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An application of a pressure-stabilized characteristics finite element scheme to the linear stability analysis of flows past a circular cylinder

Hirofumi Notsu[1] and Masahisa Tabata[2]

[1] Waseda Institute for Advanced Study, Waseda University
[2] Faculty of Science and Engineering, Waseda University

1 Introduction

In this paper we introduce a numerical scheme for the linearized Navier-Stokes equations and apply the scheme to the linear stability analysis of flows past a circular cylinder.

Let Ω be a bounded domain in $\mathbb{R}^d(d=2,3)$ with Lipschitz-continuous boundary $\Gamma\equiv\partial\Omega$ consisting of three disjoint connected components Γ_0, Γ_1 and Γ_2. We consider a boundary value problem; find $(u,p):\ \Omega\rightarrow\mathbb{R}^d\times\mathbb{R}$ such that

$$(u\cdot\nabla)u-\nabla(2\nu D(u))+\nabla p=0 \qquad \text{in } \Omega, \tag{1a}$$

$$\nabla\cdot u=0 \qquad \text{in } \Omega, \tag{1b}$$

$$u=g \qquad \text{on } \Gamma_0, \tag{1c}$$

$$\tau=0 \qquad \text{on } \Gamma_1, \tag{1d}$$

$$P_\Gamma\tau=0, \quad u\cdot n_\Gamma=0 \qquad \text{on } \Gamma_2, \tag{1e}$$

where u is the velocity, p is the pressure, $g:\ \Gamma_0\rightarrow\mathbb{R}^d$ is a given boundary velocity, $\nu(>0)$ is a viscosity, $D(u)$ is the strain-rate tensor defined by

$$D_{ij}(u)\equiv\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \quad (i,j=1,\cdots,d),$$

τ is the stress tensor defined by

$$\tau\equiv\{-pI+2\nu D(u)\}n_\Gamma$$

for the identity matrix I and the outward unit normal vector n_Γ, and $P_\Gamma\equiv I-n_\Gamma\otimes n_\Gamma$ is a projection operator. We assume $\mathrm{meas}(\Gamma_0)\neq 0$.

Let T be a positive constant. Suppose there exists a solution (u_*,p_*) of (1). Considering

$$(u,p)=(u_*,p_*)+(u',p')$$

for a perturbation (u',p') and using the same notation (u,p) as (u',p'), we set a non-stationary linearized Navier-Stokes problem around (u_*,p_*); find $(u,p):\ \Omega\times(0,T)\rightarrow\mathbb{R}^d\times\mathbb{R}$ such that

$$\frac{\partial u}{\partial t}+(u_*\cdot\nabla)u+(u\cdot\nabla)u_*-\nabla(2\nu D(u))+\nabla p=0 \qquad \text{in } \Omega\times(0,T), \tag{2a}$$

$$\nabla\cdot u=0 \qquad \text{in } \Omega\times(0,T), \tag{2b}$$

[1] h.notsu@aoni.waseda.jp

$$u = 0 \qquad \text{on } \Gamma_0 \times (0,T), \qquad (2c)$$
$$\tau = 0 \qquad \text{on } \Gamma_1 \times (0,T), \qquad (2d)$$
$$P_\Gamma \tau = 0, \quad u \cdot n_\Gamma = 0 \qquad \text{on } \Gamma_2 \times (0,T), \qquad (2e)$$
$$u = u^0 \qquad \text{in } \Omega, \text{ at } t = 0, \qquad (2f)$$

where $u^0 : \Omega \to \mathbb{R}^d$ is a given initial perturbation. We solve problem (2) by a pressure-stabilized characteristics finite element scheme for the linearized Navier-Stokes equations to see the stability of the solution (u_*, p_*) of (1).

2 A pressure-stabilized characteristics finite element scheme

In this section we introduce a pressure-stabilized characteristics finite element scheme.

We consider a general equation of (2a),

$$\frac{Du}{Dt_w} - \nabla(2\nu D(u)) + \nabla p + \sigma u = f \qquad \text{in } \Omega \times (0,T), \qquad (3)$$

where $w : \Omega \times (0,T) \to \mathbb{R}^d$ is a given velocity, D/Dt_w is a material derivation defined by

$$\frac{D}{Dt_w} \equiv \frac{\partial}{\partial t} + w \cdot \nabla,$$

$f : \Omega \times (0,T) \to \mathbb{R}^d$ is a given external force, $\sigma : \Omega \times (0,T) \to \mathbb{R}^{d \times d}$ is a given reaction function. We call problem (2) replacing (2a) with (3) problem (3).

Let $V(g) \equiv \{v \in H^1(\Omega)^d;\ v|_{\Gamma_0} = g,\ v \cdot n_\Gamma|_{\Gamma_2} = 0\}$, $V \equiv V(0)$ and $Q \equiv L^2(\Omega)$ be function spaces. We define bilinear forms a on $H^1(\Omega)^d \times H^1(\Omega)^d$, b on $H^1(\Omega)^d \times Q$ and $\mathscr{A}$ on $(H^1(\Omega)^d \times Q) \times (H^1(\Omega)^d \times Q)$ by

$$a(u,v) \equiv 2\nu\big(D(u), D(v)\big), \quad b(v,q) \equiv -(\nabla \cdot v, q),$$
$$\mathscr{A}((u,p),(v,q)) \equiv a(u,v) + b(v,p) + b(u,q),$$

respectively, where $(\cdot,\cdot)$ means $L^2(\Omega)$-inner product. Then, the weak formulation of problem (3) is to find $(u,p) : (0,T) \to V(g) \times Q$ such that, for $t \in (0,T)$,

$$\left(\frac{Du}{Dt_w}(t), v\right) + \mathscr{A}((u,p)(t),(v,q)) + (\sigma(t)u(t), v) = (f(t), v), \quad \forall (v,q) \in V \times Q, \qquad (4)$$

with $u(0) = u^0$.

Let $\mathscr{T}_h = \{K\}$ be a triangulation of $\bar{\Omega}$ $(= \bigcup_{K \in \mathscr{T}_h} K)$, h_K be a diameter of $K \in \mathscr{T}_h$, and $h \equiv \max_{K \in \mathscr{T}_h} h_K$ be the maximum element size. We define $V_h \subset V$ and $Q_h \subset Q$ by piecewise linear function spaces for the velocity and the pressure, respectively. We define function spaces X_h, Q_h, $V_h(g)$ and V_h by

$$X_h \equiv \{v_h \in C^0(\bar{\Omega}_h)^d;\ v_h|_K \in P_1(K)^d,\ \forall K \in \mathscr{T}_h\},$$
$$Q_h \equiv \{q_h \in C^0(\bar{\Omega}_h);\ q_h|_K \in P_1(K),\ \forall K \in \mathscr{T}_h\},$$

$V_h(g) \equiv X_h \cap V(g)$ and $V_h \equiv V_h(0)$, respectively, where $P_1(K)$ is a polynomial space of piecewise linear functions on $K \in \mathscr{T}_h$. Let Δt be a time increment, $N_T \equiv \lceil T/\Delta t \rceil$ be a total number of time steps, δ be a positive constant and $(\cdot,\cdot)_K$ be an inner product in $L^2(K)^d$. We define bilinear forms $\mathscr{C}_h$ on $H^1(\Omega)\times H^1(\Omega)$ and $\mathscr{A}_h$ on $(H^1(\Omega)^d\times H^1(\Omega))\times(H^1(\Omega)^d\times H^1(\Omega))$ by

$$\mathscr{C}_h(p,q)\equiv-\delta\sum_{K\in\mathscr{T}_h}h_K^2(\nabla p,\nabla q)_K,$$

$$\mathscr{A}_h((u,p),(v,q))\equiv a(u,v)+b(v,p)+b(u,q)+\mathscr{C}_h(p,q),$$

respectively. Let $X_1^n(x)$ be an upwind point of x defined by

$$X_1^n(x)\equiv x-w^n(x)\Delta t,$$

and the symbol $\circ$ mean composition of functions, i.e., for $v:\Omega\to\mathbb{R}^d$,

$$v\circ X_1^n(x)\equiv v(X_1^n(x)).$$

Let $f\in C^0([0,T];L^2(\Omega)^d)$ and $u^0\in V(g)$ be given. Let an approximate function $u_h^0\in V_h(g)$ of u^0 be given. A pressure-stabilized characteristics finite element scheme for problem (3) is to find $\{(u_h^n,p_h^n)\}_{n=1}^{N_T}\subset V_h(g)\times Q_h$ such that, for $n=1,\cdots,N_T$,

$$\left(\frac{u_h^n-u_h^{n-1}\circ X_1^n}{\Delta t},v_h\right)+\mathscr{A}_h((u_h^n,p_h^n),(v_h,q_h))=(f^n,v_h),\quad\forall(v_h,q_h)\in V_h\times Q_h.\tag{5}$$

Remark 1. *Let $Q_0\equiv L_0^2(\Omega)\equiv\{q\in L^2(\Omega);\ (q,1)=0\}$ and $Q_{0h}\equiv Q_h\cap Q_0$ be function spaces. When $\Gamma_0=\Gamma$, Q in the weak form* (4) *and Q_h in scheme* (5) *are replaced with Q_0 and Q_{0h}, respectively.*

Scheme (5) can deal with convection-dominated (small viscosity, high Reynolds number) problems by the method of characteristics. When we find (u_h^n,p_h^n) in scheme (5), the composite function $u_h^{n-1}\circ X_1^n$ is a known function and a coefficient matrix of the system of the linear equations is symmetric. The advantage, i.e., symmetry of the matrix, enables us to use linear iterative solvers for symmetric matrices, i.e., MINRES, CR [1,8] and so on. Since the coefficient matrix is independent of step number n, it is enough to make the matrix only at the first time step. The scheme employs a cheap element P1/P1, i.e., a piecewise linear approximation for both velocity and pressure, it is useful for large scale computations especially in 3D. Although P1/P1 element does not satisfy the conventional inf-sup condition [3], the scheme works by a pressure-stabilization term $\mathscr{C}_h$ introduced in [2].

Let c be a generic positive constant, independent of h and Δt. We use norms and seminorms, $\|\cdot\|_k\equiv\|\cdot\|_{H^k(\Omega)}$ $(k=0,1)$, $\|\cdot\|_{V_h}\equiv\|\cdot\|_V\equiv\|\cdot\|_1$, $\|\cdot\|_{Q_h}\equiv\|\cdot\|_Q\equiv\|\cdot\|_0$, $\|(v,q)\|_{V\times Q}\equiv\{\|v\|_V^2+\|q\|_Q^2\}^{1/2}$,

$$\|u\|_{l^\infty(X)}\equiv\max_{n=0,\cdots,N_T}\|u^n\|_X,\quad\|u\|_{l^2(X)}\equiv\left\{\Delta t\sum_{n=1}^{N_T}\|u^n\|_X^2\right\}^{1/2},$$

$$|q|_h\equiv\left\{\sum_{K\in\mathscr{T}_h}h_K^2(\nabla q,\nabla q)_K\right\}^{1/2},\quad|p|_{l^2(|\cdot|_h)}\equiv\left\{\Delta t\sum_{n=1}^{N_T}|p^n|_h^2\right\}^{1/2},$$

for $X=L^2(\Omega)$ and $H^1(\Omega)$. $\overline{D}_{\Delta t}$ is the backward difference operator defined by

$$\overline{D}_{\Delta t}a^n\equiv\frac{a^n-a^{n-1}}{\Delta t}.$$

Scheme (5) has the following stability and convergence property.

Theorem 1 (stability, [7]). *(i) Suppose $\Gamma_0 = \Gamma$, $g = 0$ and $w|_\Gamma = 0$. Suppose the given functions w, σ and f are smooth enouth. Let Δt_0 be any fixed positive number satisfying $\Delta t_0 < 1/\|w\|_{C^0(W^{1,\infty}(\Omega))}$. Then, for any $\Delta t \in (0, \Delta t_0]$ and $u_h^0 \in V_h$ there exist a unique solution (u_h, p_h) of scheme (5), and a positive constant $c = c(u_h^0, f)$ such that*

$$\|u_h\|_{l^\infty(L^2)},\ \sqrt{\nu}\|u_h\|_{l^2(H^1)},\ \sqrt{\delta}|p_h|_{l^2(|\cdot|_h)} \le c.$$

(ii) Moreover, suppose there exists $p_h^0 \in Q_h$ such that

$$b(u_h^0, q_h) + \mathscr{C}_h(p_h^0, q_h) = 0, \quad \forall q_h \in Q_h.$$

Then, there exists a positive constant $c = c(1/\nu, u_h^0, p_h^0, f)$ such that

$$\sqrt{\nu}\|u_h\|_{l^\infty(H^1)},\ \|\overline{D}_{\Delta t} u_h\|_{l^2(L^2)},\ \|p_h\|_{l^2(L^2)} \le c.$$

Theorem 2 (error estimate, [7]). *(i) Suppose that the same assumptions in Theorem 1 hold, that the solution of (4) are smooth enough, and that u^0 satisfies compatibility conditions $\nabla \cdot u^0 = 0$ and $u^0 \in V$. Suppose $\|u_h^0 - u^0\|_0 \le ch$. Then, there exists a positive constant $c' = c'(1/\nu, u, p)$ such that*

$$\|u_h - u\|_{l^\infty(L^2)},\ \sqrt{\nu}\|u_h - u\|_{l^2(H^1)},\ \sqrt{\delta}|p_h - p|_{l^2(|\cdot|_h)} \le c'(\Delta t + h).$$

(ii) Moreover, suppose u_h^0 is the first component of the Stokes projection of $(u^0, 0)$. Then, there exists a positive constant $c = c(1/\nu, u, p)$ such that

$$\sqrt{\nu}\|u_h - u\|_{l^\infty(H^1)},\ \left\|\overline{D}_{\Delta t} u_h - \frac{\partial u}{\partial t}\right\|_{l^2(L^2)},\ \|p_h - p\|_{l^2(L^2)} \le c(\Delta t + h),$$

Remark 2. *(i) The constant c in Theorem 1-(i) is independent of ν. The fact implies that scheme (5) is robust even for convection-dominated problems. (ii) The choice of u_h^0 in Theorem 2-(ii) satisfies $\|u_h^0 - u^0\|_0 \le ch$.*

3 Computation of problem (2) by scheme (5)

In this section we compute problem (2) by scheme (5) with $w \equiv u_h^*$ and $\sigma_{ij} = \partial u_{hi}^*/\partial x_j$ for a numerical stationary solution (u_h^*, p_h^*) of the Navier-Stokes equations (1).

We set $d = 2$. A quadrature formula of degree five (seven points formula) [9] is employed for computation of the integral

$$\int_K u_h^{n-1} \circ X_1^n(x) v_h(x)\, dx$$

appearing in scheme (5). Let $Re \equiv 1/\nu$ be the Reynolds number. We set $\delta = \delta_0 Re$ for a fixed positive number δ_0. $\delta_0 = 0.05$ is chosen by some numerical experience. The system of linear equations is solved by MINRES. Let

$$\Omega \equiv \{x \in \mathbb{R}^2;\ -7.5 < x_1 < 22.5,\ -7.5 < x_2 < 7.5,\ |x| > 0.5\} \tag{6}$$

be the domain and $\mathscr{T}_h$ be the triangulation of $\bar{\Omega}$. Fig. 1 shows Ω (left) and $\mathscr{T}_h$ around the cylinder. The boundary conditions for the stationary flows are also put in the left figure. The number of elements is 52,416, the number of nodes is 26,608 ($h_{\min} = 1.16 \times 10^{-2}$, $h = h_{\max} = 2.50 \times 10^{-1}$) and the number of degrees of freedom is 78,924. The triangulation $\mathscr{T}_h$ is symmetric with respect to the x_1-axis, cf. Fig. 1 (right). Let $\Omega_+ \equiv \{x \in \Omega;\ x_2 > 0\}$ be the upper half domain. We solve the nonstationary Navier-Stokes equations with the boundary conditions of Fig. 1 (left) in Ω_+ by a pressure-stabilized characteristics finite element scheme [5, 6], where symmetric boundary conditions $\tau_1 = 0$ and $u_2 = 0$ are imposed on the x_1-axis. Then, a stationary solution defined in $\bar{\Omega}_+$ is obtained for sufficiently large time, and we construct a stationary solution $(u_h^*, p_h^*) \in V_h(g) \times Q_h$ defined in $\bar{\Omega}$ which has x_1-axis symmetry (u_{h1}^*: even, u_{h2}^*: odd, p_h^*: even extensions). Fig. 2 exhibits streamlines (left) and pressure contours (right) of the stationary solution (u_h^*, p_h^*) for $Re = 10$ (top) and 100 (bottom). We set the following example.

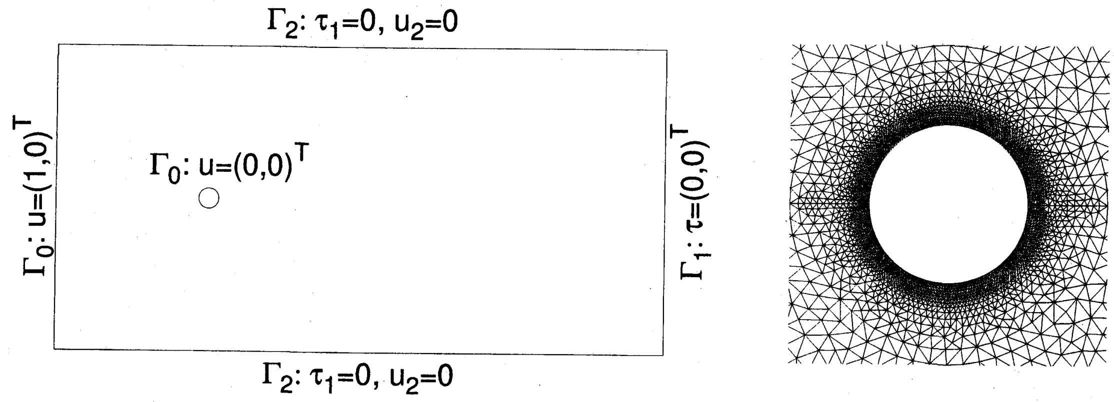

Figure 1: The domain Ω with the boundary conditions for the Navier-Stokes equations (left) and the used triangular mesh around the cylinder (right).

Example 1. *In* (2) *we set* Ω *by* (6), $T = 100$, *six values of* ν,

$$\nu = 10^{-1}, 40^{-1}, 50^{-1}, 60^{-1}, 70^{-1}, 10^{-2}\ (Re = 10, 40, 50, 60, 70, 100),$$

$w = u_h^*$, $\sigma_{ij} = \partial u_{hi}^* / \partial x_j$ $(i, j = 1, 2)$, $f = 0$ *and* $u^0 \approx 0$.

We solve Example 1 by scheme (5) with $\Delta t = 1/50$. The small perturbation $u_h^0 = (u_{h1}^0, u_{h2}^0)^T$ is set as $u_{h1}^0(P) = 0$ for all nodes P, $u_{h2}^0(P) = 0.01$ for a node $P = (-1.36, 0)^T$ and $u_{h2}^0(P) = 0$ for the other nodes P.

We compute $\|(u_h^n, p_h^n)\|_{V \times Q}$ for $n = 1, \cdots, N_T$ to see the stability of the stationary solutions (u_h^*, p_h^*). The graphs of $\|(u_h^n, p_h^n)\|_{V \times Q}$ versus t are shown in Fig. 3. For $Re = 100, 70$ and 60 the value of $\|(u_h^n, p_h^n)\|_{V \times Q}$ monotonically increases after $t = 3$, and for $Re = 50, 40$ and 10 the value of $\|(u_h^n, p_h^n)\|_{V \times Q}$ finally decreases. The results imply that the solutions (u_h^*, p_h^*) for $Re = 10, 40$ and 50 and 60, 70 and 100 are stable and unstable, respectively. Since $Re = 50$ is close to the critical Reynolds number for the onset of instability, cf. [4], it is preferable that smaller h and Δt are employed to obtain the symmetric solution of (1) and to conclude whether the solution is stable or not. Fig. 4 shows streamlines (top) and pressure (bottom) contours of the final state $(u_h, p_h)(t = 100)$ for $Re = 100$. The amplified perturbation is observed.

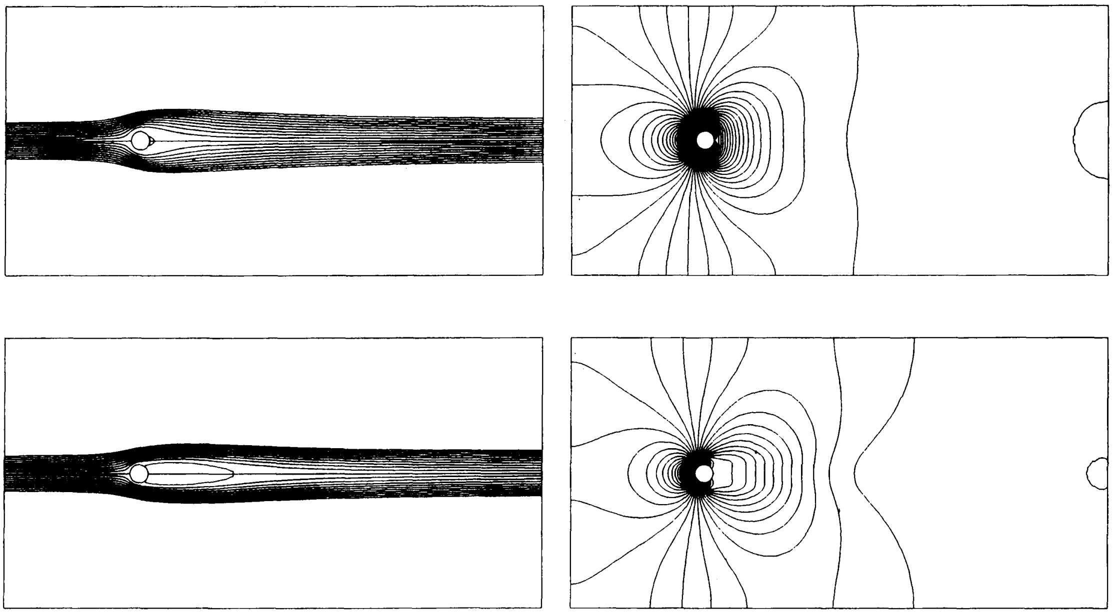

Figure 2: Streamlines (left, [-1, 1; 0.1]) and pressure contours (right, [-1, 1; 0.02]) of the stationary solution (u_h^*, p_h^*) for $Re = 10$ (top) and 100 (bottom).

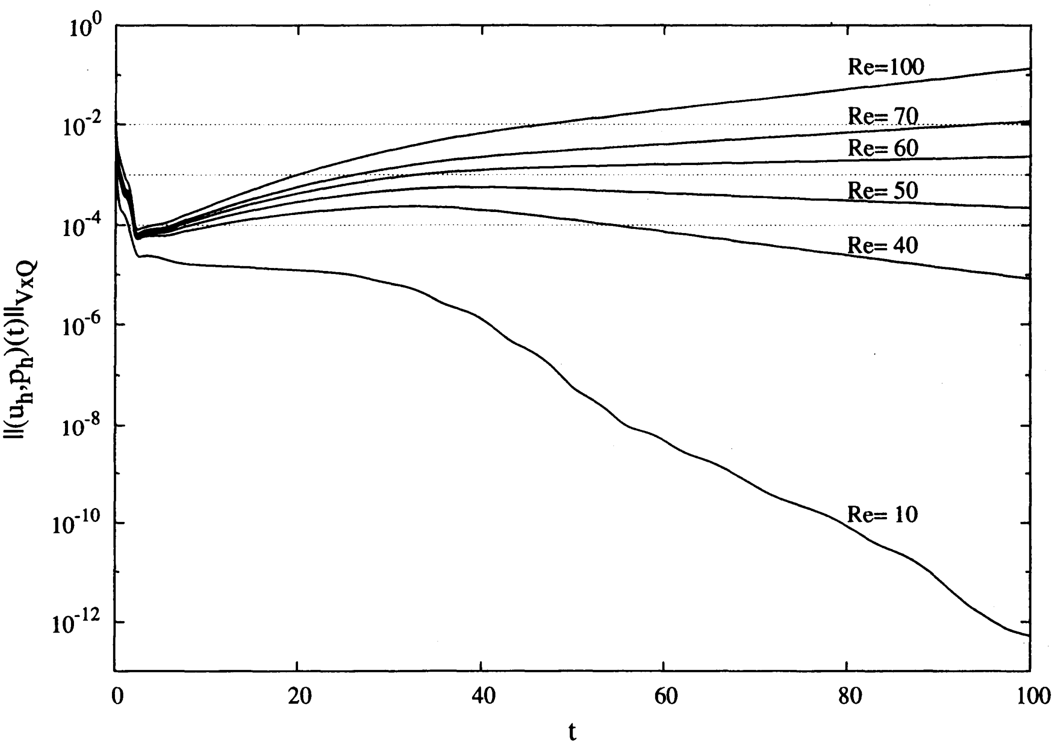

Figure 3: Graphs of $\|(u_h, p_h)(t)\|_{V\times Q}$ vs. t for $Re = 10, 40, 50, 60, 70$ and 100.

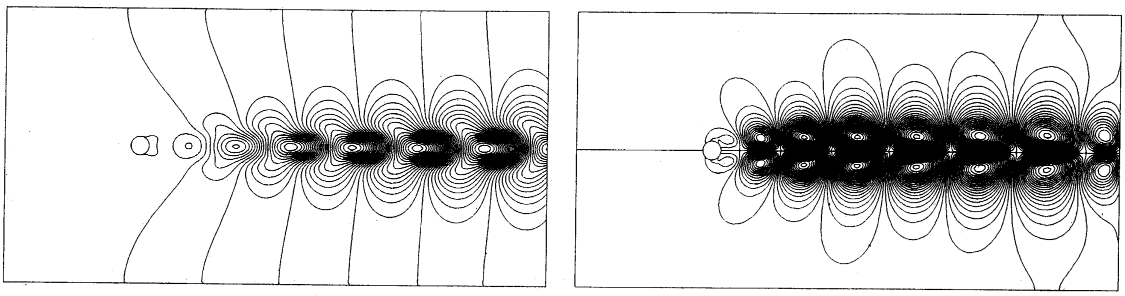

Figure 4: Streamlines (top, [−0.02, 0.02; 0.001]) and pressure contours (bottom, [−0.003, 0.003; 0.0001]) of $(u_h, p_h)(t = 100)$ for $Re = 100$.

4 Conclusions

We have introduced a pressure-stabilized characteristics finite element scheme for the linearized Navier-Stokes equations, and have applied the scheme to the linear stability analysis of flows past a circular cylinder. In order to make the stationary solutions of the Navier-Stokes equations a pressure-stabilized characteristics finite element scheme [5, 6] has been employed. The numerical results have shown that stationary solutions are stable for $Re = 10, 40$ and 50 and unstable for $Re = 60, 70$ and 100 under the computation settings (mesh and Δt). The obtained results imply that the scheme is applicable to such linear stability analysis.

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