom{21}{2} = \binom{10}{4}$, *Publ. Math. Debrecen*, 51 (1997) 175–189.

[30] Rakaczki, Cs., On the diophanzine equation $S_m(x) = g(y)$, *Publ. Math. Debrecen*, 65 (2004) 439-460.

[31] Shorey, T. N., Laishram, S., Tengely, Sz., Squares in products in arithmetic progression with at most one term omitted and common difference a prime power, *Acta Arith.*, 135 (2008) 143–158.

[32] Steiner, R., On the k-th power numerical centers, *Fibonacci Quarterly*, 16 (1978) 470–471.

[33] Stroeker, R. J., de Weger, B. M. M., Elliptic binomial diophantine equations, *Math. Comp.*, 68 (1999) 1257–1281.

[34] Szalay, L., On the resolution of simultaneous Pell equations, *Annales Mathematicae et Informaticae*, 34 (2007) 77–87.

[35] Tengely, Sz., Note on a paper "An extension of a theorem of Euler" by Hirata-Kohno et al., *Acta Arith.*, 134 (2008) 329–335.

[36] de Weger, B. M. M., A binomial Diophantine equation, *Quart. J. Math. Oxford Ser. (2)*, 47 (1996) 221–231.

[37] de Weger, B. M. M., Equal binomial coefficients: some elementary considerations, *J. Number Theory*, 63 (1997) 373–386.