数理解析研究所講究録
第 1875 巻 2014 年 84-108

Global classical solvability of an interface problem on the motion of two fluids

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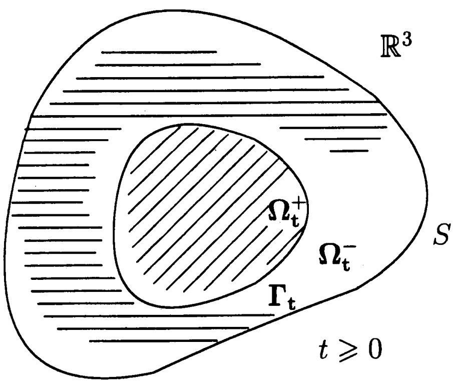
Abstract

We deal with the motion of two incompressible fluids in a container. The liquids are separated by an unknown interface on which surface tension is taken into account or ignored. Global existence theorem is proved in anisotropic Hölder classes for small smooth initial data and mass forces. We show in the case of strictly positive surface tension that fluid velocity decreases exponentially as time variable $t \to \infty$ and the interface between the liquids tends to a sphere $S^2_{R_0}(h_\infty) = \{|x - h_\infty| = R_0\}$ with a center h_∞ close to 0, the barycenter of the inner fluid at the initial moment.

The proof is based on a local existence theorem in Hölder spaces and on an exponential estimate of L_2-norms of local solutions. We follow to V. A. Solonnikov's scheme for proving global solvability of a problem on the motion of a single drop with free surface [1].

1 Statement of the problem



At the initial moment, let a fluid with the viscosity $\nu^+ > 0$ and the density $\rho^+ > 0$ occupy a bounded domain $\Omega_0^+ \subset \mathbb{R}^3$; we denote $\partial\Omega_0^+$ by Γ. And let a fluid with the viscosity $\nu^- > 0$ and the density $\rho^- > 0$ fill a domain Ω_0^- surrounding Ω_0^+. The boundary $S \equiv \partial(\Omega_0^+ \cup \Gamma \cup \Omega_0^-)$ is a given closed surface, $S \cap \Gamma = \emptyset$.

For every $t > 0$ we need to find $\Gamma_t = \partial\Omega_t^+$, velocity vector field $\mathbf{v}(x,t) = (v_1, v_2, v_3)$ and the function p, that is the deviation from the hydrostatic pressure P_0, which satisfy the following initial–boundary value problem:

$$
\begin{aligned}
&\mathcal{D}_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \nu^\pm\nabla^2\mathbf{v} + \frac{1}{\rho^\pm}\nabla p = \mathbf{f}, \ \ \nabla\cdot\mathbf{v} = 0 \quad \text{in} \ \ \Omega_t^\pm, \ t > 0,\\
&\mathbf{v}|_{t=0} = \mathbf{v}_0 \quad \text{in} \ \ \Omega_0^- \cup \Omega_0^+, \qquad \mathbf{v}|_S = 0, \qquad\qquad (1.1)\\
&[\mathbf{v}]|_{\Gamma_t} \equiv \lim_{\substack{x\to x_0\in\Gamma_t,\\ x\in\Omega_t^+}} \mathbf{v}(x) - \lim_{\substack{x\to x_0\in\Gamma_t,\\ x\in\Omega_t^-}} \mathbf{v}(x) = 0,\\
&[\mathbb{T}(\mathbf{v},p)\mathbf{n}]|_{\Gamma_t} = \sigma H\mathbf{n} \ \ \text{on} \ \Gamma_t. \qquad\qquad (1.2)
\end{aligned}
$$

Here $\mathcal{D}_t \equiv \frac{\partial}{\partial t}$, $\nabla \equiv (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$, $\mathbf{v}_0$ is the initial velocity, $\nu^\pm, \rho^\pm$ are the step functions of viscosity and density, $\mathbb{T}(\mathbf{v},p)$ is the stress tensor with the elements $T_{ik}(\mathbf{v},p) = -\delta_i^k p + \mu^\pm S_{ik}(\mathbf{v})$, $i,k = 1,2,3$, δ_i^k is the Kronecker symbol, $S_{ik}(\mathbf{v}) \equiv \partial v_i/\partial x_k + \partial v_k/\partial x_i$, are the components of the doubled strain tensor $\mathbb{S}(\mathbf{v})$; $\mu^\pm = \nu^\pm\rho^\pm$, $\sigma \geqslant 0$ is surface tension coefficient, $\mathbf{n}$ is the outward normal to Ω_t^+, $H(x,t)$ is twice the mean curvature of Γ_t ($H < 0$ at the points where Γ_t is convex towards Ω_t^-). A Cartesian coordinate system $\{x\}$ is introduced in $\mathbb{R}^3$. The centered dot denotes the Cartesian scalar product. We imply the summation from 1 to 3 with respect to repeated indexes. We mark the vectors and the vector spaces by boldface letters.

If $\sigma > 0$, we suppose the inner domain Ω_0^+ is close to a ball of its volume. In this case, we introduce a new pressure function $p_1 = p$ in Ω_t^+ and $p_1 = p + \sigma\frac{2}{R_0}$ in Ω_t^-. Then only last boundary condition (1.2) changes:

$$
[\mathbb{T}(\mathbf{v},p_1)\mathbf{n}]|_{\Gamma_t} = \sigma\Big(H + \frac{2}{R_0}\Big)\mathbf{n} \ \ \text{on} \ \Gamma_t. \tag{1.3}
$$

If $\sigma = 0$, we set $p_1 \equiv p$ in both domains.

We assume the liquids to be immiscible. A condition excluding the mass transportation through Γ_t implies that Γ_t consists of the points $x(\xi,t)$ which radius vector $\mathbf{x}(\xi,t)$ is a solution of the Cauchy problem

$$
\mathcal{D}_t\mathbf{x} = \mathbf{v}(x(\xi,t),t), \ \ \mathbf{x}(\xi,0) = \xi, \ \ \xi\in\Gamma, \ \ t > 0. \tag{1.4}
$$

Hence, $\Gamma_t = \{x(\xi,t)|\xi \in \Gamma\}$, $\Omega_t^{\pm} = \{x(\xi,t)|\xi \in \Omega_0^{\pm}\}$.

Condition (1.4) completes system (1.1), (1.3).

We transform the Eulerian coordinates $\{x\}$ into the Lagrangian ones $\{\xi\}$ by the formula

$$\mathbf{x}(\xi,t) = \xi + \int_0^t \mathbf{u}(\xi,\tau)d\tau \equiv \mathbf{X}_{\mathbf{u}}(\xi,t), \tag{1.5}$$

where $\mathbf{u}(\xi,t)$ is velocity vector field in the Lagrangian coordinates.

We separate boundary condition for stress tensor (1.3) onto the tangential and normal components. As a result , we arrive at the problem for $\mathbf{u}$, $q = p_1(X_{\mathbf{u}},t)$ with the given interface Γ. If the angle between $\mathbf{n}$ and the exterior normal $\mathbf{n}_0$ to Γ is acute, this system is equivalent to the following one:

$$\mathcal{D}_t\mathbf{u} - \nu^{\pm}\nabla_{\mathbf{u}}^2\mathbf{u} + \frac{1}{\rho^{\pm}}\nabla_{\mathbf{u}}q = \mathbf{f}(X_n,t), \quad \nabla_{\mathbf{u}}\cdot\mathbf{u} = 0 \ \text{ in } \ Q_T^{\pm} = \Omega_0^{\pm}\times(0,T),$$

$$\mathbf{u}\big|_{t=0} = \mathbf{v}_0 \ \text{ in } \ \Omega_0^-\cup\Omega_0^+, \quad \mathbf{u}\big|_{S_T} = 0, \tag{1.6}$$

$$[\mathbf{u}]\big|_{G_T} = 0, \quad [\mu^{\pm}\Pi_0\Pi\mathbb{S}_{\mathbf{u}}(\mathbf{u})\mathbf{n}]\big|_{G_T} = 0 \quad (G_T \equiv \Gamma\times(0,T)),$$

$$[\mathbf{n}_0\cdot\mathbb{T}_{\mathbf{u}}(\mathbf{u},q)\mathbf{n}]\big|_{G_T} = \sigma\Big(H(X_n) + \frac{2}{R_0}\Big)\mathbf{n}_0\cdot\mathbf{n} \quad \text{on } G_T. \tag{1.7}$$

Here $\nabla_{\mathbf{u}} = \mathbb{A}\nabla$, $\mathbb{A}$ is the matrix of co-factors A_{ij} to the elements $a_{ij}(\xi,t) = \delta_i^j + \int_0^t \frac{\partial u_i}{\partial \xi_j}\,dt'$ of the Jacobian matrix of transformation (4.2), $i,j = 1,2,3$; $\Pi\omega = \omega - \mathbf{n}(\mathbf{n}\cdot\omega)$, $\Pi_0\omega = \omega - \mathbf{n}_0(\mathbf{n}_0\cdot\omega)$ are projections of a vector ω onto the tangent planes to Γ_t and Γ, respectively;

$$\mathbb{T}_{\mathbf{u}}(\mathbf{w},q) = -\,q\mathbb{I} + \mu^{\pm}\mathbb{S}_{\mathbf{u}}(\mathbf{w}),$$

where $\mathbb{S}_{\mathbf{u}}(\mathbf{w})$ is tensor with the elements $S_{\mathbf{u}}(\mathbf{w})_{ij} = A_{ik}\partial w_j/\partial\xi_k + A_{jk}\partial w_i/\partial\xi_k$, the vector $\mathbf{n}$ is connected with $\mathbf{n}_0$ by the relation: $\mathbf{n} = \frac{\mathbb{A}\mathbf{n}_0}{|\mathbb{A}\mathbf{n}_0|}$.

We apply the well known relation for twice the mean curvature:

$$H\mathbf{n} = \Delta(t)\mathbf{x} \equiv \Delta(t)\mathbf{X}_{\mathbf{u}}, \tag{1.8}$$

where $\Delta(t)$ is the Beltrami–Laplace operator on Γ_t. In local coordinates $\{s_1,s_2\}$ on the surface Γ, it has the form:

$$\Delta(t) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial s_\alpha}g^{\alpha\beta}\sqrt{g}\frac{\partial}{\partial s_\beta} \equiv g^{\alpha\beta}\frac{\partial^2}{\partial s_\alpha\partial s_\beta} + h^\beta\frac{\partial}{\partial s_\beta},$$

where $\{g^{\alpha\beta}\}_{\alpha,\beta=1}^2$ is the inverse matrix to metric tensor matrix $\{g_{\alpha\beta}\}_{\alpha,\beta=1}^2$,

$$g_{\alpha\beta} = \frac{\partial\mathbf{X}_{\mathbf{u}}(\xi(s),t)}{\partial s_\alpha}\cdot\frac{\partial\mathbf{X}_{\mathbf{u}}(\xi(s),t)}{\partial s_\beta}, \quad g = \det\{g_{\alpha\beta}\}_{\alpha,\beta=1}^2, \quad h^\beta = \frac{\partial}{\sqrt{g}\partial s_\alpha}(g^{\alpha\beta}\sqrt{g}).$$

As a result, instead of (1.7) we arrive in (1.6) at the following interface condition:

$$[\mathbf{n}_0\cdot\mathbb{T}_{\mathbf{u}}(\mathbf{u},q)\mathbf{n}]|_{G_T}-\sigma\mathbf{n}_0\cdot\Delta(t)X_{\mathbf{u}}|_{G_T}=\sigma\frac{2}{R_0}\mathbf{n}_0\cdot\mathbf{n}\quad\text{on } G_T. \tag{1.9}$$

2 Auxiliary statements

We introduce some Hölder semi-norms.

Let Ω be a domain in $\mathbb{R}^n, n\in\mathbb{N}$; for $T>0$ we put $\Omega_T=\Omega\times(0,T)$; finally, let $\alpha\in(0,1)$.

We need the following semi-norm with $\alpha,\gamma\in(0,1)$:

$$|f|_{\Omega_T}^{(\gamma,1+\alpha)}=\langle f\rangle_{\Omega_T}^{(\gamma,1+\alpha)}+\langle f\rangle_{t,\Omega_T}^{(\frac{1+\alpha-\gamma}{2})},$$

where

$$\langle f\rangle_{\Omega_T}^{(\gamma,1+\alpha)}=\max_{t,\tau\in(0,T)}\max_{x,y\in\Omega}\frac{|f(x,t)-f(y,t)-f(x,\tau)+f(y,\tau)|}{|x-y|^{\gamma}|t-\tau|^{(1+\alpha-\gamma)/2}}.$$

It is known the estimate

$$\langle f\rangle_{\Omega_T}^{(\gamma,1+\alpha)}\leqslant c_1\langle f\rangle_{\Omega_T}^{(1+\alpha,\frac{1+\alpha}{2})}.$$

We consider that $f\in C^{(\gamma,1+\alpha)}(\Omega_T)$ if $|f|_{\Omega_T}+|f|_{\Omega_T}^{(\gamma,1+\alpha)}<\infty$.

Finally, if a function f has finite norm

$$|f|_{\Omega_T}^{(\gamma,\mu)}\equiv\langle f\rangle_{x,\Omega_T}^{(\gamma)}+|f|_{t,\Omega_T}^{(\mu)},\qquad\gamma\in(0,1),\quad\mu\in[0,1),$$

where

$$|f|_{t,\Omega_T}^{(\mu)}=\begin{cases}|f|_{\Omega_T}+\langle f\rangle_{t,\Omega_T}^{(\mu)} & \text{if } \mu>0,\\ |f|_{\Omega_T} & \text{if } \mu=0,\end{cases}$$

then it belongs to the Hölder space $C^{\gamma,\mu}(\Omega_T)$.

We suppose that a vector valued function is an element of a Hölder space if all its components belong to this space, and its norm is defined as the maximal norm of the components.

Let $T\in(0,\infty]$, $t,\tau>0$. We set $D_T\equiv\cup Q_T^{\pm}=Q_T^{-}\cup Q_T^{+}$ and $Q_T^{\pm}=\Omega^{\pm}\times(0,T)$, $Q_{(t,t+\tau)}^{\pm}=\Omega_t^{\pm}\times(t,t+\tau)$; $\Omega=\Omega_0^{-}\cup\overline{\Omega_0^{+}}\equiv\Omega_t^{-}\cup\overline{\Omega_t^{+}}$, $Q_T=\Omega\times(0,T)$, $Q_{(t,t+\tau)}=\Omega\times(t,t+\tau)$; $G_T=\Gamma\times(0,T)$.

$$|f|_{D_T}^{(k+\alpha,\frac{k+\alpha}{2})}\equiv|f|_{Q_T^-}^{(k+\alpha,\frac{k+\alpha}{2})}+|f|_{Q_T^+}^{(k+\alpha,\frac{k+\alpha}{2})},\qquad |f|_{\cup\Omega^{\pm}}^{(k+\alpha)}\equiv|f|_{\Omega^-}^{(k+\alpha)}+|f|_{\Omega^+}^{(k+\alpha)}.$$

Theorem 2.1. (Local existence) *Let* $\Gamma \in C^{3+\alpha}$, $\mathbf{f}, \mathcal{D}_x\mathbf{f} \in \mathbf{C}^{\alpha,\frac{1+\alpha-\gamma}{2}}(Q_T)$, $\mathbf{v}_0 \in \mathbf{C}^{2+\alpha}(\Omega_0^- \cup \Omega_0^+)$ *for some* $\alpha, \gamma \in (0,1)$, $T < \infty$. *Assume the compatibility conditions hold:*

$$\begin{gathered}\nabla \cdot \mathbf{v}_0 = 0, \quad [\mathbf{v}_0]\big|_\Gamma = 0, \quad [\mu^\pm \Pi_0 \mathbb{S}(\mathbf{v}_0)\mathbf{n}_0]\big|_\Gamma = 0,\\ \Big[\Pi_0\Big(\nu^\pm \nabla^2 \mathbf{v}_0 - \frac{1}{\rho^\pm}\nabla q_0\Big)\Big]\Big|_\Gamma = 0, \quad \mathbf{v}_0\big|_S = 0,\\ \Pi_S\Big(\mathbf{f}(\xi,0) - \frac{1}{\rho^-}\nabla q_0(\xi) + \nu^- \nabla^2 \mathbf{v}_0(\xi)\Big)\Big|_{\xi \in S} = 0;\end{gathered} \tag{2.1}$$

where $q_0(\xi) = p_1(\xi, 0)$ *is a solution of the following diffraction problem:*

$$\begin{aligned}&\frac{1}{\rho^\pm}\nabla^2 q_0(\xi) = \nabla \cdot (\mathbf{f}(\xi,0) - \mathcal{D}_t \mathbb{B}^*(\mathbf{v}_0)\mathbf{v}_0(\xi)), \quad \xi \in \Omega_0^- \cup \Omega_0^+,\\ &[q_0]|_\Gamma = \Big[2\mu^\pm \frac{\partial \mathbf{v}_0}{\partial \mathbf{n}_0} \cdot \mathbf{n}_0\Big]\Big|_\Gamma - \sigma\Big(H_0(\xi) + \frac{2}{R_0}\Big), \quad \xi \in \Gamma,\\ &\Big[\frac{1}{\rho^\pm}\frac{\partial q_0}{\partial \mathbf{n}_0}\Big]\Big|_\Gamma = [\nu^\pm \mathbf{n}_0 \cdot \nabla^2 \mathbf{v}_0]|_\Gamma \quad \Big(\frac{\partial}{\partial \mathbf{n}_0} = \mathbf{n}_0 \cdot \nabla\Big),\\ &\frac{1}{\rho^-}\frac{\partial q_0}{\partial \mathbf{n}_S}\Big|_S = \mathbf{n}_S \cdot \big(\nu^- \nabla^2 \mathbf{v}_0 + \mathbf{f}\big|_{t=0}\big)\Big|_S \quad \Big(\frac{\partial}{\partial \mathbf{n}_S} = \mathbf{n}_S \cdot \nabla\Big).\end{aligned} \tag{2.2}$$

(*Here* $H_0(\xi) = \mathbf{n}_0 \cdot \Delta(0)\xi\big|_\Gamma$ *is twice the mean curvature of* Γ; $\mathbb{B}^*$ *is the transpose of* $\mathbb{B} = \mathbb{A} - \mathbb{I}$, $\mathbb{I}$ *is the identity matrix*; $\Pi_S \mathbf{b} = \mathbf{b} - \mathbf{n}_S(\mathbf{n}_S \cdot \mathbf{b})$, $\mathbf{n}_S$ *is the outward normal to* S.)

Then for $\forall T < \infty$ $\exists$ *such* $\varepsilon(T)$ *that problem* (1.6), (1.9) *has a unique solution* $(\mathbf{u}, q)$: $\mathbf{u} \in \mathbf{C}^{2+\alpha, 1+\alpha/2}(D_T)$, $q \in C^{(\gamma, 1+\alpha)}(D_T)$, $\nabla q \in \mathbf{C}^{\alpha,\alpha/2}(D_T)$ *provided that*

$$|\mathbf{f}|_{Q_T}^{(\alpha,\frac{1+\alpha-\gamma}{2})} + |\mathcal{D}_x\mathbf{f}|_{Q_T}^{(\alpha,\frac{1+\alpha-\gamma}{2})} + |\mathbf{v}_0|_{\cup\Omega^\pm}^{(2+\alpha)} + \sigma\Big|H_0 + \frac{2}{R_0}\Big|_\Gamma^{(1+\alpha)} \leqslant \varepsilon(T), \tag{2.3}$$

q being unique up to a function of time. The interface Γ_t *is a surface of* $C^{3+\alpha}$. *Moreover, there holds the estimate*

$$\begin{aligned}&|\mathbf{u}|_{D_T}^{(2+\alpha,1+\alpha/2)} + |\nabla q|_{D_T}^{(\alpha,\alpha/2)} + |q|_{D_T}^{(\gamma,1+\alpha)}\\ &\qquad \leqslant c\Big\{|\mathbf{f}|_{Q_T}^{(\alpha,\frac{1+\alpha-\gamma}{2})} + |\mathcal{D}_x\mathbf{f}|_{Q_T}^{(\alpha,\frac{1+\alpha-\gamma}{2})} + |\mathbf{v}_0|_{\cup\Omega^\pm}^{(2+\alpha)} + \sigma\Big|H_0 + \frac{2}{R_0}\Big|_\Gamma^{(1+\alpha)}\Big\}.\end{aligned} \tag{2.4}$$

We note that local and global solvability of the problem governing the motion of two fluids without surface tension was obtained by the author in

[2]. In the case of strictly positive σ, Theorem 2.1 was proven in [3] for the case when S was absent and $\overline{\Omega^+ \cup \Omega^-}$ coincided with the whole space $\mathbb{R}^3$. This result was obtained in Hölder spaces with power-like weights at infinity but it is also valid in our case because the weighted spaces are equivalent to the ordinary Hölder spaces in bounded domains.

Remark 2.1. *If $\sigma = 0$, it is sufficiently to assume $\Gamma \in C^{2+\alpha}$, the interface Γ_t also will be of class $C^{2+\alpha}$.*

The proof of Theorem 2.1 is based on the solvability of the following linearized problem:

$$\begin{gathered}
\mathcal{D}_t \mathbf{w} - \nu^\pm \nabla_{\mathbf{u}}^2 \mathbf{w} + \frac{1}{\rho^\pm} \nabla_{\mathbf{u}} s = \mathbf{f}, \qquad \nabla_{\mathbf{u}} \cdot \mathbf{w} = r \quad \text{in } Q_T^\pm, \\
\mathbf{w}|_{t=0} = \mathbf{w}_0 \qquad \text{in } \Omega^- \cup \Omega^+, \\
[\mathbf{w}]|_{G_T} = 0, \quad \mathbf{w}|_S = 0, \qquad [\mu^\pm \Pi_0 \Pi \mathbb{S}_{\mathbf{u}}(\mathbf{w})\mathbf{n}]|_{G_T} = \Pi_0 \mathbf{d}, \\
[\mathbf{n}_0 \cdot \mathbb{T}_{\mathbf{u}}(\mathbf{w}, s)\mathbf{n}]|_{G_T} - \sigma \mathbf{n}_0 \cdot \Delta(t) \int_0^t \mathbf{w}\big|_\Gamma \, \mathrm{d}\tau = b + \sigma \int_0^t B \, \mathrm{d}\tau \quad \text{on } G_T.
\end{gathered} \tag{2.5}$$

Unique solvability for problem (2.5) was proved in any finite time interval.

Theorem 2.2. *Let $\alpha, \gamma \in (0,1)$, $\gamma < \alpha$, $T < \infty$. Assume that $\Gamma, S \in C^{2+\alpha}$ and that for $\mathbf{u} \in \mathbf{C}^{2+\alpha, 1+\alpha/2}(D_T)$, $[\mathbf{u}]|_{G_T} = 0$, we have*

$$(T + T^{\gamma/2}) |\mathbf{u}|_{D_T}^{(2+\alpha, 1+\alpha/2)} \leqslant \delta \tag{2.6}$$

for some sufficiently small $\delta > 0$. Moreover, we assume that the following four groups of conditions are fulfilled:

1) $\exists$ a vector $\mathbf{g} \in \mathbf{C}^{\alpha, \alpha/2}(D_T)$ and a tensor $\mathbb{G} = \{G_{ik}\}_{i,k=1}^3$ with $G_{ik} \in C^{(\gamma, 1+\alpha)}(D_T) \cap C^{\gamma, 0}(D_T)$ such that

$$\mathcal{D}_t r - \nabla_{\mathbf{u}} \cdot \mathbf{f} = \nabla \cdot \mathbf{g}, \quad g_i = \partial G_{ik} / \partial \xi_k, \quad i = 1,2,3,$$

(these equalities are understood in a weak sense) and, moreover,

$$[(\mathbf{g} + \mathbb{A}^* \mathbf{f}) \cdot \mathbf{n}_0]|_{G_T} = 0;$$

2) $\mathbf{f} \in \mathbf{C}^{\alpha, \alpha/2}(D_T)$, $r \in C^{1+\alpha, \frac{1+\alpha}{2}}(D_T)$, $\mathbf{w}_0 \in \mathbf{C}^{2+\alpha}(\Omega_0^- \cup \Omega_0^+)$, $\mathbf{d} \in \mathbf{C}^{1+\alpha, \frac{1+\alpha}{2}}(G_T)$, $b \in C^{(\gamma, 1+\alpha)}(G_T)$;

3) $\nabla\cdot\mathbf{w}_0(\xi)=r(\xi,0)=0,\quad \xi\in\Omega_0^-\cup\Omega_0^+,\quad [\mathbf{w}_0]|_\Gamma=0,\quad \mathbf{w}_0|_S=0,$

$$[\Pi_0\mathbb{T}(\mathbf{w}_0(\xi))\mathbf{n}_0]|_{\xi\in\Gamma}=\Pi_0\mathbf{d}(\xi,0),\quad \xi\in\Gamma,$$
$$\Big[\Pi_0\Big(\mathbf{f}(\xi,0)-\frac{1}{\rho^\pm}\nabla s(\xi,0)+\nu^\pm\nabla^2\mathbf{w}_0(\xi)\Big)\Big]\Big|_{\xi\in\Gamma}=0, \tag{2.7}$$
$$\Pi_S\Big(\mathbf{f}(\xi,0)-\frac{1}{\rho^-}\nabla s(\xi,0)+\nu^-\nabla^2\mathbf{w}_0(\xi)\Big)\Big|_{\xi\in S}=0;$$

4) $s_0(\xi)=s(\xi,0)$ *is a solution of the problem*

$$\frac{1}{\rho^\pm}\nabla^2 s_0(\xi)=\nabla\cdot(\mathcal{D}_t\mathbb{B}^*|_{t=0}\mathbf{w}_0(\xi)-\mathbf{g}(\xi,0))\quad\text{in}\quad \Omega_0^-\cup\Omega_0^+,$$
$$[s_0]|_\Gamma=\Big[2\mu^\pm\frac{\partial\mathbf{w}_0}{\partial\mathbf{n}_0}\cdot\mathbf{n}_0\Big]\Big|_\Gamma-b|_{t=0},$$
$$\Big[\frac{1}{\rho^\pm}\frac{\partial s_0}{\partial\mathbf{n}_0}\Big]\Big|_\Gamma=[\mathbf{n}_0\cdot(\mathbf{f}|_{t=0}+\nu^\pm\nabla^2\mathbf{w}_0)]|_\Gamma, \tag{2.8}$$
$$\frac{1}{\rho^-}\frac{\partial s_0}{\partial\mathbf{n}_S}\Big|_S=\mathbf{n}_S\cdot\big(\nu^-\nabla^2\mathbf{v}_0+\mathbf{f}\big|_{t=0}\big)\Big|_S.$$

Under all these assumptions, problem (2.5) *has a unique solution* $(\mathbf{w},s)$ *with the properties:* $\mathbf{w}\in\mathbf{C}^{2+\alpha,1+\alpha/2}(D_T)$, $s\in C^{(\gamma,1+\alpha)}(D_T)$, $\nabla s\in\mathbf{C}^{\alpha,\alpha/2}(D_T)$; *moreover, this solution satisfies the inequality*

$$\begin{aligned}&|\mathbf{w}|_{D_{t'}}^{(2+\alpha,1+\alpha/2)}+|\nabla s|_{D_{t'}}^{(\alpha,\alpha/2)}+[s]_{D_{t'}}^{(\gamma,1+\alpha)}\\ &\leqslant c_1(t')\Big\{|\mathbf{f}|_{D_{t'}}^{(\alpha,\alpha/2)}+|r|_{D_{t'}}^{(1+\alpha,\frac{1+\alpha}{2})}+|\mathbf{w}_0|_{\cup\Omega_0^\pm}^{(2+\alpha)}+|\mathbf{g}|_{D_{t'}}^{(\alpha,\alpha/2)}\\ &\quad+\langle\mathbb{G}\rangle_{D_{t'}}^{(\gamma,1+\alpha)}+|\mathbb{G}|_{D_{t'}}^{(\gamma,0)}+|\mathbf{d}|_{G_{t'}}^{(1+\alpha,\frac{1+\alpha}{2})}+|b|_{G_{t'}}+\langle b\rangle_{G_{t'}}^{(\gamma,1+\alpha)}\\ &\quad+|\nabla_\Gamma b|_{G_{t'}}^{(\alpha,\alpha/2)}+\sigma|B|_{G_{t'}}^{(\alpha,\alpha/2)}+P_{t'}[\mathbf{u}]|\mathbf{w}_0|_{\cup\Omega_0^\pm}^{(1)}\Big\},\end{aligned} \tag{2.9}$$

where $c_1(t')$ *is a monotone nondecreasing function of* $t'\leqslant T$, *and*

$$P_t[\mathbf{u}]=t^{\frac{1-\alpha}{2}}|\nabla\mathbf{u}|_{D_t}+|\nabla\mathbf{u}|_{D_t}^{(\alpha,\alpha/2)}.$$

3 Global solvability of problem (1.1), (1.3), (1.4) without surface tension

Theorem 3.1. (Global existence theorem) *Let* $\alpha,\gamma\in(0,1)$. *Assume that* $\Gamma\in C^{2+\alpha}$ *and* $\mathbf{v}_0\in C^{2+\alpha}(\Omega_0^-\cup\Omega_0^+)$, $\mathbf{f},\nabla\mathbf{f}\in C^{\alpha,\frac{1+\alpha-\gamma}{2}}(Q_\infty)$ *satisfy compatibility conditions* (2.1), *where* $q_0=p_0(\xi)\equiv p_1(\xi,0)$ *being a solution of diffraction problem* (2.2) *with* $\sigma=0$.

Moreover, we suppose that the data are small enough, i.e.

$$|\mathbf{v}_0|^{(2+\alpha)}_{\cup\Omega_0^\pm} + |\mathrm{e}^{bt}\mathbf{f}|^{(\alpha,\frac{1+\alpha-\gamma}{2})}_{Q_\infty} + |\mathrm{e}^{bt}\nabla\mathbf{f}|^{(\alpha,\frac{1+\alpha-\gamma}{2})}_{Q_\infty} + \int_0^\infty \|\mathrm{e}^{bt}\mathbf{f}\|_{2,\Omega}\,\mathrm{d}t \leqslant \varepsilon \ll 1, \quad (3.1)$$

where $b = \min\{\nu^+,\nu^-\}/(2c_0)$ *with the constant* c_0 *from inequality* (3.3).

Then problem (1.1), (1.3), (1.4) *with* $\sigma = 0$ *is uniquely solvable on infinite time interval* $t > 0$. *The solution* $(\mathbf{v}, p)$ *has the properties:* $\mathbf{v} \in \mathbf{C}^{2+\alpha,1+\alpha/2}$, $p \in C^{(\gamma,1+\alpha)}$, $\nabla p \in \mathbf{C}^{\alpha,\alpha/2}$, *the surface* Γ_t *being from* $C^{2+\alpha}$*-class. It means that for every* $t_0 \in (0,\infty)$ *the solution* $(\mathbf{u}, q)$ *and its derivatives in Lagrangian coordinates belong to the corresponding spaces over* $D_T \equiv \cup Q^\pm_{(t_0,t_0+\tau)}$ *for a small enough time interval* $(t_0, t_0+\tau)$ *and*

$$\begin{aligned} N_{(t_0-\tau_0,t_0)}[\mathbf{v},p] \;&\leqslant c(\delta,\tau_0)\mathrm{e}^{-bt_0}\Big\{|\mathbf{v}_0|^{(2+\alpha)}_{\cup\Omega_0^\pm} + \int_0^\infty \mathrm{e}^{b\tau}\|\mathbf{f}(\cdot,\tau)\|_{2,\Omega}\,\mathrm{d}\tau \\ &+|\mathrm{e}^{bt}\mathbf{f}|^{(\alpha,\frac{1+\alpha-\gamma}{2})}_{Q_\infty} + |\mathrm{e}^{bt}\nabla\mathbf{f}|^{(\alpha,\frac{1+\alpha-\gamma}{2})}_{Q_\infty}\Big\} \quad (3.2)\end{aligned}$$

We have introduced the notation: $\|\cdot\|_{2,\Omega} = \|\cdot\|_{L_2(\Omega)}$. We note that $\partial\Omega = S$ and for $\mathbf{v}$ in Ω, the Korn inequality

$$\|\mathbf{v}\|_{W_2^1(\Omega)} \leqslant c_0\|\mathbb{S}(\mathbf{v})\|_{2,\Omega} \quad (3.3)$$

holds because $\mathbf{v}\big|_S = 0$ (see [4]), $\|\mathbf{v}\|_{W_2^1(\Omega)}$ coinciding with $\|\mathbf{v}\|_{W_2^1(\Omega_t^-\cup\Omega_t^+)}$ due to $[\mathbf{v}]\big|_{\Gamma_t} = 0$.

Proposition 3.1. *Assume that a classical solution of problem* (1.1), (1.2) *with* $\sigma = 0$ *is defined in* $[0,T]$ *and* v_0 *satisfies compatibility conditions* (2.1). *Let* $\mathbf{f}(\cdot,\tau) \in L_2(\Omega)$ *and* $\int_0^\infty \|\mathrm{e}^{b\tau}\mathbf{f}(\cdot,\tau)\|_{2,\Omega}\,\mathrm{d}\tau < \infty$.

Then

$$\|\mathbf{v}(\cdot,t)\|_{2,\Omega} \leqslant \mathrm{e}^{-bt}\Big\{\|\mathbf{v}_0\|_{2,\Omega} + \int_0^t \|\mathrm{e}^{b\tau}\mathbf{f}(\cdot,\tau)\|_{2,\Omega}\,\mathrm{d}\tau\Big\}, \quad t \in (0,T], \quad (3.4)$$

$$\int_0^\infty \|\mathbf{v}(\cdot,\tau)\|_{2,\Omega}\,\mathrm{d}\tau \leqslant c\Big\{\|\mathbf{v}_0\|_{2,\Omega} + \int_0^\infty \|\mathbf{f}(\cdot,\tau)\|_{2,\Omega}\,\mathrm{d}\tau\Big\}. \quad (3.5)$$

Proof. We multiply the 1st equation in (1.6) by $\mathbf{v}$ and integrate by parts over $\Omega_t^- \cup \Omega_t^+$.

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}\|^2_{2,\Omega} + \Big\|\sqrt{\frac{\nu^\pm}{2}}\mathbb{S}(\mathbf{v})\Big\|^2_{2,\Omega} = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x.$$

If we make use of the Korn and Hölder inequalities and divide by $\|\mathbf{v}\|_{2,\Omega}$, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}\|_{2,\Omega}+b\|\mathbf{v}\|_{2,\Omega}\leqslant\|\mathbf{f}\|_{2,\Omega}$$

with $b=\min\{\nu^+,\nu^-\}/(2c_0)$. By the Gronwall lemma,

$$\|\mathbf{v}\|_{2,\Omega}\leqslant \mathrm{e}^{-bt}\|\mathbf{v}_0\|_{2,\Omega}+\int_0^t \mathrm{e}^{-b(t-\tau)}\|\mathbf{f}(\cdot,\tau)\|_{2,\Omega}\,\mathrm{d}\tau\leqslant \mathrm{e}^{-bt}\varepsilon$$

which implies (3.4). Next, we integrate over $t\in(0,\infty)$ and apply the Fubini theorem to obtain (3.5):

$$\begin{aligned}\int_0^\infty\|\mathbf{v}\|_{2,\Omega}\,\mathrm{d}t &\leqslant \frac{1}{b}\|\mathbf{v}_0\|_{2,\Omega}+\int_0^\infty\int_0^t \mathrm{e}^{-b(t-\tau)}\|\mathbf{f}(\cdot,\tau)\|_{2,\Omega}\,\mathrm{d}\tau\,\mathrm{d}t\\ &\leqslant \frac{1}{b}\|\mathbf{v}_0\|_{2,\Omega}+\int_0^\infty\int_\tau^\infty \mathrm{e}^{-b(t-\tau)}\|\mathbf{f}(\cdot,\tau)\|_{2,\Omega}\,\mathrm{d}t\,\mathrm{d}\tau\\ &\leqslant \frac{1}{b}\Big\{\|\mathbf{v}_0\|_{2,\Omega}+\int_0^\infty\|\mathbf{f}\|_{2,\Omega}\,\mathrm{d}\tau\Big\}.\end{aligned}$$

□

We cite now two lemmas from [2].

Lemma 3.1. *Let $u\in C^{0,\frac{1+\alpha}{2}}(\cup Q^{\pm}_{T_0})$, $T_0>0$, $0<r<\sqrt{T_0}$.*
Then u satisfies the inequality

$$\langle u\rangle^{(\frac{1+\alpha-\gamma}{2})}_{t,\cup Q^{\pm}_{T_0}}\leqslant 2r^\gamma\langle u\rangle^{(\frac{1+\alpha}{2})}_{t,\cup Q^{\pm}_{T_0}}+cr^{\gamma-\alpha-\frac{9}{2}}\int_0^{T_0}\|u\|_{2,\Omega}\,\mathrm{d}\tau. \tag{3.6}$$

Lemma 3.2. *For an arbitrary $u\in C^{2+\alpha,1+\frac{\alpha}{2}}(\cup Q^{\pm}_{T_0})$, $0<r<diam\{\Omega\}$, the inequality*

$$\langle u\rangle^{(\alpha,\frac{\alpha}{2})}_{\cup Q^{\pm}_{T_0}}\leqslant 2r^2\langle u\rangle^{(2+\alpha,1+\frac{\alpha}{2})}_{\cup Q^{\pm}_{T_0}}+cr^{-\alpha-\frac{7}{2}}\int_0^{T_0}\|u\|_{2,\Omega}\,\mathrm{d}\tau \tag{3.7}$$

holds.

Proposition 3.2. *Let a solution $(\mathbf{v},p)$ of problem (1.1), (1.2) with $\sigma=0$ exist on the interval $(0,T]$ and*

$$N_{(0,T)}[\mathbf{v},p]\equiv|\mathbf{u}^0|^{(2+\alpha,1+\alpha/2)}_{\cup Q^{\pm}_T}+|\nabla q^0|^{(\alpha,\alpha/2)}_{\cup Q^{\pm}_T}+[q^0]^{(\gamma,1+\alpha)}_{\cup Q^{\pm}_T}\leqslant\mu \tag{3.8}$$

hold. Here $(\mathbf{u}^0,q^0)$ is the solution of problem (1.1), (1.2) in Lagrangian coordinates.

Then

$$N_{(t_0-\tau_0,t_0)}[\mathbf{v},p] \leqslant c(\delta,\tau_0)\left\{|\mathbf{f}|^{(\alpha,\frac{1+\alpha-\gamma}{2})}_{\cup Q'_0}+|\nabla\mathbf{f}|^{(\alpha,\frac{1+\alpha-\gamma}{2})}_{\cup Q'_0}+\int_{t_0-2\tau_0}^{t_0}\|\mathbf{v}(\cdot,\tau)\|_{2,\Omega}\,\mathrm{d}\tau\right\},$$
$$\forall t_0\in(0,T], \quad (3.9)$$

where $\tau_0\in(0,t_0/2)$, τ_0 *depends on* μ, δ *is the value from* (3.13), $c(\delta,\tau_0)$ *is a non-decreasing function. Here* $\cup Q'_\beta=\cup\, Q^{\pm}_{(t_0-2\tau_0+\beta,t_0)}$.

Proof. We fixe arbitrary $t_0\in(0,T]$. Let $\tau_0\in(0,t_0/2)$ and $\eta_\lambda(t)$ be a smooth monotone function of t with $\lambda\in(0,\tau_0]$ such that

$$\eta_\lambda(t)=\begin{cases}0 & \text{if} \quad t\leqslant t_0-2\tau_0+\lambda/2,\\ 1 & \text{if} \quad t\geqslant t_0-2\tau_0+\lambda,\end{cases}$$

and for $\dot\eta_\lambda(t)\equiv\frac{\mathrm{d}\eta_\lambda(t)}{\mathrm{d}t}$ the inequalities

$$\left|\dot\eta_\lambda(t)\right|_{\mathbb{R}}\leqslant c\lambda^{-1},\quad \langle\dot\eta_\lambda(t)\rangle^{(\alpha/2)}_{\mathbb{R}}\leqslant c\lambda^{-1-\alpha/2}$$

hold.

We consider the functions $\mathbf{w}=\mathbf{v}\eta_\lambda$, $s=p\eta_\lambda$ which satisfy the system

$$\begin{aligned}
&\mathcal{D}_t\mathbf{w}+(\mathbf{v}\cdot\nabla)\mathbf{w}-\nu^{\pm}\nabla^2\mathbf{w}+\frac{1}{\rho^{\pm}}\nabla s=\mathbf{f}\eta_\lambda+\mathbf{v}\dot\eta_\lambda,\\
&\nabla\cdot\mathbf{w}=0 \quad\text{in}\quad \Omega_t^-\cup\Omega_t^+,\quad t>t_0-2\tau_0,\\
&\mathbf{w}|_{t=t_0-2\tau_0}=0 \quad\text{in}\quad \Omega^-_{t_0-2\tau_0}\cup\Omega^+_{t_0-2\tau_0},\\
&[\mathbf{w}]|_{\Gamma_t}=0,\quad [\mathbb{T}(\mathbf{w},s)\mathbf{n}]|_{\Gamma_t}=0,\quad \mathbf{w}|_S=0.
\end{aligned}\qquad(3.10)$$

Introduce the Lagrangian coordinates by the formula

$$\mathbf{x}=\xi+\int_{t_0-2\tau_0}^{t}\mathbf{u}(\xi,\tau)\,\mathrm{d}\tau\equiv\mathbf{X}(\xi,t),\quad t>t_0-2\tau_0,\qquad(3.11)$$

here $\mathbf{u}(\xi,t)=\mathbf{v}(X(\xi,t),t)$. Then we can rewrite (3.10) in the form

$$\begin{aligned}
&\mathcal{D}_t\mathbf{w}-\nu^{\pm}\nabla^2_{\mathbf{u}}\mathbf{w}+\frac{1}{\rho^{\pm}}\nabla_{\mathbf{u}}s=\mathbf{f}(X,t)\eta_\lambda+\mathbf{u}\dot\eta_\lambda,\\
&\nabla_{\mathbf{u}}\cdot\mathbf{w}=0 \qquad\text{in}\quad\cup\Omega',\quad t>t_0-2\tau_0,\\
&\mathbf{w}|_{t=t_0-2\tau_0}=0 \qquad\text{in}\quad\cup\Omega',\\
&[\mathbf{w}]|_{\Gamma'}=0,\quad \mathbf{w}|_{S_T}=0,\\
&[\Pi_0\Pi\mathbb{T}_{\mathbf{u}}(\mathbf{w})\mathbf{n}]|_{\Gamma'}=0,\qquad [\mathbf{n}_0\cdot\mathbb{T}_{\mathbf{u}}(\mathbf{w},s)\mathbf{n}]|_{\Gamma'}=0.
\end{aligned}\qquad(3.12)$$

Here $\cup\Omega' = \Omega^-_{t_0-2\tau_0} \cup \Omega^+_{t_0-2\tau_0}$, $\Gamma' = \Gamma_{t_0-2\tau_0}$, $\mathbf{n}_0$ is the outward normal to Γ', Π_0 and Π are the projections onto the tangential planes to Γ' and to Γ_t, respectively. Other notation, for example $\nabla_{\mathbf{u}}$, also corresponds the transform (3.11). The functions $\mathbf{w}$, s, $\mathbf{f}$ in the Lagrangian coordinates are denoted by the same letters.

In order to apply Theorem 2.2 (existence theorem for the linearized problem) to problem (3.12), we need to verify the hypotheses of it. To this end, we choose τ_0 so small that inequality (2.6) holds. It is enough to take τ_0 such that

$$(2\tau_0 + (2\tau_0)^{\gamma/2})\mu \leqslant \delta. \tag{3.13}$$

Since $\nabla_{\mathbf{u}} \cdot \mathbf{w} = 0$, hypothesis 1)

$$-\nabla_{\mathbf{u}} \cdot (\mathbf{f}(X,t)\eta_\lambda + \mathbf{u}\dot{\eta}_\lambda) = \nabla \cdot \mathbf{g}$$

is satisfied with $\mathbf{g} = -\mathbb{A}^*(\mathbf{f}(X,t)\eta_\lambda + \mathbf{u}\dot{\eta}_\lambda)$ due to $\nabla_{\mathbf{u}} \equiv \mathbb{A}\nabla = \nabla\mathbb{A}^*$. It is evident that

$$[(\mathbf{g} + \mathbb{A}^*(\mathbf{f}(X,t)\eta_\lambda + \mathbf{u}\dot{\eta}_\lambda)) \cdot \mathbf{n}_0]|_{\Gamma'} = 0.$$

As tensor $\mathbb{G} = \{G_{ik}\}_{i,k=1}^3$, we take the potential

$$\mathbb{G}(\xi,t) = \nabla \int_{\mathbb{R}^3} \mathcal{E}(\xi,y)\mathbb{A}^*(\mathbf{f}(X(y,t),t)\eta_\lambda + \mathbf{u}\dot{\eta}_\lambda)dy,$$

where $\mathcal{E}(x,y) = \frac{-1}{4\pi|x-y|}$ is the fundamental solution of the Laplace equation in $\mathbb{R}^3$, $\mathbf{f}$ and $\mathbf{u}$ are extended with preservation of class in the whole space $\mathbb{R}^3$ and vanish at infinity.

Condition 2) is fulfilled because $\mathbf{u} \in \mathbf{C}^{\alpha,\alpha/2}(D_T)$.

As to 3), 4), all the functions in (2.7), (2.8) with $\Omega^\pm = \Omega^\pm_{t_0-2\tau_0}$ equal zero at $t = t_0 - 2\tau_0$. Hence, by (2.9)

$$\begin{aligned}
N_{(t_0-2\tau_0+\lambda,t_0)}[\mathbf{u},q] &\leqslant N_{(t_0-2\tau_0,t_0)}[\mathbf{w},s] \\
&\leqslant c_1(2\tau_0)\Big\{ |\mathbf{f}(X,t)\eta_\lambda|^{(\alpha,\alpha/2)}_{\cup Q_0'} + |\mathbf{u}\eta_\lambda'|^{(\alpha,\frac{\alpha}{2})}_{\cup Q_0'} \\
&\qquad + |\mathbf{g}|^{(\alpha,\alpha/2)}_{\cup Q_0'} + [\mathbb{G}]^{(\gamma,1+\alpha)}_{\cup Q_0'} + |\mathbb{G}|^{(\gamma,0)}_{\cup Q_0'} \Big\}.
\end{aligned}$$

We can estimate the Hölder norm of the composite function $\mathbf{f}(X(\xi,t),t)$ as follows:

$$|\mathbf{f}(X,t)|^{(\alpha,\alpha/2)}_{\cup Q_0'} \leqslant |\mathbf{f}|^{(\alpha,\alpha/2)}_{\cup Q_0'} + |\nabla\mathbf{f}|_{\cup Q_0'}(2\tau_0|\mathbf{u}|^{(\alpha)}_{x,\cup Q_0'} + (2\tau_0)^{1-\alpha/2}|\mathbf{u}|_{\cup Q_0'})$$

since

$$\mathbf{f}(X_{\mathbf{u}},t)-\mathbf{f}(X_{\mathbf{u}'},t)=\sum_{k=1}^{3}\int_0^1 \mathbf{f}_{x_k}(X_{\mathbf{u}_z},t)\,\mathrm{d}z\int_{t_0-2\tau_0}^{t}(u_k-u_k')\,\mathrm{d}\tau.$$

For $\lambda<1$, we conclude

$$\begin{aligned} N_{(t_0-2\tau_0+\lambda,t_0)}[\mathbf{u},q] \;&\leqslant\; c_2(1+\delta)\Big\{\frac{1}{\lambda^{\frac{\alpha}{2}}}|\mathbf{f}|^{(\alpha)}_{x,\cup Q_0'}+\frac{1}{\lambda^{\frac{1+\alpha-\gamma}{2}}}\big(|\mathbf{f}|^{(\frac{1+\alpha-\gamma}{2})}_{t,\cup Q_0'}+|\nabla\mathbf{f}|_{\cup Q_0'}\big)\\ &+\frac{1}{\lambda}\big(\langle\mathbf{u}\rangle^{(\alpha,\frac{\alpha}{2})}_{\cup Q'_{\lambda/2}}+\langle\mathbf{u}\rangle^{(\frac{1+\alpha-\gamma}{2})}_{t,\cup Q'_{\lambda/2}}\big)+\frac{1}{\lambda^{\frac{3+\alpha-\gamma}{2}}}|\mathbf{u}|_{\cup Q'_{\lambda/2}}\Big\}. \end{aligned}\tag{3.14}$$

We take now $r_1=(\varepsilon\lambda)^{1/\gamma}$ and $r_2=(\varepsilon\lambda)^{1/2}$ in estimates (3.6), (3.7), respectively, and evaluate $|\mathbf{u}|_{\cup Q'_{\lambda/2}}$ in (3.14) by a way similar to Lemmas 3.1, 3.2. As a result, we obtain

$$\begin{aligned} N_{(t_0-2\tau_0+\lambda,t_0)}[\mathbf{u},q]\leqslant c_3(\delta)\Big\{&\varepsilon N_{(t_0-2\tau_0+\lambda/2,t_0)}[\mathbf{u},q]+\frac{1}{\lambda^{\frac{\alpha}{2}}}|\mathbf{f}|^{(\alpha)}_{x,\cup Q_0'}\\ &+\frac{1}{\lambda^{\frac{1+\alpha-\gamma}{2}}}\big(|\mathbf{f}|^{(\frac{1+\alpha-\gamma}{2})}_{t,\cup Q_0'}+|\nabla\mathbf{f}|_{\cup Q_0'}\big)+c(\varepsilon)\lambda^{-\varkappa}\int_{t_0-2\tau_0}^{t_0}\|\mathbf{u}(\cdot,\tau)\|_{2,\Omega}\,\mathrm{d}\tau\Big\}, \end{aligned}\tag{3.15}$$

here $\varkappa=\max\big\{\frac{9}{2\gamma}+\frac{\alpha}{\gamma},\frac{11}{4}+\frac{\alpha}{2},\frac{3+\alpha-\gamma}{2}\big(1+\frac{7}{2(1+\alpha)}\big)\big\}=\frac{9}{2\gamma}+\frac{\alpha}{\gamma}$.

Let us introduce the function

$$\Phi(\lambda)=\lambda^{\varkappa}N_{(t_0-2\tau_0+\lambda,t_0)}[\mathbf{u},q].$$

Then we can rewrite (3.15) as follows:

$$\Phi(\lambda)\leqslant c_4\varepsilon\Phi(\lambda/2)+K,\tag{3.16}$$

where $c_4=c_3(\delta)2^{\varkappa}$,

$$K=c_3(\delta)\Big\{|\mathbf{f}|^{(\alpha,\frac{1+\alpha-\gamma}{2})}_{\cup Q_0'}+|\nabla\mathbf{f}|_{\cup Q_0'}+c(\varepsilon)\int_{t_0-2\tau_0}^{t_0}\|\mathbf{u}(\cdot,\tau)\|_{2,\Omega}\,\mathrm{d}\tau\Big\}.$$

We set $\varepsilon=\frac{1}{2c_4}$ in (3.16). By iterations with $\lambda/2,\ldots,\lambda/2^k$, we deduce from inequality (3.16) in the limit $k\to\infty$ that

$$\Phi(\lambda)\leqslant 2K.$$

This inequality with $\lambda=\tau_0$ implies (3.9). $\square$

Now we can prove Theorem 3.1 (global existence theorem for $\sigma = 0$).

Proof. By Theorem 2.1 (local existence theorem), we have a solution $(\mathbf{v}, p)$ on an interval $(0, T_0]$. We can take so small data in estimate (2.3) that value T_0 will be greater than 1. The solution norm satisfies the inequality

$$N_{(0,T_0)}[\mathbf{v}, p] \leqslant \mu$$

with some $\mu > 0$. Then by Proposition 3.2, there exists $\tau_0 < T_0/2$ such that (3.13) is satisfied and for $(\mathbf{v}, p)$, T_0 estimate (3.9) holds. Together with (3.4), it implies that

$$\begin{aligned} N_{(t_0-\tau_0,t_0)}[\mathbf{v}, p] &\leqslant c_5(\delta, \tau_0)\mathrm{e}^{-bt_0}\Big\{ \|\mathbf{v}_0\|_{2,\Omega} + \int_0^\infty \mathrm{e}^{b\tau}\|\mathbf{f}(\cdot,\tau)\|_{2,\Omega}\,\mathrm{d}\tau \\ &\qquad + |\mathrm{e}^{bt}\mathbf{f}|_{Q_\infty}^{(\alpha,\frac{1+\alpha-\gamma}{2})} + |\mathrm{e}^{bt}\nabla\mathbf{f}|_{Q_\infty}^{(\alpha,\frac{1+\alpha-\gamma}{2})}\Big\} \\ &\leqslant c_5(\delta,\tau_0)\mathrm{e}^{-bt_0}\big(1 + |\Omega|^{1/2}\big)\varepsilon, \end{aligned} \tag{3.17}$$

where $|\Omega|$ is the measure of Ω, $t_0 \in (2\tau_0, T_0]$.

Thus,

$$|\mathbf{v}(\cdot, T_0)|_{\cup\Omega_{T_0}^{\pm}}^{(2+\alpha)} \leqslant \mu.$$

Now we apply Theorem 2.1 (local existence theorem) again and obtain a solution in an interval $(T_0, T_0 + T_1]$ corresponding to the initial data $\mathbf{v}(\cdot, T_0)$. Note that

$$N_{(T_0,T_0+T_1)}[\mathbf{v}, p] \leqslant \mu_1.$$

There exists $\tau_1 < T_1/2$ such that (3.13) holds for τ_1, μ_1, hence,

$$\begin{aligned} N_{(T_0+T_1-\tau_1,T_0+T_1)}[\mathbf{v}, p] &\leqslant c(\delta,\tau_1)\Big\{ |\mathbf{f}|_{Q_{(T_0+T_1-2\tau_1,T_0+T_1)}}^{(\alpha,\frac{1+\alpha-\gamma}{2})} + |\nabla\mathbf{f}|_{Q_{(T_0+T_1-2\tau_1,T_0+T_1)}}^{(\alpha,\frac{1+\alpha-\gamma}{2})} \\ &\qquad + \int_{T_0+T_1-2\tau_1}^{T_0+T_1} \|\mathbf{v}(\cdot,\tau)\|_{2,\Omega}\,\mathrm{d}\tau \Big\} \\ &\leqslant c_5(\delta,\tau_1)\mathrm{e}^{-b(T_0+T_1-2\tau_1)}\Big\{ \|\mathbf{v}_0\|_{2,\Omega} + \int_0^\infty \mathrm{e}^{b\tau}\|\mathbf{f}(\cdot,\tau)\|_{2,\Omega}\,\mathrm{d}\tau \\ &\qquad + |\mathrm{e}^{bt}\mathbf{f}|_{Q_\infty}^{(\alpha,\frac{1+\alpha-\gamma}{2})} + |\mathrm{e}^{bt}\nabla\mathbf{f}|_{Q_\infty}^{(\alpha,\frac{1+\alpha-\gamma}{2})}\Big\} \\ &\leqslant c_6(\delta,\tau_1)\mathrm{e}^{-b(T_0+T_1)}\big(1 + |\Omega|^{1/2}\big)\varepsilon. \end{aligned} \tag{3.18}$$

We take in (3.1) ε so small that $c_6(\delta,\tau_1)\big(1+|\Omega|^{1/2}\big)\varepsilon<\mu$. Hence, we have again

$$|\mathbf{v}(\cdot,T_0+T_1)|^{(2+\alpha)}_{\cup\Omega^{\pm}_{T_0+T_1}}\leqslant \mu e^{-b(T_0+T_1)}\leqslant\mu.$$

Since the norms of the data have not increased, a solution of problem (1.6), (1.9) exists in $(T_0+T_1,T_0+2T_1]$ and

$$N_{(T_0+T_1,T_0+2T_1)}[\mathbf{v},p]\leqslant\mu_1.$$

Inequality (3.18) is valid for the interval $(T_0+2T_1-\tau_1,T_0+2T_1)$. Thus, the solution of (1.1), (1.3), (1.4) with $\sigma=0$ can be extended as far as one likes.

Solution uniqueness follows from the uniqueness of local solutions at every moment of time (Theorem 2.1). □

4 Problem (1.1), (1.3), (1.4) with positive surface tension. Energy estimate

We state global existence theorem for $\sigma>0$.

Theorem 4.1. *Let the hypotheses of Theorem* 2.1 *with* $q_0=p_1(x,0)$ *hold, and for* $t=0$ *let* Γ *be given by the equation*

$$|x|=R\Big(\frac{x}{|x|},0\Big)$$

on the unit sphere S_1. *Suppose, in addition, that the initial data are small enough, i.e.*

$$|e^{b_1t}\mathbf{f}|^{(\alpha,\frac{1+\alpha-\gamma}{2})}_{Q_\infty}+|e^{b_1t}\nabla\mathbf{f}|^{(\alpha,\frac{1+\alpha-\gamma}{2})}_{Q_\infty}+\|e^{b_1t}\mathbf{f}\|_{2,Q_\infty}+$$
$$+|\mathbf{v}_0|^{(2+\alpha)}_{\cup\Omega_0^{\pm}}+|r_0|^{(3+\alpha)}_{S_1}\leqslant\varepsilon\ll1,\quad(4.1)$$

where $r_0(x/|x|)=R(x/|x|,0)-R_0$, R_0 *is the radius of the ball* B_{R_0}: $|\Omega_0^+|=4\pi R_0^3/3$.

Then problem (1.1), (1.3), (1.4) *is uniquely solvable in the whole positive half-axis* $t>0$, *and solution* $(\mathbf{v},p_1)$ *possesses the properties:* $\mathbf{v}\in\mathbf{C}^{2+\alpha,1+\alpha/2}$, $p_1\in C^{(\gamma,1+\alpha)}$, $\nabla p_1\in\mathbf{C}^{\alpha,\alpha/2}$, *the boundary* Γ_t *being given for every* t *by a function* $R(\cdot,t)$ *of the class* $C^{3+\alpha}$:

$$|x-h(t)|=R\Big(\frac{x-h}{|x-h|},t\Big)$$

(where $h(t)$ is a position of the barycenter of Ω_t^+ at the moment t), and tending to a sphere of the radius R_0 with center in a certain point h_∞, and the pressure being defined up to a bounded function of time. It means that for any $t_0 \in (0,\infty)$, the solution $(\mathbf{u}, q)$ and its derivatives in Lagrangian coordinates belong to respective Hölder spaces over $D_{(t_0,t_0+\tau)} \equiv \cup Q^\pm_{(t_0,t_0+\tau)}$ for a sufficiently small time interval $(t_0, t_0+\tau)$. Moreover, there holds the estimate

$$|\mathbf{u}|^{(2+\alpha,1+\alpha/2)}_{D_{(t_0,t_0+\tau)}} + |\nabla q|^{(\alpha,\alpha/2)}_{D_{(t_0,t_0+\tau)}} + |q|^{(\gamma,1+\alpha)}_{D_{(t_0,t_0+\tau)}} + \sup_{t\in(t_0,t_0+\tau)} |r(\cdot,t)|^{(3+\alpha)}_{S_1}$$
$$\leqslant c e^{-b_1 t_0}\{|e^{b_1 t}\mathbf{f}|^{(\alpha,\frac{1+\alpha-\gamma}{2})}_{Q_\infty} + |e^{b_1 t}\nabla\mathbf{f}|^{(\alpha,\frac{1+\alpha-\gamma}{2})}_{Q_\infty} + \|e^{b_1 t}\mathbf{f}\|_{2,Q_\infty} +$$
$$+|\mathbf{v}_0|^{(2+\alpha)}_{\cup\Omega_0^\pm} + |r_0|^{(3+\alpha)}_{S_1}\}, \qquad (4.2)$$

where $r(\omega,t) = R(\omega,t) - R_0$, $\omega \in S_1$.

This theorem for $\mathbf{f} = 0$ was proven in [5].

One can conclude from Theorem 4.1 that the trivial solution is unique when the initial velocity and mass forces are absent and the initial interface coincides with a sphere. The stability of this solution takes place in the sense that the solution differs a little from zero under a small deviation of the initial data and mass forces from zero ones, the interface tending to a sphere $S_{R_0}(h_\infty)$. However, the center h_∞ of this limit sphere may be displaced with respect to the initial barycenter of Ω_0^+ for no matter how small an initial velocity $\mathbf{v}_0$ is. This displacement is evaluated by inequality (5.5) at the end of the paper. There we also give a necessary estimate with above of initial distance between the outer boundary and fluid interface.

We need again an exponential estimate for the solution of the nonlinear problem (1.1), (1.3), (1.4) in L_2. Now we prove it by using the notion of generalized energy $\mathcal{E}$ introduced in [7, 8].

Assume the solution exists in the interval $[0,T]$. It is guaranteed by local existence theorem (Theorem 2.1). Thus, we have also barycenter trajectory of the drop Ω_t^+: $\mathbf{h}(t) = \frac{1}{|\Omega_t^+|}\int\limits_{\Omega_t^+} \mathbf{x}\,\mathrm{d}x$.

Remind that $r(\omega,t)$ is the deviation function of Γ_t from the sphere $S_{R_0}(t) \equiv S_{R_0}(h(t)) = \{|x - h(t)| = R_0\}$. Incompressibility of the fluids implies that the domains $\Omega_t^\pm$ conserve their volumes for all $t > 0$:

$$\int_{\Omega_t^+} \mathrm{d}x = \int_{\Omega_0^+} \mathrm{d}x \Rightarrow \int_{S_1} (R^3 - R_0^3)\,\mathrm{d}\omega = 0.$$

Since $r = R - R_0$, the equality

$$\int_{S_1} r\,\mathrm{d}\omega = -\frac{1}{R_0}\int_{S_1} r^2\,\mathrm{d}\omega - \frac{1}{3R_0^2}\int_{S_1} r^3\,\mathrm{d}\omega \tag{4.3}$$

holds. We are going to use it later on.

Assume (without restriction of generality) that $\mathbf{h}(0) = 0$. Moreover, assume that Γ is defined by the equation

$$x = y + \mathbf{N}(y) r_0\Big(\frac{y}{|y|}\Big), \qquad y \in S_{R_0}. \tag{4.4}$$

($\mathbf{N}(y)$ is the exterior normal to $S_{R_0} = \{|y| = R_0\}$, i.e., $\mathbf{N}(y) = \mathbf{y}/|y|$.) We assume that also for $0 < t \leqslant T$ Γ_t can be defined by

$$x = y + \mathbf{N}(y) r\Big(\frac{y}{|y|}, t\Big) + \mathbf{h}(t), \qquad y \in S_{R_0}. \tag{4.5}$$

Equations (4.4) and (4.5) are equivalent to the relations

$$|x| = R_0 + r_0\Big(\frac{x}{|x|}\Big) \quad \text{and} \quad |x - h(t)| = R_0 + r\Big(\frac{x-h}{|x-h|}, t\Big),$$

respectively.

From the kinematic condition (1.4), it follows that

$$\mathcal{D}_t\mathbf{x}\cdot\mathbf{n} = \mathbf{v}\cdot\mathbf{n}|_{\Gamma_t}.$$

Hence

$$\begin{aligned} n_{0N}(y)\mathcal{D}_t r\big|_{t=0} &= \mathbf{v}_0\cdot\mathbf{n}_0 - \mathbf{h}_t'(0)\cdot\mathbf{n}_0, \\ n_N(y)\mathcal{D}_t r &= \mathbf{v}\cdot\mathbf{n} - \mathbf{h}_t'\cdot\mathbf{n}, \quad y \in S_{R_0}, \end{aligned} \tag{4.6}$$

$n_{0N} = \mathbf{n}_0\cdot\mathbf{N}$ is a radial part of $\mathbf{n}_0$ in the coordinate system with the origin at 0, and $n_N = \mathbf{n}\cdot\mathbf{N}$ is a radial part of $\mathbf{n}$ in the coordinate system with the origin at h.

We can rewrite $\mathbf{h}(t)$ in the form:

$$\mathbf{h}(t) = \frac{1}{|\Omega_t^+|}\int_0^t\int_{\Omega_t^+} \mathbf{u}(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau \text{ and } \mathbf{h}_t'(t) = \frac{1}{|\Omega_t^+|}\int_{\Omega_t^+} \mathbf{v}(x,t)\,\mathrm{d}x. \tag{4.7}$$

Note that the barycenter of Ω_t^+ always coincides with the origin in the coordinates $\{y\}$, it means that

$$\int_{S_1} (R^4 - R_0^4)\omega_i\,\mathrm{d}\omega = 0, \tag{4.8}$$

where $\omega_i = y_i/|y|$.

Proposition 4.1. *Assume that the classical solution of the problem* (1.1), (1.3), (1.4) *is defined in* $[0,T]$ *and* $\mathbf{v}_0$ *satisfies compatibility conditions* (2.1). *In addition, let* r *be such that*

$$|r(\omega,t)|_{S_1\times(0,T)} + |\nabla_{S_1} r(\omega,t)|_{S_1\times(0,T)} \leqslant \delta_1 R_0 \ll 1. \tag{4.9}$$

Then for $\forall t\in(0,T]$

$$\|\mathbf{v}(\cdot,t)\|^2_{2,\Omega}+\|r(\cdot,t)\|^2_{W^1_2(S_1)}\leqslant c\,\mathrm{e}^{-2b_1t}\Big\{\|\mathrm{e}^{b_1\tau}\mathbf{f}\|^2_{2,Q_\infty}+\|\mathbf{v}_0\|^2_{2,\Omega}+\|r_0\|^2_{W^1_2(S_1)}\Big\}, \tag{4.10}$$

$$\int_0^T \big(\|\mathbf{v}(\cdot,\tau)\|_{2,\Omega}+\|r(\cdot,\tau)\|_{W^1_2(S_1)}\big)\,\mathrm{d}\tau \leqslant c\Big\{\|\mathrm{e}^{b_1\tau}\mathbf{f}\|_{2,Q_\infty}+\|\mathbf{v}_0\|_{2,\Omega}+\|r_0\|_{W^1_2(S_1)}\Big\}, \tag{4.11}$$

with the constants b_1 *and* c *independent of* T.

This proposition with $\mathbf{f}=0$ was proven in [5].

Proof. We multiply the first equation in (1.1) by $\rho^\pm\mathbf{v}$ and integrate by parts in $\Omega_t^-\cup\Omega_t^+$. Then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho^\pm}\mathbf{v}\|^2_{2,\Omega}+\Big\|\sqrt{\frac{\mu^\pm}{2}}\mathbb{S}(\mathbf{v})\Big\|^2_{2,\Omega}=\int\limits_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x+\sigma\int\limits_{\Gamma_t}\Big(H+\frac{2}{R_0}\Big)\mathbf{n}\cdot\mathbf{v}\,\mathrm{d}\Gamma.$$

In view of the formula (1.8) and of the fact that $\int\limits_{\Gamma_t}\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}\Gamma=0$, the integral in the right hand side has the form $\sigma\int\limits_{\Gamma_t}\mathbf{v}\cdot\Delta(t)\mathbf{x}\,\mathrm{d}\Gamma$. In [6] it is proved that

$$\sigma\int_{\Gamma_t}\mathbf{v}\cdot\Delta(t)\mathbf{x}\,\mathrm{d}\Gamma=-\sigma\frac{\mathrm{d}}{\mathrm{d}t}|\Gamma_t|.$$

Therefore we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\frac{1}{2}\|\sqrt{\rho^\pm}\mathbf{v}\|^2_{2,\Omega}+\sigma\big(|\Gamma_t|-4\pi R_0^2\big)\Big\}+\frac{1}{2}\|\sqrt{\mu^\pm}\mathbb{S}(\mathbf{v})\|^2_{2,\Omega}=\int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x.$$

Since $\mathbf{v}\big|_S=0$, the vector field $\mathbf{v}$ satisfies the Korn inequality (3.3) in Ω. Consequently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\frac{1}{2}\|\rho^\pm\mathbf{v}\|^2_{2,\Omega}+\sigma\big(|\Gamma_t|-4\pi R_0^2\big)\Big\}+c_1\|\mathbf{v}\|^2_{W^1_2(\Omega)}\leqslant c\|\mathbf{f}\|^2_{2,\Omega}. \tag{4.12}$$

In order to be able to use the Gronwall inequality for the generalized energy mentioned above, we should try to add in the left hand side the term of the type $|\Gamma_t| - 4\pi R_0^2$. To this end, we construct an auxiliary vector-valued function $\mathbf{W}(x,t)$, $x \in \Omega$.

Let the function $f_0(z)$ be defined on the sphere S_{R_0} and $\int\limits_{S_{R_0}} f_0 \, \mathrm{d}S = 0$. We define $\mathbf{W}_0(z)$ as a solenoidal vector field in the whole space $\mathbb{R}^3$, such that

$$\mathbf{W}_0\big|_{S_{R_0}} = \mathbf{N} f_0(z).$$

We also assume that $\operatorname{supp}\mathbf{W}_0$ is contained in the ball $B_{R_0+a} = \{|y| \leqslant R_0 + a\}$, where $a > 0$ is not very large. In addition,

$$\|\mathbf{W}_0\|_{W_2^1(\mathbb{R}^3)} \leqslant c\|f_0\|_{W_2^{1/2}(S_{R_0})}, \quad \|\mathbf{W}_0\|_{2,\mathbb{R}^3} \leqslant c\|f_0\|_{2,S_{R_0}}, \tag{4.13}$$

and if $f_0 = f_0(z,t)$, then

$$\|\mathcal{D}_t \mathbf{W}_0\|_{2,\mathbb{R}^3} \leqslant c\|\mathcal{D}_t f_0\|_{2,S_{R_0}}. \tag{4.14}$$

Such $\mathbf{W}_0$ can be constructed (see, for instance, [9], Ch. I).

Further we set

$$f_0(y,t) = \widetilde{r}(y,t) \equiv r(y,t) - \overline{r}(t),$$

where $\overline{r}(t) = \frac{1}{4\pi R_0^2} \int\limits_{S_{R_0}} r(y,t) \, \mathrm{d}S_{R_0}$. For a and $\mathbf{h}$ sufficiently small the vector field $\mathbf{W}_0$ vanishes near S for all $t \leqslant T$.

Now we make the following coordinate transformation:

$$x = y + \mathbf{N}^*(y) r^*(y,t) + \mathbf{h}(t) = e_r(y,t), \quad y \in \Omega,$$

where $\mathbf{N}^*$ is the extension of $\mathbf{N}$ in Ω, and $r^*(y,t)$ is the extension of $\widetilde{r}(y)$ with the support in the neighborhood of S_{R_0} and $r^* = 0$ near S. We note that for small $r^*(y,t)$ and $\mathbf{h}$ this transformation is invertible, and it maps S_{R_0} on Γ_t. The vector field $\mathbf{W}$ can be defined as

$$\mathbf{W}(x,t) = \frac{\mathcal{L}(y,t)}{L(y,t)} \mathbf{W}_0(y,t)\big|_{y = e_r^{-1}(x)}.$$

Here $\mathcal{L}$ is the Jacobi matrix of the transformation e_ρ:

$$\mathcal{L}(y,t) = \Big\{\frac{\partial e_r(y,t)}{\partial y}\Big\} = \Big\{\delta_j^i + \frac{\partial (N_i^* r^*)}{\partial y_j}\Big\}_{i,j=1}^3,$$

and L =det $\mathcal{L}$. Let $\widehat{\mathcal{L}}$ be a co-factors matrix of $\mathcal{L}$, ,i.e., $\widehat{\mathcal{L}}(y,t) = L\mathcal{L}^{-1}(y,t)$. The vector field $\mathbf{W}$ is divergence free as the function of x. Indeed,

$$\nabla_x \cdot \mathbf{W} = \mathcal{L}^{-1T}\nabla_y \cdot \mathbf{W} = L^{-1}\nabla_y \cdot (\widehat{\mathcal{L}}\mathbf{W}) = L^{-1}\nabla_y \cdot \widehat{\mathcal{L}}\frac{\mathcal{L}}{L}\mathbf{W}_0 = 0$$

in view of the identity $\nabla_y \cdot \widehat{\mathcal{L}} = 0$, that holds for the co-factors matrix of arbitrary coordinate transformation. In addition, $\mathbf{W}(x,t) = 0$ for $x \in S$ and

$$\mathbf{W} \cdot \mathbf{n}\big|_{\Gamma_t} = \mathbf{W} \cdot \frac{\widehat{\mathcal{L}}^T\mathbf{N}}{|\widehat{\mathcal{L}}^T\mathbf{N}|} = \frac{\mathbf{W}_0 \cdot \mathbf{N}}{|\widehat{\mathcal{L}}^T\mathbf{N}|} = \frac{\widetilde{r}}{|\widehat{\mathcal{L}}^T\mathbf{N}|}\bigg|_{y=e_r^{-1}(x)}. \tag{4.15}$$

From (4.13), (4.14) the estimates

$$\begin{aligned} \|\mathbf{W}(\cdot,t)\|_{W_2^1(\Omega)} &\leqslant c\|\mathbf{W}_0(\cdot,t)\|_{W_2^1(B_{R_0+a})} \\ &\leqslant c\|r(\cdot,t)\|_{W_2^{1/2}(S_{R_0})}, \\ \|\mathbf{W}(\cdot,t)\|_{2,\Omega} &\leqslant c\|r(\cdot,t)\|_{2,S_{R_0}}, \end{aligned} \tag{4.16}$$

follow.

In [5], due to (4.6), it was also proved the inequality

$$\|\mathcal{D}_t\mathbf{W}(\cdot,t)\|_{2,\Omega} \leqslant c(|\mathbf{v}|^{(1)}_{\cup\Omega_t^{\pm}})\{\|\mathbf{v}\|_{W_2^1(\Omega)} + \|r\|_{W_2^{1/2}(S_1)}\}. \tag{4.17}$$

Now we multiply the first equation in (1.1) by $\rho^{\pm}\mathbf{W}$ and integrate by parts over $\Omega_t^- \cup \Omega_t^+$:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega} \rho^{\pm}\mathbf{v}\cdot\mathbf{W}\,\mathrm{d}x + \int\limits_{\Omega} \rho^{\pm}\mathbf{v}\cdot(\mathcal{D}_t\mathbf{W} + (\mathbf{v}\cdot\nabla)\mathbf{W})\,\mathrm{d}x \\ &\quad + \int\limits_{\Omega} \frac{\mu^{\pm}}{2}\mathbb{S}(\mathbf{v}) :\ \mathbb{S}(\mathbf{W})\,\mathrm{d}x - \sigma\int\limits_{\Gamma_t}\Big(H + \frac{2}{R_0}\Big)\mathbf{n}\cdot\mathbf{W}\,\mathrm{d}\Gamma = \int\limits_{\Omega}\mathbf{f}\cdot\mathbf{W}\,\mathrm{d}x, \end{aligned} \tag{4.18}$$

where $\mathbb{S}(\mathbf{v}) : \mathbb{S}(\mathbf{W}) = S_{ij}(\mathbf{v})S_{ij}(\mathbf{W})$. We add equation (4.18) multiplied by a small γ to (4.12). Using (4.15) and the Korn inequality (3.3), we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\frac{1}{2}\|\rho^{\pm}\mathbf{v}\|_{2,\Omega}^2 + \gamma\int_{\Omega}\rho^{\pm}\mathbf{v}\cdot\mathbf{W}\,\mathrm{d}x + \sigma\big(|\Gamma_t| - 4\pi R_0^2\big)\Big\} + c_1\|\mathbf{v}\|^2_{W_2^1(\Omega)} \\ &- \gamma\int_{\Omega}\rho^{\pm}\mathbf{v}\cdot(\mathcal{D}_t\mathbf{W} + (\mathbf{v}\cdot\nabla)\mathbf{W})\,\mathrm{d}x + \gamma\int_{\Omega}\frac{\mu^{\pm}}{2}\mathbb{S}(\mathbf{v}) : \mathbb{S}(\mathbf{W})\,\mathrm{d}x \\ &- \gamma\sigma\int_{S_{R_0}}\Big(H + \frac{2}{R_0}\Big)\widetilde{r}\,\mathrm{d}S_{R_0} \leqslant c(\varepsilon_1)\|\mathbf{f}\|_{2,\Omega}^2 + \varepsilon_1\gamma\|\mathbf{W}\|_{2,\Omega}^2. \end{aligned} \tag{4.19}$$

(We have used here the equality $d\Gamma(x) = |\widehat{\mathcal{L}}^T \mathbf{N}|\, dS_{R_0}(y)$, [8, p. 227].) For arbitrary small γ the expression under the sign of derivative, that can be called a generalized energy $\mathcal{E}(t)$, is positive. Indeed, by theorem 3 in [6], under conditions (4.3), (4.8), (4.9) the inequality

$$E_1(R, R_0) \equiv |\Gamma_t| - 4\pi R_0^2 \geqslant c_2 \|r\|^2_{W_2^1(S_1)}, \tag{4.20}$$

holds with the constant c_2 independent of δ_1 and R_0.

Now we make the formula

$$E_1(R, R_0) = \int_{S_1} \left(R\sqrt{R^2 + |\nabla_{S_1} R|^2} - R_0^2 \right) d\omega$$

our starting point. By decomposition of $E_1(R, R_0)$ in the Taylor series at the point R_0, it was shown in [5] that $E_1(R, R_0)$ depends only on r and ∇r , therefore we write $E_1(r) \equiv E_1(R, R_0)$. Under condition (4.9), it holds

$$E_1(r) \leqslant c_3 \|r\|^2_{W_2^1(S_1)}. \tag{4.21}$$

Next, it was proved that the surface integral

$$E_2(r) \equiv - \int\limits_{S_{R_0}} \left(H + \frac{2}{R_0} \right) \widetilde{r}\, dS_{R_0}$$

was also positive definite, the well known formula for twice the mean curvature

$$H[R] = \frac{1}{R} \nabla_{S_1} \cdot \frac{\nabla_{S_1} R}{\sqrt{g}} - \frac{2}{\sqrt{g}},$$

being used in the spherical coordinates, $g = R^2 + |\nabla_{S_1} R|^2$,

$$\nabla_{S_1} R = R'_\theta e_\theta + \frac{1}{\sin\theta} R'_\varphi e_\varphi, \quad \nabla_{S_1} \cdot \mathbf{u} = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} (\sin\theta\, u_\theta) + \frac{1}{\sin\theta} \frac{\partial u_\varphi}{\partial\varphi}.$$

Thus, we have

$$E_2(r) \geqslant c \|r\|^2_{W_2^1(S_1)}. \tag{4.22}$$

And we set

$$\mathcal{E}(t) = \frac{1}{2} \|\rho^\pm \mathbf{v}\|^2_{2,\Omega} + \gamma \int_\Omega \rho^\pm \mathbf{v} \cdot \mathbf{W}\, dx + \sigma E_1(r).$$

On the basis of (4.16), (4.20), (4.21) we conclude that for sufficiently small γ

$$c_4 (\|\mathbf{v}\|^2_{2,\Omega} + \|r\|^2_{W_2^1(S_1)}) \leqslant \mathcal{E}(t) \leqslant c_5 (\|\mathbf{v}\|^2_{2,\Omega} + \|r\|^2_{W_2^1(S_1)}). \tag{4.23}$$

Let us denote by $\mathcal{E}_1(t)$ the terms outside the derivative $\mathcal{D}_t\mathcal{E}(t)$ in (4.19). It is easily seen, in view of (4.16), (4.17), (4.22), (4.23), that for small γ there exists such a constant $b_1 > 0$ that

$$\mathcal{E}_1(t) \geqslant 2b_1\mathcal{E}(t),$$

which implies the estimate

$$\mathcal{D}_t\mathcal{E}(t) + 2b_1\mathcal{E}(t) \leqslant c\|\mathbf{f}\|^2_{2,\Omega}.$$

Then from the Gronwall lemma it follows that

$$\begin{aligned}\mathcal{E}(t) &\leqslant \mathrm{e}^{-2b_1t}\mathcal{E}(0) + \int_0^t \mathrm{e}^{-2b_1(t-\tau)}\|\mathbf{f}(\cdot,\tau)\|^2_{2,\Omega}\mathrm{d}\tau \\ &\leqslant c\,\mathrm{e}^{-2b_1t}(\|\mathbf{v}_0\|^2_{2,\Omega} + \|r_0\|^2_{W^1_2(S_1)} + \|\mathrm{e}^{b_1\tau}\mathbf{f}\|^2_{2,Q_t})\end{aligned}$$

and (4.10), due to (4.23). By integrating from 0 to T, we obtain inequality (4.11). □

Corollary 4.1. *The coordinates of the barycenter of Ω_t^+ satisfy the inequality*

$$|h(t)| \leqslant c\Big\{\|\mathrm{e}^{b_1\tau}\mathbf{f}\|_{2,Q_\infty} + \|\mathbf{v}_0\|_{2,\Omega} + \|r_0\|_{W^1_2(S_1)}\Big\}, \quad \forall t \in [0,T]. \tag{4.24}$$

Proof. From formula (4.7) it follows the estimate

$$|h(t)| \leqslant \frac{1}{|\Omega_t^+|^{1/2}}\int_0^t \|\mathbf{v}(\cdot,\tau)\|_{2,\Omega_t^+}\,\mathrm{d}\tau,$$

which, together with (4.11) implies inequality (4.24). □

5 Proof of the solvability of problem (1.1), (1.3), (1.4) with $\sigma > 0$ in global

Proposition 5.1. *Let the solution of problem* (1.1), (1.3), (1.4) *be defined in the interval* $(0,T]$ *and let the estimate*

$$N_{(0,T)}[\mathbf{v},p_1] \equiv |\mathbf{u}|^{(2+\alpha,1+\alpha/2)}_{D_T} + |\nabla q|^{(\alpha,\alpha/2)}_{D_T} + |q|^{(\gamma,1+\alpha)}_{D_T} \leqslant \mu.$$

hold, where $(\mathbf{u},q)$ *is a solution of problem* (1.1), (1.3), (1.4) *written as a function of the Lagrangian coordinates.*

Then

$$N_{(t_0-\tau_0,t_0)}[\mathbf{v},p_1,r] \equiv N_{(t_0-\tau_0,t_0)}[\mathbf{v},p_1] + \sup_{t_0-\tau_0<\tau<t_0} |r(\cdot,\tau)|_{S_1}^{(3+\alpha)} \tag{5.1}$$
$$\leqslant c_1(\delta,\tau_0)\Big\{|\mathbf{f}|_{\cup Q_0'}^{(\alpha,\frac{1+\alpha-\gamma}{2})} + |\nabla\mathbf{f}|_{\cup Q_0'}^{(\alpha,\frac{1+\alpha-\gamma}{2})}$$
$$+\int_{t_0-2\tau_0}^{t_0} \big(\|\mathbf{v}(\cdot,\tau)\|_{2,\Omega} + \|r(\cdot,\tau)\|_{W_2^1(S_1)}\big)\,\mathrm{d}\tau\Big\},$$

where $t_0 \in (0,T]$, $\tau_0 \in (0,t_0/2)$, τ_0 *depends on* μ *and on the constant* δ *in* (3.13), $\cup Q_0' = \cup\, Q^{\pm}_{(t_0-2\tau_0,t_0)}$.

This proposition was demonstrated in [5] for $\mathbf{f} = 0$ in a similar way as Proposition 3.2.

Lemma 5.1. *Let* $r_0 \in C^{1+\alpha}(S_1)$ *and* $\mathbf{u} \in C^{1+\alpha,0}(D_{T_0})$, $\alpha \in (0,1)$.
Then $r(\cdot,t) \in C^{1+\alpha}(S_1)$ *for arbitrary* $t \in (0,T_0)$ *and the inequality*

$$|r(\cdot,t)|_{S_1}^{(1+\alpha)} \leqslant c_2\big(|r_0|_{S_1}^{(1+\alpha)} + t|\mathbf{u}|_{\xi,D_t}^{(1+\alpha)}\big), \tag{5.2}$$

holds, if the norms r_0 *and* $\mathbf{u}$ *are small.*

This lemma is proved by the passage to the Lagrangian coordinates [5].

Proof of Theorem 4.1. By Theorem 2.1, there exists a local solution $(\mathbf{v},p_1)$ in $(0,T_0]$, $T_0 > 1$ when ε in (4.1) is small enough. For $(\mathbf{v},p_1)$, estimate (2.4) holds therefore

$$N_{(0,T_0)}[\mathbf{v},p_1] \leqslant c\Big(|\mathbf{f}|_{Q_{T_0}}^{(\alpha,\frac{1+\alpha-\gamma}{2})} + |\mathcal{D}_x\mathbf{f}|_{Q_{T_0}}^{(\alpha,\frac{1+\alpha-\gamma}{2})} + |\mathbf{v}_0|_{\cup\Omega_0^{\pm}}^{(2+\alpha)} + |H_0 + \frac{2}{R_0}|_{\Gamma}^{(1+\alpha)}\Big)$$
$$\leqslant c\Big(|\mathbf{f}|_{Q_{T_0}}^{(\alpha,\frac{1+\alpha-\gamma}{2})} + |\mathcal{D}_x\mathbf{f}|_{Q_{T_0}}^{(\alpha,\frac{1+\alpha-\gamma}{2})} + |\mathbf{v}_0|_{\cup\Omega_0^{\pm}}^{(2+\alpha)} + |r_0|_{S_1}^{(3+\alpha)}\Big) \leqslant c_3\varepsilon \equiv \mu.$$

By Prop. 5.1, there exists $\tau_0 < T_0/2$ such that (3.13) is satisfied and for $(\mathbf{v},p_1)$, T_0 estimate (5.1) holds. Lemma 5.1 guarantees the inequality

$$|r|_{S_1\times(0,T_0)}^{(1+\alpha,0)} \leqslant c_2\big(|r_0|_{S_1}^{(1+\alpha)} + c_3\varepsilon T_0\big) \leqslant \delta_1 R_0$$

($|r|_{S_1\times(0,T_0)}^{(1+\alpha,0)} \equiv \sup_{0<\tau<T_0} |r(\cdot,\tau)|_{S_1}^{(1+\alpha)}$), when ε is sufficiently small. This permits us to apply Prop. 4.1, and from (4.10), (5.1) we obtain

$$N_{(t_0-\tau_0,t_0)}[\mathbf{v},p_1,r] \leqslant c_4\,\mathrm{e}^{-b_1(t_0-2\tau_0)}\Big\{|\mathrm{e}^{b_1 t}\mathbf{f}|_{\cup Q_0'}^{(\alpha,\frac{1+\alpha-\gamma}{2})} + |\mathrm{e}^{b_1 t}\nabla\mathbf{f}|_{\cup Q_0'}^{(\alpha,\frac{1+\alpha-\gamma}{2})}$$
$$+\|\mathrm{e}^{b_1\tau}\mathbf{f}\|_{\cup Q_0'}^2 + \|\mathbf{v}_0\|_{2,\Omega} + \|r_0\|_{W_2^1(S_1)}\Big\}$$
$$\leqslant c_5(\tau_0)\mathrm{e}^{-b_1 t_0}\big(|\Omega|^{\frac{1}{2}} + 2\pi^{\frac{1}{2}}\big)\varepsilon,$$

where $|\Omega|$ is the measure of Ω, and $t_0 \in (2\tau_0, T_0]$.

For $t_0 = T_0$, we have the estimate

$$|\mathbf{v}(\cdot, T_0)|^{(2+\alpha)}_{\cup\Omega^{\pm}_{T_0}} + |r(\cdot, T_0)|^{(3+\alpha)}_{S_1} \leqslant \mu.$$

Next, we use Theorem 2.1 again to obtain solution in $(T_0, T_0 + T_1]$ for the initial data $\mathbf{v}(\cdot, T_0)$, $|r(\cdot, T_0)$. The norm of the solution

$$N_{(T_0,T_0+T_1)}[\mathbf{v}, p_1] \leqslant \mu_1$$

Due to Prop. 5.1, we can find $0 < \tau_1 < T_1/2$ such that satisfies (3.13) and

$$N_{(T_0+T_1-\tau_1,T_0+T_1)}[\mathbf{v}, p_1, r] \leqslant c(\delta, \tau_1)\Big\{ |\mathbf{f}|^{(\alpha,\frac{1+\alpha-\gamma}{2})}_{Q_{(T_0+T_1-2\tau_1,T_0+T_1)}} + |\nabla\mathbf{f}|^{(\alpha,\frac{1+\alpha-\gamma}{2})}_{Q_{(T_0+T_1-2\tau_1,T_0+T_1)}} + \int\limits_{T_0+T_1-2\tau_1}^{T_0+T_1} \big(\|\mathbf{v}(\cdot,\tau)\|_{2,\Omega} + \|r(\cdot,0)\|_{W^1_2(S_1)}\big)\, \mathrm{d}\tau \Big\}. \tag{5.3}$$

Since in view of (5.2)

$$|r|^{(1+\alpha,0)}_{S_1\times(T_0,T_0+T_1)} \leqslant c_2\big(|r(\cdot,0)|^{(1+\alpha)}_{S_1} + T_1|\mathbf{u}|^{(1+\alpha)}_{\xi,D_{(T_0,T_0+T_1)}}\big) \leqslant c_2(c_1\varepsilon + T_1\mu_1) \leqslant \delta_1 R_0,$$

similarly to (3.18) we continue (5.3) by Prop. 4.1 as follows

$$N_{(T_0+T_1-\tau_1,T_0+T_1)}[\mathbf{v}, p_1, r] \leqslant c_6(\delta, \tau_1)\mathrm{e}^{-b_1(T_0+T_1)}\big(1 + |\Omega|^{1/2} + 2\pi^{1/2}\big)\varepsilon.$$

Choose ε so small that $c_6(\delta, \tau_1)\big(1 + |\Omega|^{1/2} + 2\pi^{1/2}\big)\varepsilon \leqslant \mu$.

Hence,

$$|\mathbf{v}(\cdot, T_0 + T_1)|^{(2+\alpha)}_{\cup\Omega^{\pm}_{T_0+T_1}} + |r(\cdot, T_0 + T_1)|^{(3+\alpha)}_{S_1} \leqslant \mu\mathrm{e}^{-b_1(T_0+T_1)}.$$

Thus, the norms of the initial data do not increase. Therefore we can extend the solution in the interval $(T_0 + T_1, T_0 + 2T_1]$. This procedure may be repeated again and again as long as we like.

By repeating our argument, we should pass to the Lagrangian coordinates according to the formula

$$\mathbf{X} = \boldsymbol{\xi}^{(1)} + \int_{T_0}^{t} \widetilde{\mathbf{u}}(\xi^{(1)}, \tau)\, \mathrm{d}\tau, \qquad \xi^{(1)} \in \cup\Omega^{\pm}_{T_0}, \quad t \in (T_0, T_0 + T_1). \tag{5.4}$$

In fact, due to the additivity of the integral, (5.4) coincides with (4.2):

$$\mathbf{X}(\xi, t) = \boldsymbol{\xi} + \int_0^{T_0} \mathbf{u}(\xi, \tau)\, \mathrm{d}\tau + \int_{T_0}^{t} \mathbf{u}(\xi, \tau)\, \mathrm{d}\tau, \qquad \xi \in \cup\Omega^{\pm}_0,\ t \in (T_0, T_0 + T_1),$$

because $\widetilde{\mathbf{u}}(\xi^{(1)}, \tau) = \mathbf{u}(\xi, \tau)$.

The same remark is valid for the coordinates of inner fluid barycenter, since the volume of the fluid is conserved:

$$\begin{aligned}\mathbf{h}(t) &= \mathbf{h}(T_0) + \int_{T_0}^{t} \frac{1}{|\Omega_t^+|} \int_{\Omega_t^+} \mathbf{v}(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau = \int_0^{T_0} \frac{1}{|\Omega_t^+|} \int_{\Omega_t^+} \mathbf{v}(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \\ &+ \int_{T_0}^{t} \frac{1}{|\Omega_t^+|} \int_{\Omega_t^+} \mathbf{v}(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau = \frac{3}{4\pi R_0^3} \int_0^t \int_{\Omega_t^+} \mathbf{v}(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau, \quad t > T_0.\end{aligned}$$

Hence, one can conclude that

$$|\mathbf{h}(t)| \leqslant a, \qquad t \leqslant T_0 + T_1,$$

where

$$a = \frac{c_8}{|\Omega_0^+|^{1/2} b_1} \Big\{ \|\mathrm{e}^{b_1 t} \mathbf{f}\|_{2,Q_\infty} + \|\mathbf{v}_0\|_{2,\Omega} + \|r_0\|_{W_2^1(S_1)} \Big\}.$$

The solution of the system (1.1), (1.3), (1.4) can be extended in this way with respect to t as far as necessary and it will satisfy the inequality (4.2) .

The limiting position of the barycenter is estimated from Corollary 4.1:

$$|h_\infty| \leqslant a \leqslant c_7 \Big(\|\mathrm{e}^{b_1 t} \mathbf{f}\|_{2,Q_\infty} + |\mathbf{v}_0|_{\cup \Omega_0^\pm}^{(2+\alpha)} + |r_0|_{S_1}^{(3+\alpha)} \Big) \leqslant c_7 \varepsilon. \qquad (5.5)$$

Inequality (5.5) implies that the initial distance between the surfaces Γ and S should be strictly larger than $c_7 \Big(\|\mathrm{e}^{b_1 t} \mathbf{f}\|_{2,Q_\infty} + |\mathbf{v}_0|_{\cup \Omega_0^\pm}^{(2+\alpha)} + |r_0|_{S_1}^{(3+\alpha)} \Big) + \delta_1 R_0$ with δ_1, R_0 from (4.9), to exclude their intersection in the future.

The uniqueness of a solution follows from the local existence and uniqueness theorem.

Remark 5.1. *If* $\sigma = 0$, *we need evaluate the integral* $I \equiv \int_0^\infty |\mathbf{u}(\cdot, t)|_{\Omega^+}\,\mathrm{d}t$ *in order to estimate the distance between the solid boundary and the interface. In virtue of* (3.2),

$$|\mathbf{u}(\cdot, t_0)|_{\Omega^+} \leqslant c(\delta, T_0) \mathrm{e}^{-bt_0} \varepsilon. \qquad (5.6)$$

By using (2.4) *and integrating* (5.6) *with respect to* $t_0 > T_0/2$, *we arrive at*

$$I \leqslant \frac{T_0}{2} |\mathbf{u}|_{Q_{T_0/2}^+} + \int_{T_0/2}^\infty |\mathbf{u}(\cdot, t)|_{\Omega^+}\,\mathrm{d}t \leqslant \frac{T_0}{2}\varepsilon + c(\delta, T_0)\frac{1}{b}\varepsilon \leqslant c_8 \varepsilon.$$

Thus, if the initial distance between the surfaces is greater than $c_8\varepsilon$, Γ_t *will never intersect* S.

Acknowledgements The author expresses her gratitude to Professor Y. Shibata for fruitful discussions concerning the subject of this paper.

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