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GLOBAL STRONG SOLUTION WITH VACUUM TO THE 2D DENSITY-DEPENDENT NAVIER-STOKES SYSTEM

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1. INTRODUCTION

The Navier-Stokes equations are usually used to describe the motion of fluids. In particular, for the study of multiphase fluids without surface tension, the following density-dependent Navier-Stokes equations acts as a model on some bounded domain $\Omega \subset R^N (N=2,3)$,

$$
(1.1) \qquad \begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \quad \text{in } \Omega \times (0,T], \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu(\rho) d) + \nabla P = 0, \quad \text{in } \Omega \times (0,T], \\ \operatorname{div} u = 0, \quad \text{in } \Omega \times [0,T], \\ u = 0, \quad \text{on } \partial\Omega \times [0,T], \\ \rho|_{t=0} = \rho_0, \quad u|_{t=0} = u_0, \quad \text{in } \Omega. \end{cases}
$$

Here ρ, u, and P denote the density, velocity and pressure of the fluid, respectively. $d = \frac{1}{2}\left[\nabla u + (\nabla u)^T\right]$ is the deformation tensor. $\mu = \mu(\rho)$ states the viscosity and is a function of ρ, which is assumed to satisfy

$$
(1.2) \quad \mu \in C^1[0,\infty), \quad \text{and} \quad \mu \geq \underline{\mu} > 0 \quad \text{on } [0,\infty) \quad \text{for some positive constant } \underline{\mu}.
$$

In this paper, we study the two-dimensional initial boundary value problem for the system (1.1)-(1.2) .

Let us recall some known results for this system (1.1). The mathematical study for nonhomogeneous incompressible flow was initiated by the Russian school. They studied the case that $\mu(\rho)$ is a constant and the initial density ρ_0 is bounded away from 0. In the absence of vacuum, global existence of weak solutions was established by Kazhikov [17], see also [2]. Later, Antontsev-Kazhikov-Monakhov [3] gave the first result on local existence and uniqueness of strong solutions. Moreover, the unique local strong solution is proved to be global in 2D, see also [16, 18, 21].

On the other hand, when the initial density allows vacuum in some region and $\mu(\rho)$ is still a constant, Simon [22] proved the global existence of weak solutions. For strong solutions, to treat the possible degeneracy near vacuum, Choe-Kim [5] proposed a compatibility condition, which is the original form of (1.4) below. Under such a compatibility condition, local existence of strong solutions was established. Global strong solution with vacuum in 2D was recently derived by the authors [15]. Meanwhile, some global solutions in 3D with small critical norms have been constructed, refer to the results in [1, 6, 7, 20] and references therein.

Finally, we come to the most general case: viscosity $\mu(\rho)$ depends on density ρ. Global weak solutions were derived by the revolutionary work [9, 19] of DiPerna and Lions. Later, Desjardins [8] proved the global weak solution with more regularity for the two-dimensional case provided that the viscosity function $\mu(\rho)$ is a small pertubation of a positive constant in L^∞- norm. Regarding the strong solution away from vacuum, Gui-Zhang [12] proved global well-posdness with ρ_0 is a small perturbation of a constant in H^s, $s \geq 2$. To deal with the possible presence of vacuum, Cho-Kim [4] generalized the compatibility condition in [5] and constructed the local strong solution. Their result is stated as follows(2D Version):

Theorem 1.1. *Assume that the initial data (ρ_0, u_0) satisfies the regularity condition*

$$0 \leq \rho_0 \in W^{1,q}, \quad 2 < q < \infty, \quad u_0 \in H^1_{0,\sigma} \cap H^2, \tag{1.3}$$

and the compatibility condition

$$-\mathrm{div}\left(\mu(\rho_0)\left[\nabla u_0 + (\nabla u_0)^T\right]\right) + \nabla P_0 = \rho^{\frac{1}{2}} g\ , \tag{1.4}$$

for some $(P_0, g) \in H^1 \times L^2$. Then there exists a small time T and a unique strong solution (ρ, u, P) to the initial boundary value problem (1.1) *such that*

$$\begin{aligned} &\rho \in C([0,T];\ W^{1,q}), && \nabla u, P \in C([0,T];\ H^1) \cap L^2(0,T;\ W^{1,r}),\\ &\rho_t \in C([0,T]; L^q), && \sqrt{\rho} u_t \in L^\infty(0,T; L^2), \quad u_t \in L^2(0,T; H^1_0), \end{aligned}$$

for any r with $1 \leq r < q$. Furthermore, if T^* is the maximal existence time of the local strong solution (ρ, u) , then either $T^* = \infty$ or

$$\sup_{0 \leq t < T^*} \left(\|\nabla \rho(t)\|_{L^q} + \|\nabla u(t)\|_{L^2}\right) = \infty. \tag{1.5}$$

It is worth noting that the blowup criterion (1.5) involves both $\|\nabla\rho\|_{L^q}$ and $\|\nabla u\|_{L^2}$. Motivated by the global existence result [15] for the special case that μ is a constant, we aim to remove the second part in (1.5). In fact, we find that the boundedness for $\|\nabla\mu(\rho)\|_{L^q}$ implies that for $\|\nabla u\|_{L^2}$, which is true at least for 2D case. More precisely,

Theorem 1.2. *Assume that the initial data (ρ_0, u_0) satisfies the regularity condition* (1.3) *and the compatibility condition* (1.4), *as in Theorem 1.1, and $0 \leq \rho_0 \leq \bar{\rho}$. Suppose (ρ, u, P) is the unique local strong solution derived in Theorem 1.1, and T^* is the maximal existence time for the solution, then*

$$\sup_{0 \leq t < T^*} \|\nabla \mu(\rho)\|_{L^p} = \infty, \tag{1.6}$$

for every $2 < p \leq q$.

Corollary 1.3. *If μ is a constant, then $\nabla\mu(\rho)$ is always 0, which implies that the strong solution to the system* (1.1) *will exist globally. This is recently proved by the authors [15].*

Our second result proves the existence of global strong solution under the condition that $\|\nabla\mu(\rho_0)\|_{L^q}$ is small.

Theorem 1.4. *Assume that the initial data (ρ_0, u_0) satisfies* (1.3) *and* (1.4), *and*

$$0 \leq \rho_0 \leq \bar{\rho}, \quad \|u_0\|_{H^1} \leq K, \quad \underline{\mu} \leq \mu(\rho) \leq \overline{\mu} \text{ on } [0, \bar{\rho}]. \tag{1.7}$$

Then there exists some small positive constant ϵ_0, depending only on Ω, q, $\underline{\mu}$, $\overline{\mu}$, $\bar{\rho}$ and K, such that if

$$\|\nabla\mu(\rho_0)\|_{L^q} \le \epsilon_0, \tag{1.8}$$

then there is a unique global strong solution (ρ, u) of the density-dependent equations (1.1) *with the following regularity*

$$\begin{aligned} &\rho \in C([0,\infty);\ W^{1,q}), && \nabla u, P \in C([0,\infty);\ H^1) \cap L^2_{loc}(0,\infty;\ W^{1,r}),\\ &\rho_t \in C([0,\infty); L^q), && \sqrt{\rho}u_t \in L^\infty_{loc}(0,\infty; L^2), \quad u_t \in L^2_{loc}(0,\infty; H^1_0), \end{aligned} \tag{1.9}$$

for any r with $1 \le r < q$.

Remark 1.1. *Compared to Gui-Zhang [12]'s global well-posedness result, our result does not require that density is a small perturbation of a positive constant. In fact it allows for the presence of regions of vacuum. The smallness assumption is made on $\nabla\mu(\rho_0)$, instead of ρ_0. So Theorem 1.4 also implies global strong solution for the case $\mu(\rho) = constant$.*

The main idea for proving Theorem 1.4 is similar to that in [6, 14], and partly due to Hoff [13]. The proof is a sort of energy estimate method and utilizes the parabolic property of the equations.

First we assume $\|\nabla\mu(\rho)\|_{L^q} \le 1$ on $[0, T]$, then we prove that there exists a positive constant ϵ_0 as stated in Theorem 1.4 such that $\|\nabla\mu(\rho)\|_{L^q} \le \frac{1}{2}$ on $[0, T]$ provided $\|\nabla\mu(\rho_0)\|_{L^q} \le \epsilon_0 \le \frac{1}{2}$. So if $\|\nabla\mu(\rho)\|_{L^q}$ are initially less than ϵ_0, then it is always less than $\frac{1}{2}$. On the other hand, as proved in Theorem 1.2, the boundedness of $\|\nabla\mu(\rho)\|_{L^q}$ leads to uniform estimates for other higher order quantities of the density and velocity, which guarantees the extension of local strong solutions.

The rest of the paper is organized as follows: Section 2 consists of some notations, definitions, and basic lemmas. We give the proof for Theorems 1.2, 1.4 in Sections 3 and 4 respectively.

2. Preliminaries

In this paper Ω is a bounded smooth domain in $\mathbb{R}^2$. Denote

$$\int f\,dx = \int_\Omega f\,dx.$$

For $1 \le r \le \infty$ and $k \in \mathbb{N}$, the Sobolev spaces are defined in a standard way,

$$L^r = L^r(\Omega), \qquad W^{k,r} = \left\{ f \in L^r : \quad \nabla^k f \in L^r \right\},$$

$$H^k = W^{k,2}, \quad C^\infty_{0,\sigma} = \{ f \in C^\infty_0 : \ \operatorname{div} f = 0 \text{ in } \Omega \}.$$

$$H^1_0 = \overline{C^\infty_0}, \quad H^\infty_{0,\sigma} = \overline{C^\infty_{0,\sigma}}, \text{ closure in the norm of } H^1.$$

High-order a priori estimates rely on the following regularity results for the Stokes equations.

The following lemma plays an important role in the whole analysis.

Lemma 2.1. *Assume that $\rho \in W^{1,p}$, $2 < p < \infty$, $0 \le \rho \le \bar{\rho}$, and $\underline{\mu} \le \mu(\rho) \le \bar{\mu}$ on $[0, \bar{\rho}]$. Let $(u, P) \in H_0^1 \times L^2$ be the unique weak solution to the boundary value problem*

$$-\text{div }(2\mu(\rho)d) + \nabla P = F, \quad \text{div } u = 0 \ \textit{ in } \Omega, \ \textit{ and } \int P dx = 0, \tag{2.10}$$

where $d = \frac{1}{2}\left[\nabla u + (\nabla u)^T\right]$ and μ satisfies (1.2). Then we have the following regularity results:

(1) If $F \in L^2$, then $(u, P) \in H^2 \times H^1$ and

$$\begin{aligned} \|u\|_{H^2} &\le C\|F\|_{L^2} \left(1 + \|\nabla\mu(\rho)\|_{L^p}\right)^{\frac{p}{p-2}} , \\ \|P\|_{H^1} &\le C\|F\|_{L^2} \left(1 + \|\nabla\mu(\rho)\|_{L^p}\right)^{\frac{2p-2}{p-2}} . \end{aligned} \tag{2.11}$$

(2) If $F \in L^r$ for some $r \in (2, p)$, then $(u, P) \in W^{2,r} \times W^{1,r}$ and

$$\begin{aligned} \|u\|_{W^{2,r}} &\le C\|F\|_{L^r} \left(1 + \|\nabla\mu(\rho)\|_{L^p}\right)^{\frac{pr}{2(p-r)}} , \\ \|P\|_{W^{1,r}} &\le C\|F\|_{L^r} \left(1 + \|\nabla\mu(\rho)\|_{L^p}\right)^{1+\frac{pr}{2(p-r)}} . \end{aligned} \tag{2.12}$$

Here the constant C in (2.11) and (2.12) depends on Ω, $\bar{\rho}$, $\underline{\mu}$, $\bar{\mu}$.

The proof of Lemma 2.1 is a slight variation of the version in [4]. We sketch it here for completeness.

Proof. For the existence and uniqueness of the solution, please refer to Giaquinta-Modica [11]. We give the a priori estimates here. Assume that $F \in L^2$. Multiply (2.10) by u and integrate over Ω,

$$2\int \mu(\rho)|d|^2\, dx = \int F \cdot u\, dx \le C\|F\|_{L^2}\|\nabla u\|_{L^2} . \tag{2.13}$$

Since $\mu(\rho) \ge \underline{\mu}$ and $2\int |d|^2\, dx = \int |\nabla u|^2\, dx$, (2.13) implies that

$$\|\nabla u\|_{L^2} \le C\|F\|_{L^2} .$$

Choose some function $v \in H_0^1$, such that $P = \text{div} v$ and $\|v\|_{H^1} \le C\|P\|_{L^2}$, then

$$\int |P|^2\, dx = -\int \nabla P \cdot v\, dx = \int \left(2\mu(\rho)d : \nabla v - F \cdot v\right)\, dx \le C\|F\|_{L^2}\|\nabla v\|_{L^2}.$$

Hence, $\|P\|_{L^2} \le C\|F\|_{L^2}$.
For higher-order estimates, we make use of the classical theory for Stokes system. Rewrite (2.10) as

$$-\Delta u + \nabla\tilde{P} = \mu^{-1}\left(F + 2\nabla\mu \cdot d - \tilde{P}\nabla\mu\right), \ \text{ and } \ \text{div } u = 0, \tag{2.14}$$

where $\tilde{P} = P/\mu$. It follows the well-known regularity results for Stokes system [10] that

$$\begin{aligned} \|u\|_{H^2} + \|\tilde{P}\|_{H^1} &\le C\left(\|F\|_{L^2} + \||\nabla\mu(\rho)||\nabla u|\|_{L^2} + \|\tilde{P}\nabla\mu(\rho)\|_{L^2} + \|\tilde{P}\|_{L^2}\right) \\ &\le C\|F\|_{L^2} + C\|\nabla\mu(\rho)\|_{L^p}\|\nabla u\|_{L^{\frac{2p}{p-2}}} + C\|\nabla\mu(\rho)\|_{L^p}\|\tilde{P}\|_{L^{\frac{2p}{p-2}}}. \end{aligned}$$

By Gagliardo-Nirenberg inequality,

$$\begin{aligned}&\|u\|_{H^2}+\|\tilde{P}\|_{H^1}\\&\le C\|F\|_{L^2}+C\|\nabla\mu(\rho)\|_{L^p}\|u\|_{H^2}^{\frac{2}{p}}\|\nabla u\|_{L^2}^{1-\frac{2}{p}}+C\|\nabla\mu(\rho)\|_{L^p}\|\tilde{P}\|_{H^1}^{\frac{2}{p}}\|\tilde{P}\|_{1-\frac{2}{p}}\,,\end{aligned}$$

which together with Young's inequality proves that

$$\|u\|_{H^2}+\|\tilde{P}\|_{H^1}\le C\|F\|_{L^2}\left(1+\|\nabla\mu(\rho)\|_{L^p}\right)^{\frac{p}{p-2}}\,. \tag{2.15}$$

Hence by Poincaré's inequality,

$$\begin{aligned}\|P\|_{H^1}\le C\|\nabla P\|_{L^2}&\le C\|\nabla\tilde{P}\|_{L^2}+C\|\tilde{P}\|_{H^1}\|\nabla\mu(\rho)\|_{L^p}\\&\le C\|F\|_{L^2}\left(1+\|\nabla\mu(\rho)\|_{L^p}\right)^{\frac{2p-2}{p-2}}\,.\end{aligned} \tag{2.16}$$

Similarly, using the $W^{2,r}$-regularity theory for Stokes system, we have

$$\|u\|_{W^{2,r}}+\|\tilde{P}\|_{W^{1,r}}\le C\|F\|_{L^r}\left(1+\|\nabla\mu(\rho)\|_{L^p}\right)^{\frac{pr}{2(p-r)}}\,, \tag{2.17}$$

and

$$\|P\|_{W^{1,r}}\le C\|F\|_{L^r}\left(1+\|\nabla\mu(\rho)\|_{L^p}\right)^{1+\frac{pr}{2(p-r)}}\,. \tag{2.18}$$

(2.15)-(2.18) complete the proof for Lemma 2.1.

□

Next, for $u\in H_0^1(\Omega)$, by Gagliardo-Nirenberg inequality, we have in 2D

$$\|u\|_{L^4}^2\le C\|u\|_{L^2}\|\nabla u\|_{L^2}\,. \tag{2.19}$$

However, to deal with nonhomogeneous problem with vacuum, some interpolation inequality for u with degenerate weight like $\sqrt{\rho}$ is required. We look for a similar estimate for $\sqrt{\rho}u$ as in (2.19). Here we will use a lemma first established by Desjardins [8] which reads as follows,

Lemma 2.2. *Suppose that* $0\le\rho\le\bar{\rho}$, $u\in H_0^1$, *then*

$$\|\sqrt{\rho}u\|_{L^4}^2\le C(\bar{\rho},\Omega)\left(1+\|\rho u\|_{L^2}\right)\|\nabla u\|_{L^2}\sqrt{\log\left(2+\|\nabla u\|_{L^2}^2\right)}\,. \tag{2.20}$$

3. Proof of Theorem 1.2

Let T^* be the maximum time for the existence of strong solution (ρ,u,P) to the system (1.1). Suppose that the opposite of (1.6) holds, that is,

$$\sup_{0\le t<T^*}\|\nabla\mu(\rho)(t)\|_{L^p}=M<+\infty\,, \tag{3.21}$$

with some p satisfying $2<p\le q$.

In this section, without special claim, C denotes some positive constant which may depend on Ω, μ, $\bar{\rho}$, the initial data, T^* and M.

Under the assumption (3.21), we will show that

$$\begin{aligned}&\sup_{0<t<T^*}\left(\|\rho(t)\|_{W^{1,q}}+\|\rho_t(t)\|_{L^q}+\|\nabla u(t)\|_{H^1}+\|\sqrt{\rho}u_t\|_{L^2}\right)\le C\,,\\&\sup_{0<t<T^*}\left(\int_0^t\|\nabla u\|_{W^{1,r}}^2\,ds+\int_0^t\|\nabla u_t\|_{L^2}^2\,ds\right)\le C\quad\text{for }1\le r<q,\end{aligned} \tag{3.22}$$

which can guarantee the extension of local strong solution. So the whole proof of Theorem 1.2 consists of a priori estimates of different levels.

3.1. **Energy level estimates.** First, as the density satisfies the transport equation $(1.1)_1$ and making use of $(1.1)_3$, one has the following lemma.

Lemma 3.1. *Suppose (ρ, u, P) is a strong solution to* (1.1) *on* $[0, T^*)$. *Then for every* $t \in [0, T^*)$,

$$\|\rho(t)\|_{L^\infty} = \|\rho_0\|_{L^\infty} \le \bar{\rho} .$$

Next, the basic energy inequality of the system (1.1) reads

Lemma 3.2. *Suppose (ρ, u, P) is a strong solution to* (1.1) *on* $[0, T^*)$. *Then for every* $t \in [0, T^*)$,

$$\frac{1}{2}\int \rho|u(t)|^2\,dx + 2\int_0^t\int \mu(\rho)|d|^2\,dxds \le \frac{1}{2}\int \rho_0|u_0|^2\,dx . \tag{3.23}$$

Since $\mu(\rho) \ge \underline{\mu}$, *and* $2\int |d|^2 dx = \int |\nabla u|^2 dx$, *owing to* $\operatorname{div} u = 0$, *then* (3.23) *implies*

$$\int_0^t\int |\nabla u|^2\,dxds \le C\int \rho_0|u_0|^2\,dx . \tag{3.24}$$

Before proceeding to higher order estimates, we insert one lemma for further use.

Lemma 3.3. *Suppose (ρ, u, P) is a strong solution to* (1.1) *on* $[0, T^*)$. *Under the assumption* (3.21), *it holds that for every* $t \in [0, T^*)$

$$\|\nabla u\|_{H^1} \le C\|\rho u_t\|_{L^2} + C\|\rho u\|_{L^4}^2\|\nabla u\|_{L^2} , \tag{3.25}$$

and consequently by Sobolev embedding,

$$\|\nabla u\|_{H^1} \le C\|\rho u_t\|_{L^2} + C\|\nabla u\|_{L^2}^3 . \tag{3.26}$$

Proof. According to Lemma 2.1 and Gagliardo-Nirenberg inequality,

$$\begin{aligned}\|\nabla u\|_{H^1} \le& C\left(\|\rho u_t\|_{L^2} + \|\rho u\cdot\nabla u\|_{L^2}\right)\cdot\left(1 + \|\nabla\mu(\rho)\|_{L^p}\right)^{\frac{p}{p-2}}\\ \le& C\|\rho u_t\|_{L^2} + C\|\rho u\|_{L^4}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{H^1}^{\frac{1}{2}}\\ \le& C\|\rho u_t\|_{L^2} + C\|\rho u\|_{L^4}^2\|\nabla u\|_{L^2} + \frac{1}{2}\|\nabla u\|_{H^1} ,\end{aligned}$$

which verifies (3.25). □

Now we are ready to estimate $\|\nabla u\|_{L^\infty(0,t;L^2)}$, which is one of the key steps in the blow-up criterion (1.5). More precisely, we have the following lemma.

Lemma 3.4. *Suppose (ρ, u, P) is a strong solution to* (1.1) *on* $[0, T^*)$. *Under the assumption* (3.21), *there exists a generic positive constant C such that*

$$\sup_{0\le t<T^*}\left[\|\nabla u(t)\|_{L^2}^2 + \int_0^t \|\sqrt{\rho}u_t\|_{L^2}^2\,ds\right] \le C . \tag{3.27}$$

Proof. Multiply the momentum equation $(1.1)_2$ by u_t and integrate over Ω, then

$$\begin{aligned}&\int \rho|u_t|^2\,dx + \frac{d}{dt}\int \mu(\rho)|d|^2\,dx\\ \le& \left|\int \rho u\cdot\nabla u\cdot u_t\,dx\right| + C\int |\nabla\mu(\rho)|\cdot|u|\cdot|\nabla u|^2\,dx .\end{aligned} \tag{3.28}$$

Here we have used the renormalized mass equation for $\mu(\rho)$,

$$\partial_t [\mu(\rho)] + u \cdot \nabla \mu(\rho) = 0, \tag{3.29}$$

which is derived due to the fact div $u = 0$.

Applying Gagliardo-Nirenberg inequality and Lemma 3.3, we get

$$\begin{aligned} &\left| \int \rho u \cdot \nabla u \cdot u_t \, dx \right| \\ &\leq \frac{1}{8} \|\sqrt{\rho} u_t\|_{L^2}^2 + C \|\sqrt{\rho} u\|_{L^4}^2 \|\nabla u\|_{L^4}^2 \\ &\leq \frac{1}{8} \|\sqrt{\rho} u_t\|_{L^2}^2 + C \|\sqrt{\rho} u\|_{L^4}^2 \|\nabla u\|_{L^2} \|\nabla u\|_{H^1} \\ &\leq \frac{1}{4} \|\sqrt{\rho} u_t\|_{L^2}^2 + C \|\sqrt{\rho} u\|_{L^4}^4 \|\nabla u\|_{L^2}^2 \ . \end{aligned} \tag{3.30}$$

By Sobolev embedding theorem and Lemma 3.3,

$$\begin{aligned} &\int |\nabla \mu(\rho)| \cdot |u| \cdot |\nabla u|^2 \, dx \\ &\leq C \|\nabla \mu(\rho)\|_{L^p} \|u\|_{L^{p^*}} \|\nabla u\|_{L^4}^2 \quad \text{for } 1/p + 1/p^* = 1/2 \\ &\leq C \|\nabla u\|_{L^2}^2 \|\nabla u\|_{H^1} \\ &\leq C \|\nabla u\|_{L^2}^2 \|\rho u_t\|_{L^2} + C \|\rho u\|_{L^4}^2 \|\nabla u\|_{L^2}^3 \\ &\leq \frac{1}{4} \|\sqrt{\rho} u_t\|_{L^2}^2 + C \|\rho u\|_{L^4}^4 \|\nabla u\|_{L^2}^2 + C \|\nabla u\|_{L^2}^4 \ . \end{aligned} \tag{3.31}$$

Note that Lemma 2.2 tells

$$\begin{aligned} \|\sqrt{\rho} u\|_{L^4}^4 &\leq C \left(1 + \|\rho u\|_{L^2}^2\right) \|\nabla u\|_{L^2}^2 \cdot \log\left(2 + \|\nabla u\|_{L^2}^2\right) \\ &\leq C \|\nabla u\|_{L^2}^2 \log\left(2 + \|\nabla u\|_{L^2}^2\right) \ . \end{aligned} \tag{3.32}$$

Insert the estimates (3.30)-(3.32) into (3.28),

$$\frac{1}{2} \int \rho |u_t|^2 \, dx + \frac{d}{dt} \int \mu(\rho) |d|^2 \, dx \leq C \|\nabla u\|_{L^2}^4 \left(1 + \log\left(2 + \|\nabla u\|_{L^2}^2\right)\right) . \tag{3.33}$$

The proof of Lemma 3.4 is finished after applying Gronwall's inequality to (3.33).

□

3.2. **Higher order level estimates.** Now we are ready to derive the higher order derivatives estimates of the density and velocity.

Lemma 3.5. *Suppose (ρ, u, P) is a strong solution to* (1.1) *on $[0, T^*)$. Under the assumption* (3.21), *there exists a generic positive constant C such that*

$$\sup_{0 \leq T < T^*} \left(\|u\|_{L^2(0,T;L^\infty)} + \|u\|_{L^4(0,T;L^\infty)} \right) \leq C \ . \tag{3.34}$$

Proof. By Gagliardo-Nirenberg inequality and Lemma 3.3, we have

$$\begin{aligned} \int_0^T \|u\|_{L^\infty}^4 \, dt &\leq C \int_0^T \|u\|_{L^2}^2 \|\nabla u\|_{H^1}^2 \, dt \\ &\leq C \int_0^T \left(\|\nabla u\|_{L^2}^2 \|\rho u_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^8 \right) dt \ , \end{aligned}$$

which completes the proof for (3.34), owing to Lemma 3.4. □

The next lemma is crucial to derive the second order derivatives of the velocity.

Lemma 3.6. *Suppose (ρ, u, P) is a strong solution to* (1.1) *on* $[0, T^*)$. *Under the assumption* (3.21), *there exists a generic positive constant C such that*

$$\sup_{0\le t<T^*}\left[\|\sqrt{\rho}u_t(t)\|_{L^2}^2+\int_0^t\|\nabla u_t\|_{L^2}^2\,ds\right]\le C\,. \tag{3.35}$$

Proof. Take t-derivative of the momentum equation,

$$\rho u_{tt}+(\rho u)\cdot\nabla u_t-\text{div }(2\mu(\rho)d_t)+\nabla P_t=-\rho_t u_t-(\rho u)_t\cdot\nabla u+\text{div }(2\mu(\rho)_t d)\,. \tag{3.36}$$

Multiplying (3.36) by u_t and integrating over Ω, we get after integration by parts that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\rho|u_t|^2\,dx+2\int\mu(\rho)|d_t|^2\,dx\\&=-\int\rho_t|u_t|^2\,dx-\int(\rho u)_t\cdot\nabla u\cdot u_t\,dx-\int 2\mu(\rho)_t d\cdot\nabla u_t\,dx\\&\triangleq\sum_{i=1}^{3}I_i.\end{aligned} \tag{3.37}$$

Let us estimate each term I_i step by step.

First, utilizing the mass equation, one has

$$\begin{aligned}I_1&=-2\int\rho u\cdot\nabla u_t\cdot u_t\,dx\\&\le C\|u\|_{L^\infty}\|\nabla u_t\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}\\&\le\frac{1}{8}\int\mu(\rho)\,|d_t|^2\,dx+C\|u\|_{L^\infty}^2\|\sqrt{\rho}u_t\|_{L^2}^2\,,\end{aligned} \tag{3.38}$$

where for the last inequality we used the fact $\int|\nabla u_t|^2\,dx=2\int|d_t|^2\,dx$.

Seconly, utilizing the renormalized mass equation (3.29) for $\mu(\rho)$,

$$\begin{aligned}I_3&=-\int 2\mu(\rho)_t\cdot d\cdot\nabla u_t\,dx\\&\le C\int|u|\cdot|\nabla\mu(\rho)|\cdot|d|\cdot|\nabla u_t|\,dx\\&\le C\|u\|_{L^\infty}\|\nabla\mu(\rho)\|_{L^p}\|d\|_{L^{p^*}}\|\nabla u_t\|_{L^2},\quad\text{for }1/p+1/p^*=1/2\\&\le\frac{1}{8}\int\mu(\rho)|d_t|^2\,dx+C\|u\|_{L^\infty}^2\|\nabla\mu(\rho)\|_{L^p}^2\|\nabla u\|_{L^{p^*}}^2\\&\le\frac{1}{8}\int\mu(\rho)|d_t|^2dx+C\|u\|_{L^\infty}^2\|\nabla u\|_{H^1}^2\\&\le\frac{1}{8}\int\mu(\rho)|d_t|^2dx+C\|u\|_{L^\infty}^2\|\sqrt{\rho}u_t\|_{L^2}^2+C\|u\|_{L^\infty}^2\|\nabla u\|_{L^2}^6\,.\end{aligned} \tag{3.39}$$

Finally, taking into account the mass equation again, we arrive at

$$
\begin{aligned}
I_2 &= -\int (\rho u)_t \cdot \nabla u \cdot u_t \, dx \\
&= -\int \rho u \cdot \nabla \left[u \cdot \nabla u \cdot u_t \right] dx - \int \rho u_t \cdot \nabla u \cdot u_t \, dx \\
&\leq \int \rho |u| \cdot |\nabla u|^2 \cdot |u_t| \, dx + C \int \rho |u|^2 \cdot |\nabla^2 u| \cdot |u_t| \, dx \\
&\quad + \int \rho |u|^2 \cdot |\nabla u| \cdot |\nabla u_t| \, dx + \int \rho |u_t|^2 \cdot |\nabla u| \, dx \\
&\triangleq \sum_{i=1}^{4} J_i.
\end{aligned} \tag{3.40}
$$

Herein, it follows from Sobolev embedding theorem, Gagliardo-Nirenberg inequality, and Lemma 3.3 that

$$
\begin{aligned}
J_1 &= \int \rho |u| \cdot |\nabla u|^2 \cdot |u_t| \, dx \\
&\leq \|u\|_{L^\infty} \|\nabla u\|_{L^4}^2 \|\sqrt{\rho} u_t\|_{L^2} \\
&\leq C \|u\|_{L^\infty}^2 \|\sqrt{\rho} u_t\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 \|\nabla u\|_{H^1}^2 \\
&\leq C \|u\|_{L^\infty}^2 \|\sqrt{\rho} u_t\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 \|\sqrt{\rho} u_t\|_{L^2}^2 + C \|\nabla u\|_{L^2}^8
\end{aligned} \tag{3.41}
$$

and

$$
\begin{aligned}
J_2 &= \int \rho |u|^2 \cdot |\nabla^2 u| \cdot |u_t| \, dx \\
&\leq C \|u\|_{L^\infty}^2 \|\nabla^2 u\|_{L^2} \|\sqrt{\rho} u_t\|_{L^2} \\
&\leq C \|u\|_{L^\infty}^2 \|\sqrt{\rho} u_t\|_{L^2} \left(\|\rho u_t\|_{L^2} + \|\nabla u\|_{L^2}^3 \right) \\
&\leq C \|u\|_{L^\infty}^2 \|\sqrt{\rho} u_t\|_{L^2}^2 + C \|u\|_{L^\infty}^2 \|\nabla u\|_{L^2}^6 .
\end{aligned} \tag{3.42}
$$

Owing to the fact that $2\int |d_t|^2 \, dx = \int |\nabla u_t|^2 \, dx$,

$$
\begin{aligned}
J_3 &= \int \rho |u|^2 \cdot |\nabla u| \cdot |\nabla u_t| \, dx \\
&\leq C \|u\|_{L^\infty}^2 \|\nabla u\|_{L^2} \|\nabla u_t\|_{L^2} \\
&\leq \frac{1}{8} \int \mu(\rho) |d_t|^2 \, dx + C \|u\|_{L^\infty}^4 \|\nabla u\|_{L^2}^2 .
\end{aligned} \tag{3.43}
$$

Recall Lemma 3.3 and Sobolev embedding theorem again, one deduces that

$$
\begin{aligned}
J_4 &= \int \rho |u_t|^2 \cdot |\nabla u| \, dx \\
&\leq C \|\sqrt{\rho} u_t\|_{L^2} \|u_t\|_{L^4} \|\nabla u\|_{L^4} \\
&\leq C \|\sqrt{\rho} u_t\|_{L^2} \|\nabla u_t\|_{L^2} \|\nabla u\|_{H^1} \\
&\leq \frac{1}{8} \int \mu(\rho) |d_t|^2 \, dx + C \|\sqrt{\rho} u_t\|_{L^2}^2 \|\nabla u\|_{H^1}^2 \\
&\leq \frac{1}{8} \int \mu(\rho) |d_t|^2 \, dx + C \|\sqrt{\rho} u_t\|_{L^2}^4 + C \|\sqrt{\rho} u_t\|_{L^2}^2 \|\nabla u\|_{L^2}^6 .
\end{aligned} \tag{3.44}
$$

Inserting the estimates (3.38)-(3.44) into (3.37), we obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho|u_t|^2\,dx + \int \mu(\rho)|d_t|^2\,dx \\
(3.45)\qquad &\le C\|u\|_{L^\infty}^2\|\sqrt{\rho}u_t\|_{L^2}^2 + C\left(\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^6\right)\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\sqrt{\rho}u_t\|_{L^2}^4 \\
&\quad + C\|u\|_{L^\infty}^2\|\nabla u\|_{L^2}^6 + C\|u\|_{L^\infty}^4\|\nabla u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^8 .
\end{aligned}$$

Consequently, it follows from Gronwall's inequality and Lemmas 3.4,3.5 that

$$\sup_{0\le t<T^*}\left[\|\sqrt{\rho}u_t(t)\|_{L^2}^2 + \int_0^t \|\nabla u_t\|_{L^2}^2\,ds\right] \le C .$$

□

Now we are ready to estimate $\|\nabla u\|_{H^1}$.

Lemma 3.7. *Suppose (ρ, u, P) is a strong solution to (1.1) on $[0, T^*)$. Under the assumption (3.21), there exists a generic positive constant C such that*

$$\sup_{0\le t<T^*}\|\nabla u(t)\|_{H^1} \le C .$$

Proof. By Lemma 3.3,

$$\|\nabla u\|_{H^1} \le C\|\rho u_t\|_{L^2} + C\|\nabla u\|_{L^2}^3 , \tag{3.46}$$

which proves Lemma 3.7 with the aid of Lemmas 3.4 and 3.6. □

Furthermore, one has

Lemma 3.8. *Suppose (ρ, u, P) is a strong solution to (1.1) on $[0, T^*)$. Under the assumption (3.21), there exists a generic positive constant C such that*

$$\sup_{0\le T<T^*}\left(\int_0^T \|\nabla u\|_{L^\infty}\,dt\right) \le C . \tag{3.47}$$

Proof. Choose some r, with $2 < r < \min\{p, 4\}$, by Sobolev embedding theorem and Lemma 2.1,

(3.48)

$$\begin{aligned}
\|\nabla u\|_{L^1(0,T;L^\infty)} &\le C\|\nabla u\|_{L^1(0,T;W^{1,r})} \\
&\le C\|\rho u_t\|_{L^1(0,T;L^4)} + C\|\rho u\cdot\nabla u\|_{L^1(0,T;L^4)} \\
&\le C\|\nabla u_t\|_{L^1(0,T;L^2)} + C\|\nabla u\|_{L^2(0,T;H^1)}^2 \\
&\le C\|\nabla u_t\|_{L^1(0,T;L^2)} + C\|\rho u_t\|_{L^2(0,T;L^2)}^2 + C\|\nabla u\|_{L^6(0,T;L^2)}^6 ,
\end{aligned}$$

which completes the proof for (3.47), with the aid of Lemmas 3.4 and 3.6. □

With the help of Lemma 3.8, we are in a position to close the first order derivative estimates for the density.

Lemma 3.9. *Suppose (ρ, u, P) is a strong solution to (1.1) on $[0, T^*)$. Under the assumption (3.21), there exists a generic positive constant C such that*

$$\sup_{0\le t<T^*}\left(\|\rho(t)\|_{W^{1,q}} + \|\rho_t(t)\|_{L^q}\right) \le C . \tag{3.49}$$

Proof. Consider the x_i-derivative of the mass equation, $i = 1, 2$,

$$(\partial_i \rho)_t + (u \cdot \nabla)\partial_i \rho + (\partial_i u \cdot \nabla)\rho = 0 \, .$$

It implies that for every $t \in [0, T^*)$,

$$\|\nabla \rho(t)\|_{L^q} \le C \|\nabla \rho_0\|_{L^q} \exp\left\{ \int_0^t \|\nabla u(s)\|_{L^\infty} \, ds \right\} . \tag{3.50}$$

Hence, by Lemma 3.8, we finish the proof for the first part of (3.49).

It follows from the mass equation and Sobolev embedding theorem that

$$\|\rho_t\|_{L^q} \le \|u \cdot \nabla \rho\|_{L^q} \le \|u\|_{L^\infty} \|\nabla \rho\|_{L^q} \le \|\nabla u\|_{H^1} \|\nabla \rho\|_{L^q} \, ,$$

which together with (3.50) and Lemma 3.7 completes the proof for the second part of (3.49).

□

In addition, one has the following regularity.

Lemma 3.10. *Suppose (ρ, u, P) is a strong solution to (1.1) on $[0, T^*)$. Under the assumption (3.21), it holds that for $2 \le r < q$,*

$$\sup_{0 \le T < T^*} \int_0^T \left(\|\nabla u\|_{W^{1,r}}^2 + \|P\|_{W^{1,r}}^2 \right) dt \le C \, . \tag{3.51}$$

Proof. By Lemma 2.1, Lemma 3.9 and Sobolev embedding theorem,

$$\begin{aligned} \|\nabla u\|_{W^{1,r}} + \|P\|_{W^{1,r}} &\le C \left(\|\rho u_t\|_{L^r} + \|\rho u \cdot \nabla u)\|_{L^r} \right) \left(1 + \|\nabla \mu(\rho)\|_{L^q}\right)^{1 + qr/2(q-r)} \\ &\le C \left(\|\nabla u_t\|_{L^2} + \|\nabla u\|_{H^1}^2 \right) \cdot \left(1 + \|\nabla \rho\|_{L^q}\right)^{1 + qr/2(q-r)} . \end{aligned}$$

Hence, (3.51) is proved with the aid of Lemmas 3.6, 3.7 and 3.9. □

Now, combining all the estimates derived in Theorems 3.4-3.10, we finish all the estimates mentioned in (3.22), and hence completes the proof for Theorem 1.2.

4. Proof of Theorem 1.4

The proof of Theorem 1.4 consists of two parts. The first part is devoted to proving that $\|\nabla \mu(\rho)\|_{L^q}$ is always less than $\frac{1}{2}$ provided that the initial data $\nabla \mu(\rho_0)$ is small enough. Based on these estimates, the second part aims to extend the local strong solution to global one.

4.1. **A Priori Estimates.** In this subsection, we establish some a priori time-weighted estimates independent of time interval. The idea is based on the parabolic property of the system.

In this subsection, the constant C will denote some positive constant which depends only on Ω, q, $\bar{\rho}$, $\underline{\mu}$, $\overline{\mu}$, $\|\nabla u_0\|_{L^2}$ but independent of time T.

First, just same as Lemma 3.1, one has

Lemma 4.1. *Suppose (ρ, u, P) is the unique local strong solution to (1.1) on $[0, T]$, with the initial data (ρ_0, u_0), it holds that*

$$0 \le \rho(x, t) \le \bar{\rho}, \qquad \textit{for every } (x, t) \in \Omega \times [0, T].$$

Next, the basic energy estimate reads

Lemma 4.2. *Suppose (ρ, u, P) is the unique local strong solution to (1.1) on $[0,T]$, with the initial data (ρ_0, u_0), it holds that*

$$\int \rho|u(t)|^2\,dx + \int_0^t \int |\nabla u|^2\,dxds \le C\int \rho_0|u_0|^2\,dx\,, \quad \textit{for every } t\in[0,T], \tag{4.52}$$

Furthermore,

$$\sup_{t\in[0,T]} t\|\sqrt{\rho}u(t)\|_{L^2}^2 + \int_0^T t\|\nabla u\|_{L^2}^2\,dt \le C\int \rho_0|u_0|^2\,dx\,. \tag{4.53}$$

Proof. The proof of (4.52) is same as Lemma 3.2. It only remains to prove (4.53).
First, one has

$$\frac{1}{2}\frac{d}{dt}\int \rho|u|^2 dx + 2\int \mu(\rho)|d|^2 dx = 0. \tag{4.54}$$

Since Ω is a bounded domain, one can deduce from Poincaré's inequality that

$$\frac{1}{2}\int \rho|u|^2\,dx \le C\|u\|_{L^2}^2 \le C\|\nabla u\|_{L^2}^2 \le C\int \mu(\rho)|d|^2\,dx\,, \tag{4.55}$$

where the fact $\mu(\rho) \ge \underline{\mu} > 0$ is used. Combining (4.54) and (4.55), we obtain

$$\int \rho|u(t)|^2\,dx \le Ce^{-Ct}\int \rho_0|u_0|^2\,dx\,. \tag{4.56}$$

Multiplying the equality (4.54) by t and integrating over Ω, one has

$$\frac{d}{dt}\int \frac{t}{2}\rho|u|^2\,dx + 2t\int \mu(\rho)|d|^2\,dx = \frac{1}{2}\int \rho|u|^2\,dx\,,$$

which together with (4.56) implies

$$\sup_{t\in[0,T]} t\|\sqrt{\rho}u(t)\|_{L^2}^2 + \int_0^T t\|\nabla u\|_{L^2}^2\,dt \le C\int_0^T\int \rho|u|^2\,dxdt \le C\int \rho_0|u_0|^2\,dx\,.$$

□

The next lemma is exactly the same as Lemma 3.3 which will be used later. We write down here without proof.

Lemma 4.3. *Suppose (ρ, u, P) is the unique local strong solution to (1.1) on $[0,T]$ and*

$$\sup_{t\in[0,T]} \|\nabla\mu(\rho(t))\|_{L^q} \le 1.$$

then

$$\|\nabla u\|_{H^1} \le C\|\rho u_t\|_{L^2} + C\|\rho u\|_{L^4}^2\|\nabla u\|_{L^2}\,. \tag{4.57}$$

Now, we are ready to get some time-weighted estimates for $\|\nabla u\|_{L^2}$.

Lemma 4.4. *Suppose (ρ, u, P) is the unique local strong solution to (1.1) on $[0,T]$ and satisfies*

$$\sup_{t\in[0,T]} \|\nabla\mu(\rho(t))\|_{L^q} \le 1 \ .$$

Then

$$\sup_{t\in[0,T]} t^\alpha \|\nabla u\|_{L^2}^2 + \int_0^T \int t^\alpha \rho |u_t|^2 \, dxdt \le C(\alpha) \ , \quad \text{for every } \alpha \in [0,2], \tag{4.58}$$

where $C(\alpha)$ is a positive constant depending on $\alpha, \Omega, q, \bar{\rho}, \underline{\mu}, \|u_0\|_{H^1}$.

Proof. It suffices to verify (4.58) for $\alpha = 0$ and $\alpha = 2$.

When $\alpha = 0$, the proof is exactly the same as Lemma 3.4. Indeed, we get from (3.33) that

$$\begin{aligned} &\frac{1}{2}\int \rho |u_t|^2 \, dx + \frac{d}{dt}\int \mu(\rho)|d|^2 \, dx \\ &\le C\left(1 + \|\rho u\|_{L^2}\right) \|\nabla u\|_{L^2}^4 \left(1 + \log\left(2 + \|\nabla u\|_{L^2}^2\right)\right) \ . \end{aligned} \tag{4.59}$$

When $\alpha = 2$, multiplying (4.59) by t^2 arrives at

$$\begin{aligned} &\frac{1}{2} t^2 \int \rho |u_t|^2 \, dx + \frac{d}{dt}\int t^2 \mu(\rho)|d|^2 \, dx \\ &\le 2t \int \mu(\rho)|d|^2 \, dx + Ct^2 \|\nabla u\|_{L^2}^4 \left(1 + \log\left(2 + \|\nabla u\|_{L^2}^2\right)\right) \\ &\le 2t \int \mu(\rho)|d|^2 \, dx + Ct^2 \|\nabla u\|_{L^2}^4 \ , \end{aligned} \tag{4.60}$$

which together with Lemma 4.2 completes the proof for (4.58) with $\alpha = 2$. Hence, the proof for Lemma 4.4 is complete.

□

We insert a lemma before the estimates for $\|\sqrt{\rho} u_t\|_{L^2}$.

Lemma 4.5. *Suppose (ρ, u, P) is the unique local strong solution to (1.1) on $[0,T]$ and satisfies*

$$\sup_{t\in[0,T]} \|\nabla\mu(\rho(t))\|_{L^q} \le 1 \ .$$

Then

$$\|u\|_{L^2(0,T;L^\infty)} + \|\nabla u\|_{L^2(0,T;L^3)} \le C \ , \tag{4.61}$$

and also

$$\int_0^T t\|u\|_{L^\infty}^4 \, dt + \int_0^T t^2 \|u\|_{L^\infty}^4 \, dt \le C \ . \tag{4.62}$$

Proof. It follows from Sobolev embedding theorem and Lemma 4.3 that

$$\begin{aligned} \|u\|_{L^2(0,T;L^\infty)} + \|\nabla u\|_{L^2(0,T;L^3)} &\le C\|\nabla u\|_{L^2(0,T;H^1)} \\ &\le C\|\rho u_t\|_{L^2(0,T;L^2)} + C\|u\|_{L^4(0,T;L^4)}^2 \|\nabla u\|_{L^\infty(0,T;L^2)} \\ &\le C\|\rho u_t\|_{L^2(0,T;L^2)} + C\|\nabla u\|_{L^4(0,T;L^2)}^2 \|\nabla u\|_{L^\infty(0,T;L^2)} \\ &\le C\|\rho u_t\|_{L^2(0,T;L^2)} + C\|\nabla u\|_{L^2(0,T;L^2)} \|\nabla u\|_{L^\infty(0,T;L^2)}^2 \ , \end{aligned}$$

which together with Lemmas 4.2 and 4.4 completes the proof for (4.61).

By Gagliardo-Nirenberg inequality and Lemma 4.3,

$$
\begin{aligned}
\int_0^T t\|u\|_{L^\infty}^4\,dt &\le C\int_0^T t\|u\|_{L^2}^2\|\nabla u\|_{H^1}^2\,dt \\
&\le C\int_0^T t\|\nabla u\|_{L^2}^2\|\rho u_t\|_{L^2}^2\,dt \\
&\quad + C\int_0^T t\|u\|_{L^2}^2\|u\|_{L^4}^4\|\nabla u\|_{L^2}^2\,dt \\
&\le C\|\nabla u\|_{L^\infty(0,T;L^2)}^2\int_0^T t\|\sqrt{\rho}u_t\|_{L^2}^2\,dt \\
&\quad + C\|\nabla u\|_{L^\infty(0,T;L^2)}^6\int_0^T t\|\nabla u\|_{L^2}^2\,dt\,.
\end{aligned}
\tag{4.63}
$$

Also, upon the estimates in (4.63), we have

$$
\begin{aligned}
\int_0^T t^2\|u\|_{L^\infty}^4\,dt &\le C\|\nabla u\|_{L^\infty(0,T;L^2)}^2\int_0^T t^2\|\sqrt{\rho}u_t\|_{L^2}^2\,dt \\
&\quad + C\|\nabla u\|_{L^\infty(0,T;L^2)}^4\|t\nabla u\|_{L^\infty(0,T;L^2)}^2\int_0^T \|\nabla u\|_{L^2}^2\,dt\,.
\end{aligned}
\tag{4.64}
$$

Applying Lemmas 4.2 and 4.4 to (4.63)-(4.64) finishes the proof for (4.62). □

Lemma 4.6. *Suppose (ρ, u, P) is the unique local strong solution to (1.1) on $[0,T]$ and satisfies*

$$\sup_{t\in[0,T]}\|\nabla\mu(\rho(t))\|_{L^q}\le 1\,.$$

Then

$$\sup_{t\in[0,T]} t^\beta\int \rho|u_t|^2\,dx + \int_0^T t^\beta\|\nabla u_t\|_{L^2}^2\,dt \le C(\beta)\,, \quad \textit{for every } \beta\in[1,2], \tag{4.65}$$

where $C(\beta)$ is a positive constant depending on $\beta, \Omega, q, \bar{\rho}, \underline{\mu}, \overline{\mu}, \|u_0\|_{H^1}$.

The proof of Lemma 4.6 is almost the same as that of Lemma 3.6.

Proof. It suffices to verify (4.65) for $\beta = 1$ and $\beta = 2$.

When $\beta = 1$, multiplying the inequality (3.45) in Section 3 by t, we obtain that

$$
\begin{aligned}
&\frac{d}{dt}\int \frac{t}{2}\rho|u_t|^2\,dx + t\int \mu(\rho)|d_t|^2\,dx \\
&\le \frac{1}{2}\int \rho|u_t|^2\,dx + Ct\|u\|_{L^\infty}^2\|\sqrt{\rho}u_t\|_{L^2}^2 + Ct\left(\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^6\right)\|\sqrt{\rho}u_t\|_{L^2}^2 \\
&\quad + Ct\|\sqrt{\rho}u_t\|_{L^2}^4 + Ct\|u\|_{L^\infty}^2\|\nabla u\|_{L^2}^6 + Ct\|u\|_{L^\infty}^4\|\nabla u\|_{L^2}^2 + Ct\|\nabla u\|_{L^2}^8\,.
\end{aligned}
$$

Gronwall's inequality gives that

$$\sup_{t\in[0,T]} t\|\sqrt{\rho}u_t\|_{L^2}^2 + \int_0^T t\|\nabla u_t\|_{L^2}^2\,dt \le C\,, \tag{4.66}$$

owing to the estimates in Lemmas 4.2, 4.4 and4.5 .

When $\beta = 2$, multiplying (3.45) by t^2 gives

$$\begin{aligned}
&\frac{d}{dt}\int \frac{t^2}{2}\rho|u_t|^2\,dx + t^2\int \mu(\rho)|d_t|^2\,dx\\
&\le t\int \rho|u_t|^2\,dx + Ct^2\|u\|_{L^\infty}^2\|\sqrt{\rho}u_t\|_{L^2}^2 + Ct^2\left(\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^6\right)\|\sqrt{\rho}u_t\|_{L^2}^2\\
&\quad + Ct^2\|\sqrt{\rho}u_t\|_{L^2}^4 + Ct^2\|u\|_{L^\infty}^2\|\nabla u\|_{L^2}^6 + Ct^2\|u\|_{L^\infty}^4\|\nabla u\|_{L^2}^2 + Ct^2\|\nabla u\|_{L^2}^8\,.
\end{aligned}$$

Again, utilizing Gronwall's inequality, Lemmas 4.2, 4.4 and 4.5, we prove (4.65) for $\beta = 2$. Hence Lemma 4.6 is proved. □

The next lemma is crucial to derive the higher order estimates for the density.

Lemma 4.7. *Suppose (ρ, u, P) is the unique local strong solution to (1.1) on $[0,T]$ and satisfies*

$$\sup_{t\in[0,T]}\|\nabla\mu(\rho(t))\|_{L^q} \le 1\,.$$

Then there exists a generic positive constant C independent of time T, such that

$$\|\nabla u\|_{L^1(0,T;L^\infty)} \le C\,. \tag{4.67}$$

Proof. Choose some r, with $2 < r < \min\{q, 3\}$, by Lemma 2.1,

$$\begin{aligned}
\|\nabla u\|_{L^1(0,T;L^\infty)} &\le C\|\nabla u\|_{L^1(0,T;W^{1,r})}\\
&\le C\|\rho u_t\|_{L^1(0,T;L^3)} + C\|\rho u\cdot\nabla u\|_{L^1(0,T;L^3)}\,.
\end{aligned} \tag{4.68}$$

Herein, by Gagliardo-Nirenberg inequality and Poincaré's inequality,

$$\|\rho u_t\|_{L^3} \le C\|\rho u_t\|_{L^2}^{\frac12}\|\rho u_t\|_{L^6}^{\frac12} \le C\|\sqrt{\rho}u_t\|_{L^2}^{\frac12}\|\nabla u_t\|_{L^2}^{\frac12}\,,$$

which implies

$$\begin{aligned}
\int_0^T \|\rho u_t\|_{L^3}\,dt &\le C\int_0^T t^{-\frac38}\|\sqrt{\rho}u_t\|_{L^2}^{\frac12}\cdot t^{\frac38}\|\nabla u_t\|_{L^2}^{\frac12}\,dt\\
&\le C\left[\int_0^T t^{-\frac12}\|\sqrt{\rho}u_t\|_{L^2}^{\frac23}\,dt\right]^{\frac34}\cdot\left[\int_0^T t^{\frac32}\|\nabla u_t\|_{L^2}^2\,dt\right]^{\frac14}\,.
\end{aligned}$$

If $T \le 1$, taking $\beta = 1$ in Lemma 4.6, one has

$$\int_0^T \|\rho u_t\|_{L^3}\,dt \le C\left[\int_0^T t^{-\frac12}\cdot t^{-\frac12\cdot\frac23}\,dt\right]^{\frac34}\cdot\left[\int_0^T t^{\frac32}\|\nabla u_t\|_{L^2}^2\,dt\right]^{\frac14} \le C\,.$$

If $T > 1$, taking $\beta = 2$ in Lemma 4.6 again to get

$$\begin{aligned}
\int_0^T \|\rho u_t\|_{L^3}\,dt \le & \int_0^1 \|\rho u_t\|_{L^3}\,dt + \int_1^T \|\rho u_t\|_{L^3}\,dt \\
\le & C\left[\int_0^1 t^{-\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^2}^{\frac{2}{3}}\,dt\right]^{\frac{3}{4}} \cdot \left[\int_0^1 t^{\frac{3}{2}}\|\nabla u_t\|_{L^2}^2\,dt\right]^{\frac{1}{4}} \\
& + C\left[\int_1^T t^{-\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^2}^{\frac{2}{3}}\,dt\right]^{\frac{3}{4}} \cdot \left[\int_1^T t^{\frac{3}{2}}\|\nabla u_t\|_{L^2}^2\,dt\right]^{\frac{1}{4}} \\
\le & C\left[\int_0^1 t^{-\frac{1}{2}} \cdot t^{-\frac{1}{2}\cdot\frac{2}{3}}\,dt\right]^{\frac{3}{4}} \cdot \left[\int_0^1 t^{\frac{3}{2}}\|\nabla u_t\|_{L^2}^2\,dt\right]^{\frac{1}{4}} \\
& + C\left[\int_1^T t^{-\frac{1}{2}} \cdot t^{-\frac{2}{3}}\,dt\right]^{\frac{3}{4}} \cdot \left[\int_1^T t^{\frac{3}{2}}\|\nabla u_t\|_{L^2}^2\,dt\right]^{\frac{1}{4}} \\
\le & C\,.
\end{aligned}$$

Hence, we have $\int_0^T \|\rho u_t\|_{L^3}\,dt \le C$, no matter $T \le 1$ or $T \ge 1$, and we remark again that C is independent of T.

On the other hand, by Lemma 4.5,

$$\int_0^T \|(\rho u \cdot \nabla)u\|_{L^3}\,dt \le C\|u\|_{L^2(0,T;L^\infty)} \cdot \|\nabla u\|_{L^2(0,T;L^3)} \le C\,.$$

Now we can conclude that $\int_0^T \|\nabla u\|_{L^\infty} dt \le C$, which completes the proof for Lemma 4.7. □

Finally, let's close the estimates for $\nabla\mu(\rho)$.

Lemma 4.8. *Suppose (ρ, u, P) is the unique local strong solution to (1.1) on $[0,T]$ and*

$$\sup_{t\in[0,T]} \|\nabla\mu(\rho)\|_{L^q} \le 1\,.$$

There exists a positive number ϵ_0 depending only on Ω, q, $\bar{\rho}$, $\underline{\mu}$, $\overline{\mu}$, $\|u_0\|_{H^1}$, such that if

$$\|\nabla\mu(\rho_0)\|_{L^q} \le \epsilon_0\,,$$

then

$$\sup_{t\in[0,T]} \|\nabla\mu(\rho)\|_{L^q} \le \frac{1}{2}\,.$$

Note that ϵ_0 is independent of T.

Proof. Consider the x_i-derivative of the renormalized mass equation for $\mu(\rho)$,

$$(\partial_i\mu(\rho))_t + (\partial_i u \cdot \nabla)\mu(\rho) + u \cdot \nabla\partial_i\mu(\rho) = 0\,.$$

It implies that for every $t \in [0,T]$,

$$\begin{aligned}
\|\nabla\mu(\rho)(t)\|_{L^q} &\le C\|\nabla\mu(\rho_0)\|_{L^q} \cdot \exp\left\{\int_0^t \|\nabla u\|_{L^\infty}\,ds\right\} \\
&\le C_2\|\nabla\mu(\rho_0)\|_{L^q}\,,
\end{aligned} \tag{4.69}$$

where we used Lemma 4.7 and C_2 is a constant independent of T.

Hence, if we set $\epsilon_0 = \frac{1}{2C_2}$, then Lemma 4.8 is proved. □

Now we plan to get high order estimates. The proof is the same as in Section 3. We will omit the details for brevity and just write down the lemma.

Lemma 4.9. *Suppose (ρ, u, P) is the unique local strong solution to* (1.1) *on* $[0,T]$, *and*

$$\sup_{t\in[0,T]} \|\nabla\mu(\rho)\|_{L^q} \le 1 .$$

Then it holds that

$$\begin{aligned} &\sup_{0<t\le T} (\|\rho(t)\|_{W^{1,q}} + \|\rho_t(t)\|_{L^q} + \|\nabla u(t)\|_{H^1} + \|\sqrt{\rho}u_t(t)\|_{L^2}) \le \bar{C} , \\ &\int_0^T (\|\nabla u\|_{W^{1,r}}^2 + \|\nabla u_t\|_{L^2}^2)\, dt \le \bar{C} . \end{aligned} \tag{4.70}$$

Here $\bar{C}$ is a positive constant, which may depend on T, μ, and the initial data.

4.2. **Proof of Theorem 1.4.** With the a priori estimates in Subsection 4.1 in hand, we are prepared for the proof of Theorem 1.4.

Proof. According to Theorem 1.1, there exists a $T_* > 0$ such that the density-dependent Navier-Stokes system (1.1) has a unique local strong solution (ρ, u, P) on $[0, T_*]$, and T_* depends on $\|\rho_0\|_{W^{1,q}}$, $\|\nabla u_0\|_{H^1}$, $\|g\|_{L^2}$ and μ, where g is the function in the compatibility condition (1.4). We plan to extend the local solution to a global one.

Since $\|\nabla\mu(\rho_0)\|_{L^q} \le \epsilon_0 \le \frac{1}{2}$ and due to the continuity of $\nabla\mu(\rho)$ in L^q, there exists a $T_1 \in (0, T_*)$ such that $\sup_{0\le t\le T_1} \|\nabla\mu(\rho)(t)\|_{L^q} \le 1$. Set

$$T^* = \sup\{T|\ (\rho, u, P) \text{ is a strong solution on } [0,T]\} .$$

$$T_1^* = \sup\left\{T|\ (\rho, u, P) \text{ is a strong solution on } [0,T] \text{ and } \sup_{0\le t\le T} \|\nabla\mu(\rho)\|_{L^q} \le 1\right\} .$$

Then $T_1^* \ge T_1 > 0$. Recall Lemma 4.8, it's easy to verify

$$T^* = T_1^*. \tag{4.71}$$

We claim that $T^* = \infty$. Otherwise, assuming that $T^* < \infty$. By Lemma 4.9, for every $t \in [0, T^*)$, there exists a uniform generic constant $\bar{C}(T^*)$, such that

$$\|\rho(t)\|_{W^{1,q}} + \|\nabla u(t)\|_{H^1} \le \bar{C}(T^*) . \tag{4.72}$$

which contradicts to the blowup criterion (1.5). Hence we complete the proof for Theorem 1.4.

□

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