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Leray's problem on D-solutions to the stationary Navier-Stokes equations past an obstacle

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Introduction.

Let Ω be an exterior domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega \in C^\infty$. We consider the stationary Navier-Stokes equations in Ω:

$$\text{(N-S)} \qquad \begin{cases} -\Delta u + u\cdot\nabla u + \nabla p = f & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u(x) \to u^\infty & \text{as } |x| \to \infty, \end{cases}$$

where $u = u(x) = (u_1(x), u_2(x), u_3(x))$ and $p = p(x)$ denote the unknown velocity vector and the unknown pressure at $x = (x_1, x_2, x_3) \in \Omega$, while $f = f(x) = (f_1(x), f_2(x), f_3(x))$ is the given external force, and $u^\infty = (u_1^\infty, u_2^\infty, u_3^\infty)$ is the prescribed constant vector in $\mathbb{R}^3$ at infinity. In the pioneer work of Leray [14], it was shown that for every $f \in \dot{H}^{-1,2}(\Omega) \equiv \dot{H}_0^{1,2}(\Omega)^*$ and for every $u^\infty \in \mathbb{R}^3$, there exists at least one weak solution u of (N-S) with $\int_\Omega |\nabla u(x)|^2 dx < \infty$ such that

$$\int_\Omega |u(x) - u^\infty|^6 dx < \infty.$$

Here and in what follows, $\dot{H}_0^{1,q}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ with respect to the homogeneous norm $\|\nabla u\|_{L^q}$ for $1 < q < \infty$. Leray named such a weak solution u *D-solution* of (N-S) because it has a finite Dirichlet integral in Ω. The asymptotic behavior of D-solution u at infinity had been improved by Finn [3], Fujita [4] and Ladyzhenskaya [13] in such a way that

$$u(x) \to u^\infty \quad \text{uniformly as } |x| \to \infty,$$

provided f has a compact support in Ω. In his paper [14], Leray proposed the problem whether every D-solution u satisfies the energy identity

$$\text{(EI)} \qquad \int_\Omega \nabla u \cdot \nabla(u-a)dx + \int_\Omega u \cdot \nabla a \cdot (u-a)dx = \langle f, u-a \rangle$$

for all $a \in C^1(\bar{\Omega})$ such that div $a = 0$ in Ω, $a|_{\partial\Omega} = 0$, $a(x) \equiv u^\infty$ for all $x \in \Omega$ satisfying $|x| \geq R$ with some large $R > 0$. Here $\langle \cdot, \cdot \rangle$ denotes the duality pairing of $\dot{H}^{-1,2}(\Omega)$ and $\dot{H}_0^{1,2}(\Omega)$. The second important question is a uniqueness problem of D-solutions. It is still an open question whether there exists a small constant δ such that if $\|f\|_{\dot{H}^{1,2}} + |u^\infty| \leq \delta$, then the D-solution u of (N-S) is unique. This is so-called a uniqueness theorem of D-solutions for arbitrary small given data $f \in \dot{H}^{-1,2}(\Omega)$ and $u^\infty \in \mathbb{R}^3$.

In this article, we shall give final affirmative answers to these two questions provided $u^\infty \neq 0$. It should be noted that the corresponding results to those in the case $u^\infty = 0$ are still open questions. See e.g., Nakatsuka [15]. There is another notion of physically reasonable(PR) solutions introduced by Finn [2], [3]. We call the solution u of (N-S) physically reasonable if it holds

$$\text{(PR)} \qquad u(x) - u^\infty = O(|x|^{-\alpha}) \quad \text{as } |x| \to \infty$$

for some $\alpha > 1/2$. If u is a PR-solution of (N-S) with $f \in C_0^\infty(\Omega)$, then u behaves like

$$\text{(WR)} \qquad u(x) - u^\infty = O(|x|^{-1}(1+s_x)^{-1}), \quad s_x \equiv |x| - \frac{x \cdot u^\infty}{|u^\infty|} \quad \text{as } |x| \to \infty,$$

which exhibits a parabolic wake region behind the obstacle. It had been shown by Finn [3] that in the case when $f \in C_0^\infty(\Omega)$, every PR-solution u becomes necessarily a D-solution. The converse assertion was treated by Babenko [1] who proved that if $f \equiv 0$, then every D-solution u of (N-S) satisfies (PR) with $\alpha = 1$. As a result, it turns out that every D-solution with $f \equiv 0$ has a parabolic wake region such as (WR). Later on, Galdi [6], [7], [8], [9] and Farwig [5] succeeded to handle more general f by introducing anisotropic weight functions, and obtained more precise asymptotic behavior of u than (WR) in the class of PR-solutions. Furthermore, Kobayashi-Shibata [11] showed the stability of PR-solutions for small f and u^∞ in terms of the Oseen semi-group in L^p-spaces.

1 Results.

Before stating our results, let us introduce some notation and then give our definition of D-solutions of (N-S). $C_{0,\sigma}^\infty(\Omega)$ is the set of all C^∞-vector functions $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ with compact support in Ω, such that div $\varphi = 0$. For $1 < q < \infty$, $L^q(\Omega)$ stands for all L^q-summabel vector functions on Ω with the norm $\| \cdot \|_{L^q}$. We denote by $(\cdot, \cdot)$ the duality paring between $L^q(\Omega)$ and $L^{q'}(\Omega)$, where $1/q + 1/q' = 1$. $\dot{H}_0^{1,q}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ with respect to the homogeneous norm $\|\nabla\varphi\|_{L^q}$, where $\nabla\varphi = \left(\dfrac{\partial\varphi_i}{\partial x_j}\right)$, $i, j = 1, 2, 3$. $\dot{H}^{-1,q}(\Omega)$ is the dual space of $\dot{H}_0^{1,q'}(\Omega)$, and $\langle f, \phi \rangle$ denotes the duality pairing between $f \in \dot{H}^{-1,q}(\Omega)$ and $\phi \in \dot{H}_0^{1,q'}(\Omega)$. Finally, for $u^\infty \in \mathbb{R}^3$, we define the space $A(u^\infty)$ by

$$A(u^\infty) \equiv \{a \in C^1(\bar{\Omega}); \text{div } a = 0, a|_{\partial\Omega} = 0, a(x) \equiv u^\infty \quad \text{for all } x \in \mathbb{R}^3 \text{ satisfying } |x| > R\}$$

with some $R > 0$.

Our definition of D-solutions to (N-S) reads as follows.

Definition. Let $f \in \dot{H}^{-1,2}(\Omega)$ and $u^\infty \in \mathbb{R}^3$. A measurable function u on Ω is called a D-solution of (N-S) if the following conditions (i), (ii) and (iii) are satisfied.

(i) $\nabla u \in L^2(\Omega)$ with div $u = 0$ in Ω and $u = 0$ on $\partial\Omega$;

(ii) $u(\cdot) - u^\infty \in L^6(\Omega)$;

(iii) it holds that

$$\text{(E)} \qquad (\nabla u, \nabla\varphi) + (u \cdot \nabla u, \varphi) = \langle f, \varphi\rangle \quad \text{for all } \varphi \in C_{0,\sigma}^\infty(\Omega).$$

Remark. For every D-solution u of (N-S), there exists a unique scalar function $p \in L^2_{loc}(\Omega)$ up to an additive constant such that

$$\text{(E')} \qquad (\nabla u, \nabla\phi) + (u \cdot \nabla u, \phi) + (p, \text{div } \phi) = \langle f, \phi\rangle \quad \text{for all } \phi \in C_0^\infty(\Omega).$$

Our first result on the energy identity (EI) now reads:

Theorem 1.1 *Assume that $f \in \dot{H}^{-1,2}(\Omega)$ and $u^\infty \in \mathbb{R}^3$ with $u^\infty \neq 0$. Then every D-solution u of (N-S) satisfies*

$$(1.1) \qquad (\nabla u, \nabla u) - (\nabla u, \nabla a) + (u \cdot \nabla a, u - a) = \langle f, u - a\rangle \quad \textit{for all } a \in A(u^\infty).$$

Moreover, if in addition $f \in \dot{H}^{-1,2}(\Omega) \cap L^q(\Omega)$ for some $1 < q < 2$, then it holds that

$$(1.2) \qquad \int_\Omega |\nabla u|^2 dx + u^\infty \cdot \int_{\partial\Omega} T(u,p) \cdot \nu dS = \langle f, u - u^\infty\rangle,$$

where $T(u,p) \equiv \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \delta_{ij}p\right)_{1\le i,j\le 3}$ denotes the stress tensor and where ν is the unit outer normal to $\partial\Omega$.

Remarks. (i) Galdi [8] and Farwig [5] showed a similar result to that of Theorem 1.1 under the assumption that $f \in \dot{H}^{-1,2}(\Omega) \cap L^{\frac{4}{3}}(\Omega) \cap L^{\frac{3}{2}}(\Omega)$. On the other hand, for the validity of the energy identity (1.1), we do not need any condition on f except for $f \in \dot{H}^{-1,2}(\Omega)$.

(ii) The corresponding problem for $u^\infty = 0$ is still open. Indeed, up to the present, the energy identity (1.1) is shown under the hypothesis that $u \in \dot{H}^{1,2}(\Omega) \cap L^{3,\infty}(\Omega)$, where $L^{q,r}(\Omega)$ denotes the Lorentz space on Ω. For instance, see Kozono-Yamazaki [12].

Next, we consider the uniqueness of D-solutions under the smallness assumption on the given data.

Theorem 1.2 *There is a constant $\delta_1 = \delta_1(\Omega) > 0$ such that if $u^\infty \neq 0$ and $f \in \dot{H}^{-1,2}(\Omega)$ satisfy*

$$(1.3) \qquad \|f\|_{\dot{H}^{-1,2}} + |u^\infty| \le \delta_1 |u^\infty|^{\frac{1}{2}},$$

then there exists a unique D-solution u of (N-S). Moreover, such a solution u is necessarily subject to the estimate

$$(1.4) \qquad |u^\infty|^{\frac{1}{4}}\|u-u^\infty\|_{L^4}+\|\nabla u\|_{L^2}\le C(\|f\|_{\dot H^{-1,2}}+|u^\infty|),$$

where $C=C(\Omega)$.

Remarks. (i) Galdi [8] showed that if $u^\infty\neq 0$ and $f\in L^{\frac{6}{5}}(\Omega)\cap L^{\frac{3}{2}}(\Omega)$ satisfy

$$\|f\|_{L^{\frac{6}{5}}}+|u^\infty|\le\delta_1$$

then there exists a unique D-solution. Since $L^{\frac{6}{5}}(\Omega)\subset\dot H^{-1,2}(\Omega)$, our result covers that of Galdi [8]. Furthermore, we do not need any redundant assumption such as $f\in L^{\frac{3}{2}}(\Omega)$. Hence, Theorem 1.2 seems to be a final answer to Leray's question on uniqueness of D-solutions for small data.

(ii) The case when $u^\infty=0$, such a uniqueness result as in Theorem 1.2 is known in more restrictive situations. For instance, Nakatsuka [15] treated the case $u^\infty=0$, and proved that for every $3<r<\infty$ there is a constant $\delta=\delta(r)>0$ such that if $\{u,p\}$ and $\{v,q\}$ with $\nabla u,\nabla v,p,q\in L^{\frac{3}{2},\infty}(\Omega)$ satisfy (E') and if

$$\|u\|_{L^{3,\infty}}\le\delta,\quad v\in L^3(\Omega)+L^r(\Omega),$$

then it holds that

$$\{u,p\}=\{v,q\}.$$

In his result, it is necessary to assume the smallness of one solution u and some redundant regularity on another solution v. It is still an open question whether any norm of solutions u of (N-S) with $u^\infty=0$ can be controlled by f. For details, we refer to Kim-Kozono [10].

2 Oseen equations.

In this section, we investigate the following Oseen equations.

$$(\text{Os}) \qquad \begin{cases} -\Delta v+u^\infty\cdot\nabla v+\nabla\pi=f & \text{in } \Omega,\\ \text{div } v=0 \quad \text{in } \Omega,\\ v=0 \quad \text{on } \partial\Omega,\\ v(x)\to 0 \quad \text{as } |x|\to\infty. \end{cases}$$

Let us introduce the two function spaces $\tilde H^{1,q}(\Omega)$ and $\tilde H^{2,q}(\Omega)$ defined by

$$\tilde H^{1,q}(\Omega)\equiv\{v\in L^{\frac{4q}{4-q}}(\Omega);\nabla v\in L^q(\Omega)\},\quad 1<q<4,$$
$$\tilde H^{2,q}(\Omega)\equiv\{v\in \tilde H^{1,\frac{4q}{4-q}}(\Omega);\nabla^2 v\in L^q(\Omega)\},\quad 1<q<2.$$

Then we have the following results on unique solvability of (Os).

Lemma 2.1 *Let* $u^\infty\neq 0$. *Assume that* $1<q_1,q_2<4$. *The solution* $\{v,\pi\}\in\tilde H^{1,q_1}(\Omega)+\tilde H^{1,q_2}(\Omega)\times L^1_{loc}(\Omega)$ *of (Os) is unique.*

Lemma 2.2 (i) *For* $f \in \dot{H}^{-1,q}(\Omega)$ *with* $\frac{3}{2} < q < 4$, *there exists a unique solution* $\{v, \pi\} \in \tilde{H}^{1,q}(\Omega) \times L^q(\Omega)$ *of (Os). Moreover, for every* $\frac{3}{2} < q < 3$ *and every* $M > 0$ *there is a constant* $C = C(q, M, \Omega)$ *such that if* $\{v, \pi\} \in \tilde{H}^{1,q}(\Omega) \times L^q(\Omega)$ *is a solution of (Os) with* $|u^\infty| \le M$, *then it holds that*

$$k_1 \|v\|_{L^{\frac{4q}{4-q}}} + \|\nabla v\|_{L^q} + \|\pi\|_{L^q} \le C\|f\|_{\dot{H}^{-1,q}},$$

where $k_1 \equiv min.\{1, |u^\infty|^{\frac{1}{4}}\}$.

(ii) *For every* $f \in L^q(\Omega)$ *with* $1 < q < 2$, *there exists a unique solution* $\{v, \pi\} \in \tilde{H}^{2,q}(\Omega) \times L^{q_*}(\Omega)$ *of (Os) with* $\nabla\pi \in L^q(\Omega)$, *where* $\frac{1}{q_*} = \frac{1}{q} - \frac{1}{3}$. *Moreover, for every* $1 < q < \frac{3}{2}$ *and every* $M > 0$ *there is a constant* $C = C(q, M, \Omega)$ *such that if* $\{v, \pi\} \in \tilde{H}^{1,q}(\Omega) \times L^q(\Omega)$ *is a solution of (Os) with* $|u^\infty| \le M$, *then it holds that*

$$k_2 \|v\|_{L^{\frac{2q}{2-q}}} + k_1 \|\nabla v\|_{L^{\frac{4q}{4-q}}} + \|\nabla^2 v\|_{L^q} + \|\pi\|_{L^{q_*}} + \|\pi\|_{L^q} \le C\|f\|_{L^q},$$

where $k_2 = k_1^2 \equiv min.\{1, |u^\infty|^{\frac{1}{2}}\}$.

3 Proof of Theorems.

The following lemma is based on Lemma 2.2 and plays a key role for the proof of Theorem 1.1.

Lemma 3.1 *Let* $u^\infty \neq 0$ *and* $f \in \dot{H}^{-1,2}(\Omega)$. *Let* u *be a D-solution of (N-S).*

(i) *If in addition* $f \in \dot{H}^{-1,2}(\Omega) \cap \dot{H}^{-1,q}(\Omega)$ *for* $\frac{4}{3} < q < 4$, *then it holds that*

$$\begin{aligned} &u - u^\infty \in L^{\frac{4q}{4-q}}(\Omega), \quad u^\infty \cdot \nabla u \in \dot{H}^{-1,q}(\Omega), \\ &\nabla u \in L^q(\Omega), \quad p - p_\infty \in L^q(\Omega) \quad \textit{for some constant } p_\infty. \end{aligned}$$

(ii) *If in addition* $f \in \dot{H}^{-1,2}(\Omega) \cap L^q(\Omega)$ *for* $1 < q < 2$, *then it holds that*

$$\begin{aligned} &u - u^\infty \in L^{\frac{2q}{2-q}}(\Omega), \quad \nabla u \in L^{\frac{4q}{4-q}}(\Omega) \cap L^{\frac{3q}{3-q}}(\Omega), \\ &p - p_\infty \in L^{\frac{3q}{3-q}}(\Omega) \quad \textit{for some constant } p_\infty, \\ &\nabla^2 u, \nabla p, u^\infty \cdot \nabla u \in L^q(\Omega). \end{aligned}$$

By taking $q = 2$ in this lemma, we have

Corollary 3.1 *Every D-solution* u *of (N-S) with* $u^\infty \neq 0$ *and* $f \in \dot{H}^{-1,2}(\Omega)$ *satisfies*

$$u - u^\infty \in L^4(\Omega), \quad u^\infty \cdot \nabla u \in \dot{H}^{-1,2}(\Omega), \quad p - p_\infty \in L^2(\Omega)$$

for some constant p_∞.

To deal with the nonlinear term, we need

Proposition 3.1 *Let* $v, w \in \dot{H}_0^{1,2}(\Omega) \cap L^4(\Omega)$.

(i) *If* $u \in L^4(\Omega)$ *with* div $u = 0$ *in* Ω, *then it holds that*

$$(u \cdot \nabla v, w) = -(u \cdot \nabla w, v).$$

(ii) *If* $u^\infty \cdot \nabla v \in \dot{H}^{-1,2}(\Omega)$ *and* $u^\infty \cdot \nabla w \in \dot{H}^{-1,2}(\Omega)$, *then it holds that*

$$\begin{aligned} &\langle u^\infty \cdot \nabla v, w\rangle = -\langle u^\infty \cdot \nabla w, v\rangle, \\ &\langle a \cdot \nabla v, w\rangle = -\langle a \cdot \nabla w, v\rangle \quad \textit{for all } a \in A(u^\infty). \end{aligned}$$

3.1 Proof of Theorem 1.1.

By Definition of D-solutions, we have

$$\begin{aligned}\langle f, \phi\rangle &= (\nabla u, \nabla \phi) + (u\cdot\nabla u, \phi) - (p, \text{div } \phi)\\ &= (\nabla u, \nabla \phi) + ((u-a)\cdot\nabla u, \phi) + \langle a\cdot \nabla u, \phi\rangle - (p - p_\infty, \text{div } \phi) \qquad (3.1)\end{aligned}$$

for all $\phi \in C_0^\infty(\Omega)$ Since $C_0^\infty(\Omega)$ is dense in $\dot{H}_0^{1,2}(\Omega)\cap L^4(\Omega)$, we have

$$\langle f, \phi\rangle = (\nabla u, \nabla \phi) + ((u-a)\cdot\nabla u, \phi) + \langle a\cdot \nabla u, \phi\rangle - (p - p_\infty, \text{div } \phi) \qquad (3.2)$$

for all $\phi \in \dot{H}_0^{1,2}(\Omega)\cap L^4(\Omega)$. By Corollary 3.1 it holds that $u - a = u - u^\infty + u^\infty - a \in \dot{H}_0^{1,2}(\Omega)\cap L^4(\Omega)$. Hence, taking $\phi = u - a$ in (3.2), we have

$$\langle f, u-a\rangle = (\nabla u, \nabla(u-a)) + ((u-a)\cdot\nabla u, u-a) + \langle a\cdot\nabla u, u-a\rangle. \qquad (3.3)$$

Furthermore by Proposition 3.1, it holds that

$$\begin{aligned}&((u-a)\cdot\nabla u, u-a) + \langle a\cdot\nabla u, u-a\rangle\\ =\ &((u-a)\cdot\nabla(u-a), u-a) + \langle a\cdot\nabla(u-a), u-a\rangle\\ &+((u-a)\cdot\nabla a, u-a) + \langle a\cdot\nabla a, u-a\rangle\\ =\ &(u\cdot\nabla a, u-a),\end{aligned}$$

from which and (3.3) we obtain

$$\|\nabla u\|_{L^2}^2 - (\nabla u, \nabla a) + (u\cdot\nabla a, u-a) = \langle f, u-a\rangle.$$

This proves (1.1).

Assume in addition that $f \in \dot{H}^{-1,2}(\Omega)\cap L^q(\Omega)$ for some $1 < q < 2$. By Lemma 3.1 (ii), we have

$$-\Delta u + u\cdot\nabla u + \nabla p = f \quad \text{a.e. in } \Omega.$$

Note that

$$a - u^\infty \in C_{0,\sigma}^\infty(\mathbb{R}^3), \quad a - u^\infty = 0 \quad \text{on } \partial\Omega.$$

By integration by parts, we have

$$\begin{aligned}(f, a-u^\infty) &= (-\Delta u + u\cdot\nabla u + \nabla p, a - u^\infty)\\ &= (-\text{div } (T(u,p), a-u^\infty) + (u\cdot\nabla u, a-u^\infty)\\ &= (\nabla u, \nabla a) + u^\infty\cdot\int_{\partial\Omega} T(u,p)\cdot\nu dS - (u\cdot\nabla a, u). \qquad (3.4)\end{aligned}$$

Addition of (3.4) and (1.1) yields that

$$\|\nabla u\|_{L^2}^2 + u^\infty\cdot\int_{\partial\Omega} T(u,p)\cdot\nu dS - (u\cdot\nabla a, a) = \langle f, u - u^\infty\rangle. \qquad (3.5)$$

Since supp ∇a is compact, we see easily

$$(u\cdot\nabla a, a) = 0,$$

from which and (3.5) we obtain the desired identity (1.2). This proves Theorem 1.1.

3.2 Proof of Theorem 1.2.

Step 1. We first show that there are constants $\delta_* = \delta_*(\Omega)$ and $C_*(\Omega) > 0$ such that if

$$\|f\|_{\dot{H}^{-1,2}} + |u^\infty| \le \delta_* |u^\infty|^{\frac{1}{2}}, \tag{3.6}$$

then every D-solution u of (N-S) satisfies

$$|u^\infty|^{\frac{1}{4}} \|u - a\|_{L^4} + \|\nabla u\|_{L^2} \le C_*(\|f\|_{\dot{H}^{-1,2}} + |u^\infty|) \tag{3.7}$$

for some $a \in A(u^\infty)$. Indeed, taking $0 < R_0 < R_1 < \infty$ and $a \in A(u^\infty)$ in such a way that

$$\Omega^c = \mathbb{R}^3 \setminus \Omega \subset B_{R_0}(0), \quad \text{supp } \nabla a \subset \{R_0 < |x| < R_1\}.$$

we have

$$\|a\|_{L^\infty} + \|\nabla a\|_{L^1 \cap L^\infty} \le C|u^\infty| \quad \text{with } C = C(\Omega). \tag{3.8}$$

By (1.1), we see that

$$\|\nabla u\|_{L^2}^2 = \langle f, u - a \rangle + (\nabla u, \nabla a) + (u \cdot \nabla a, u - a),$$

from which and (3.8) with the aid of the Young inequality it follows that

$$\|\nabla u\|_{L^2}^2 \le \left(\frac{1}{2} + C|u^\infty|\right) \|\nabla u\|_{L^2}^2 + C\|f\|_{\dot{H}^{-1,2}}^2 + C(|u^\infty|^2 + |u^\infty|^4).$$

Hence, under the assumption

$$|u^\infty| \le \delta_*^{(1)} \equiv \min.\{1, \frac{1}{4C}\}, \tag{3.9}$$

we have

$$\begin{aligned} \frac{1}{4}\|\nabla u\|_{L^2}^2 &\le C\|f\|_{\dot{H}^{-1,2}}^2 + C(|u^\infty|^2 + |u^\infty|^4) \\ &\le C(\|f\|_{\dot{H}^{-1,2}}^2 + |u^\infty|^2), \end{aligned}$$

which yields that

$$\|\nabla u\|_{L^2} \le C(\|f\|_{\dot{H}^{-1,2}} + |u^\infty|). \tag{3.10}$$

Next, we show the bound of $\|u - a\|_{L^4}$. Define $v = u - a$, and we have by (3.8) and (3.9) that

$$v \in \dot{H}_0^{1,2}(\Omega), \quad \|\nabla v\|_{L^2} \le C(\|f\|_{\dot{H}^{-1,2}} + |u^\infty|), \tag{3.11}$$

and that

$$\begin{cases} -\Delta v + u^\infty \cdot \nabla v + \nabla \pi = f - Q(v) & \text{in } \Omega, \\ \text{div } v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \\ v(x) \to 0 & \text{as } |x| \to \infty, \end{cases}$$

where

$$Q(v) \equiv v \cdot \nabla v + (a - u^\infty) \cdot \nabla v + v \cdot \nabla a - \Delta a + a \cdot \nabla a.$$

By (3.8) and (3.11), it holds that

$$\begin{aligned}
&\|v\cdot\nabla v\|_{L^{\frac{4}{3}}} \le \|v\|_{L^4}\|\nabla v\|_{L^2} \le C(\|f\|_{\dot H^{-1,2}} + |u^\infty|)\|v\|_{L^4}\\
&\|Q(v) - v\cdot\nabla v\|_{\dot H^{-1,2}}\\
&= \|(a-u^\infty)\cdot\nabla v + v\cdot\nabla a - \Delta a + a\cdot\nabla a\|_{\dot H^{-1,2}}\\
&\le C(\|\nabla v\|_{L^2} + |u^\infty|)\\
&\le C(\|f\|_{\dot H^{-1,2}} + |u^\infty|).
\end{aligned}$$

Hence, it follows from Lemma 2.1 and Lemma 2.2 with $q=2$ in (i) and with $q=\frac{4}{3}$ in (ii) that

$$\begin{aligned}
\|v\|_{L^4} &\le C\left(\frac{1}{k_1}\|f - Q(v) - v\cdot\nabla v\|_{\dot H^{-1,2}} + \frac{1}{k_2}\|v\cdot\nabla v\|_{L^{\frac{4}{3}}}\right)\\
&\le C\left(\frac{1}{k_1}(\|f\|_{\dot H^{-1,2}} + |u^\infty|) + \frac{1}{k_2}(\|f\|_{\dot H^{-1,2}} + |u^\infty|)\|v\|_{L^4}\right). \qquad (3.12)
\end{aligned}$$

Hence, under the assumption

$$\frac{1}{k_2}(\|f\|_{\dot H^{-1,2}} + |u^\infty|) \le \delta_* \equiv \min.\{\delta_*^{(1)}, \frac{1}{2C}\}, \qquad (3.13)$$

we have

$$\|u-a\|_{L^4} = \|v\|_{L^4} \le \frac{C}{k_1}(\|f\|_{\dot H^{-1,2}} + |u^\infty|). \qquad (3.14)$$

Since the assumption (3.13) necessarily implies the assumption (3.9), we see by (3.10) and (3.14) that if

$$\|f\|_{\dot H^{-1,2}} + |u^\infty| \le \delta_*|u^\infty|^{\frac{1}{2}},$$

then it holds that

$$|u^\infty|^{\frac{1}{4}}\|u-a\|_{L^4} + \|\nabla u\|_{L^4} \le (\|f\|_{\dot H^{-1,2}} + |u^\infty|),$$

which implies (3.7)

Step 2. We next show uniqueness. Let u_1 and u_2 be two D-solutions of (N-S). Define $v_1 = u_1 - a$ and $v_2 = u_2 - a$ with $a \in A(u^\infty)$ as in Step1. Then $v \equiv v_1 - v_2 = u_1 - u_2$ fulfills

$$\begin{cases}
-\Delta v + u^\infty\cdot\nabla v + \nabla\pi = -v_1\cdot\nabla v - v\cdot\nabla u_2 & \text{in } \Omega,\\
\operatorname{div} v = 0 \quad \text{in } \Omega,\\
v = 0 \quad \text{on } \partial\Omega,\\
v(x)\to 0 \quad \text{as } |x|\to\infty,
\end{cases}$$

Hence it follows from Lemmata 2.1 and 2.1 with

$$\begin{aligned}
&f = -v_1\cdot\nabla v = \operatorname{div}(v_1\otimes v) \quad \text{for } q=2 \text{ in (i)},\\
&f = -v\cdot\nabla u_2 \quad \text{for } q=\tfrac{4}{3} \text{ in (ii)}
\end{aligned}$$

that

$$\begin{aligned}\|v\|_{L^4} &\le C\left(\frac{1}{k_1}\|\mathrm{div}\,(v_1\otimes v)\|_{\dot{H}^{-1,2}} + \frac{1}{k_2}\|v\cdot\nabla u_2\|_{L^{\frac{4}{3}}}\right)\\ &\le C\left(\frac{1}{k_1}\|v_1\otimes v\|_{L^2} + \frac{1}{k_2}\|v\|_{L^4}\|\nabla u_2\|_{L^2}\right)\\ &\le C\left(\frac{1}{k_1}\|v_1\|_{L^4} + \frac{1}{k_2}\|\nabla u_2\|_{L^2}\right)\|v\|_{L^4}. \qquad (3.15)\end{aligned}$$

By Step1, under the assumption

$$\|f\|_{\dot{H}^{-1,2}} + |u^\infty| \le \delta_*|u^\infty|^{\frac{1}{2}},$$

we have

$$\|v_1\|_{L^4} \le \frac{C}{k_1}(\|f\|_{\dot{H}^{-1,2}} + |u^\infty|), \quad \|\nabla u_2\|_{L^2} \le C(\|f\|_{\dot{H}^{-1,2}} + |u^\infty|),$$

from which and (3.15) with $k_1^2 = k_2$ it follows that

$$\|v\|_{L^4} \le \frac{C}{k_2}(\|f\|_{\dot{H}^{-1,2}} + |u^\infty|)\|v\|_{L^4}. \qquad (3.16)$$

Now, define $\delta_1 = \delta_1(\Omega)$ so that

$$\delta_1 \equiv \min.\{\delta_*, \frac{1}{2C}\}.$$

Then under the assumption

$$\|f\|_{\dot{H}^{-1,2}} + |u^\infty| \le \delta_1|u^\infty|^{\frac{1}{2}},$$

it follows from (3.16) with the aid of the relation $k_2 = \min.\{1, |u^\infty|^{\frac{1}{2}}\}$ that

$$\|v\|_{L^4} \le 0,$$

which yields the desired uniqueness result. This completes the proof of Theorem 1.2.

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