数理解析研究所講究録
第 1874 巻 2014 年 210-214

Continued fractions and Dedekind sums for function fields

Yoshinori Hamahata
Institute for Teaching and Learning
Ritsumeikan University

1 Introduction

For coprime integers a and $c>0$, the classical Dedekind sum $d(a,c)$ is defined by

$$d(a,c)=\frac{1}{4c}\sum_{k=1}^{c-1}\cot\left(\frac{\pi k}{c}\right)\cot\left(\frac{\pi ka}{c}\right). \tag{1}$$

For coprime positive integers a and c, it holds that

$$d(a,c)+d(c,a)=\frac{1}{12}\left(\frac{a}{c}+\frac{c}{a}+\frac{1}{ac}-3\right);$$

this is called the reciprocity law. The value of $d(a,c)$ has been investigated. Rewriting (1) in terms of the sawtooth function, we can easily see that $d(a,c)$ is a rational number. Rademacher [4] proved that $d(a,c)$ is not bounded above and below in the neighborhood of each a/c. Rademacher and Grosswald [5] posed the following two questions:

1. Is $\{(a/c,d(a,c))\mid a/c\in \mathbb{Q}^{*}\}$ dense in $\mathbb{R}^{2}$?
2. Is $\{d(a,c)\mid a/c\in \mathbb{Q}^{*}\}$ dense in $\mathbb{R}$?

Hickerson [3] answered them using the theory of continued fractions.

As is well known, there is an analogy between algebraic number fields and function fields. For example, $A:=\mathbb{F}_{q}[T]$, $K:=\mathbb{F}_{q}(T)$, and $K_{\infty}:=\mathbb{F}_{q}((1/T))$ are similar to $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, respectively. Each A-lattice is an analog of a lattice in $\mathbb{C}$. In [1, 2], we introduced Dedekind sums and their higher-dimensional generalization for a given A-lattice in a function field, and we established the reciprocity law. The A-lattice L corresponding to the Carlitz module defines the Dedekind sum $s(a,c)$ (see Section 2), which is very similar to $d(a,c)$. In this report, we answer the analogous questions for $s(a,c)$.

2 Dedekind sums

2.1 A-lattices and Drinfeld modules

Let C_{∞} be the completion of an algebraic closure of K_{∞}; it is an analog of $\mathbb{C}$. A rank r A-lattice is a finitely generated A-module of rank r such that it is discrete in

C_∞. For such an A-lattice Λ, we define the infinite product $e_\Lambda(z)$ by

$$e_\Lambda(z) = z \prod_{0 \neq \lambda \in \Lambda} \left(1 - \frac{z}{\lambda}\right).$$

This product uniformly converges at a bounded set in C_∞, and defines a map $e_\Lambda : C_\infty \to C_\infty$. The function $e_\Lambda(z)$ has the following properties:

(E1) $e_\Lambda(z)$ is entire in the sense of rigid analysis;

(E2) $e_\Lambda : C_\infty \to C_\infty$ is surjective $\mathbb{F}_q$-linear, and Λ-periodic;

(E3) e_Λ has a simple zero at each point in Λ, and no further zeros;

(E4) $de_\Lambda(z)/dz = e'_\Lambda(z) = 1$.

For $a \in A$, there exists a unique polynomial $\phi_a(z) = \phi_a^\Lambda(z) = \sum l_i(\phi_a) z^{q^i}$ such that $\phi_a(e_\Lambda(z)) = e_\Lambda(az)$ holds. Let $\tau : z \mapsto z^q$ be the Frobenius map, and let $C_\infty\{\tau\}$ be a non-commutative ring in τ with the commutation rule $c^q\tau = \tau c \ (c \in C_\infty)$. There exists a unique positive integer r such that for any $a \in A \setminus \{0\}$,

$$\phi_a = \sum_{i=0}^{r \deg a} l_i(a)\tau^i \qquad (l_0(a) = a).$$

Then, the map $\phi : A \to C_\infty\{\tau\}$, $a \mapsto \phi_a$ is called a rank r Drinfeld module over C_∞. The map ϕ is an $\mathbb{F}_q$-algebra homomorphism; hence, the values $\phi_a (a \in A)$ are determined by ϕ_T. The rank 1 Drinfeld module ρ with $\rho_T(z) = Tz + z^q$ is called the Carlitz module. The Carlitz module and a Drinfeld module of rank ≥ 2 are similar to the multiplicative group $\mathbb{G}_m$ and an elliptic curve, respectively. There exists a bijection between the set of rank r A-lattices and the set of rank r Drinfeld modules over C_∞, defined by $\phi_a(e_\Lambda(z)) = e_\Lambda(az)$ $(a \in A)$. The A-lattice L corresponding to ρ is similar to $2\pi i$, and each A-lattice of rank ≥ 2 is similar to a lattice in $\mathbb{C}$.

2.2 Dedekind sums

Let L be the A-lattice corresponding to the Carlitz module ρ. For coprime $a, c \in A \setminus \{0\}$, we define the inhomogeneous Dedekind sum $s(a,c)$ by

$$s(a,c) = \frac{1}{c} \sum_{0 \neq \ell \in L/cL} e_L\left(\frac{a\ell}{c}\right)^{-1} e_L\left(\frac{\ell}{c}\right)^{-1}.$$

When $L/cL = 0$, $s(a,c)$ is defined to be zero. Using the Galois theory, we see that $s(a,c) \in K$. By (E2), it holds that $s(a,c) = 0$ if $q > 3$. Thus, henceforth, we assume that $q = 3$ or 2. The reciprocity law for $s(a,c)$ is as follows.

Theorem 2.1 (Reciprocity law) *For coprime $a, c \in A$, we have*

$$s(a,c)+s(c,a)=\begin{cases}\dfrac{1}{T^3-T}\left(\dfrac{a}{c}+\dfrac{c}{a}+\dfrac{1}{ac}\right) & \text{if } q=3,\\[2mm] \dfrac{1}{T^4+T^2}\left(\dfrac{a}{c}+\dfrac{c}{a}+\dfrac{1}{a}+\dfrac{1}{c}+\dfrac{1}{ac}+1\right) & \text{if } q=2.\end{cases}$$

This result follows from the fact that the sum of all residues of $1/(z\rho_a(z)\rho_c(z))$ is zero.

2.3 Continued fractions

Since the value $s(a,c)$ depends on a/c, we write $s(a/c)=s(a,c)$. Then $s(a/c+b)=s(a/c)$ is valid. For $x=a/c \in K$, we define the sequence $(x_n)_{n\geq 0}$ by $x_0=x$, $x_{n+1}=1/(x_n-a_n)$, where a_n is the polynomial part $\sum_{i=0}^{k} A_iT^i$ of the Laurent expansion $x_n=\sum_{i=-\infty}^{k} A_iT^i$. This sequence yields the continued fraction development of x:

$$x=[a_0,a_1,\dots,a_n]:=a_0+\cfrac{1}{a_1+\cfrac{1}{\cdots+\cfrac{1}{a_{n-1}+\cfrac{1}{a_n}}}},$$

where a_i $(i\geq 1)$ are non-constant. Note that if $x\in K_\infty\setminus K$, x is an infinite continued fraction. The following theorem gives us the value of $s(a/c)$.

Theorem 2.2 *(i) If $q=3$, then*

$$s([a_0,\dots,a_r])=\begin{cases}\dfrac{1}{T^3-T}([0,a_1,\dots,a_r]+(-1)^{r+1}[0,a_r,\dots,a_1] \\ \quad +a_1-a_2+\cdots+(-1)^{r+1}a_r) & \text{if } r\geq 1,\\ 0 & \text{if } r=0.\end{cases}$$

(ii) If $q=2$, then

$$s([a_0,\dots,a_r])=\begin{cases}\dfrac{1}{T^4+T^2}([0,a_1,\dots,a_r]+(-1)^{r+1}[0,a_r,\dots,a_1] \\ \quad +\prod_{i=1}^{r}[0,a_i,\dots,a_r]+a_1-a_2+\cdots+(-1)^{r+1}a_r+r-1) & \text{if } r\geq 1,\\ 0 & \text{if } r=0.\end{cases}$$

We can prove this by induction on r by using Theorem 2.1.

Remark 2.3 Hickerson [3] proved the following result for $d(a/c):=d(a,c)$:

$$d([a_0,\dots,a_r])=\begin{cases}\frac{-1+(-1)^r}{8}+\frac{1}{12}([0,a_1,\dots,a_r]+(-1)^{r+1}[0,a_r,\dots,a_1] \\ +a_1-a_2+\cdots+(-1)^{r+1}a_r) & \text{if } r\geq 1,\\ 0 & \text{if } r=0.\end{cases}$$

3 Density theorem

As an analog of Hickerson's result, the following two theorems are obtained.

Theorem 3.1 *If* $q = 3$ *or* 2, *then* $\{(a/c, s(a/c)) \mid a/c \in K^*\}$ *is dense in* K_∞^2.

Theorem 3.2 *If* $q = 3$ *or* 2, *then* $\{s(a/c) \mid a/c \in K^*\}$ *is dense in* K_∞.

Outline of proof of Theorems 3.1, 3.2. We consider the case $q = 3$. Since $(K_\infty \setminus K) \times K$ is dense in K_∞^2, it suffices to prove that for any $(x, y) \in K_\infty \setminus K$ and for $\epsilon > 0$, there exists $a/c \in K^*$ such that $|x - a/c| < \epsilon$, $|y - s(a/c)| < 2\epsilon$. We write $x = [b_0, b_1, \ldots]$. Take any element $\alpha \in K_\infty^*$. For any $\epsilon > 0$, taking fully large s, $|x - [b_0, \ldots, b_{s-1}, \alpha]| < \epsilon$ holds. Similarly, we write $x - (T^3 - T)y = [d_0, d_1, \ldots]$. Taking fully large t, $|x - (T^3 - T)y - [d_0, \ldots, d_{t-1}, \alpha]| < \epsilon$ holds. Suppose that $s + t$ is even. There exits $m, n \in A \setminus \mathbb{F}_q$ such that

$$-b_0 + b_1 - b_2 + \cdots + (-1)^s b_{s-1} + (-1)^{t-1} d_{t-1} + \cdots - d_1 + d_0 = (-1)^s (m - n).$$

Putting

$$a/c = [b_0, \ldots, b_{s-1}, m, n, d_{t-1}, \ldots, d_1], \quad \alpha = [m, n, d_{t-1}, \ldots, d_1],$$

we have $|x - a/c| < \epsilon$. By Theorem 2.2 (i), we obtain

$$\begin{aligned} s(a/c) = \frac{1}{T^3 - T}([0, b_1, \ldots, b_{s-1}, m, n, d_{t-1}, \ldots, d_1] \\ - [0, d_1, \ldots, d_{t-1}, n, m, b_{s-1}, \ldots, b_1] \\ + b_1 - b_2 + \cdots + (-1)^s b_s + (-1)^{s+1} m + (-1)^{s+2} n \\ + (-1)^{t-1} d_{t-1} + \cdots + -d_1), \end{aligned}$$

which yields $|y - s(a/c)| < 2\epsilon$. Theorem 3.2 follows from Theorem 3.1. The case $q = 2$ can be proved in the same way.

References

[1] A. Bayad and Y. Hamahata, Higher dimensional Dedekind sums in function fields, Acta Arithmetica **152** (2012), 71–80.

[2] Y. Hamahata, Dedekind sums in function fields, Monatshefte für Mathematik **167** (2012), 461–480.

[3] D. Hickerson, Continued fractions and density results for Dedekind sums, J. Reine Angew. Math. **290** (1977), 113–116.

[4] H. Rademacher, Zur Theorie der Dedekindschen Summen, Math. Z. **63** (1956), 445-463.

[5] H. Rademacher and E. Grosswald, Dedekind sums, Math. Assoc. Amer., Washington, DC, 1972.

Institute for Teaching and Learning
Ritsumeikan University
1-1-1 Noji-higashi, Kusatsu, Shiga 525-8577
JAPAN
E-mail address: hamahata@fc.ritsumei.ac.jp

立命館大学・教育開発推進機構　浜畑 芳紀