数理解析研究所講究録
第 1876 巻 2014 年 22-27

HOMOTOPY COMMUTATIVITY IN LOCALIZED GAUGE GROUPS

DAISUKE KISHIMOTO

1. Introduction and statement of the result

This is a survey the paper [KKTh] written with Akira Kono and Stephen Theriault.

Throughout the paper, we only consider the Lie group $G = \mathrm{SU}(n)$ for simplicity, while most results hold for other simply connected, simple Lie groups. Let us recall p-local properties of G.

Theorem 1.1 (Mimura, Nishida and Toda [MNT]). *There exist p-local spaces $B_1, \ldots, B_{p-1}$ satisfying*

$$G_{(p)} \simeq B_1 \times \cdots \times B_{p-1},$$

where the mod p cohomology of B_i is given by

$$H^*(B_i; \mathbb{Z}/p) = \Lambda(x_{2i+1+2k(p-1)} \,|\, 0 \le k < \tfrac{n-i-1}{p-1}), \quad |x_j| = j.$$

This is called the mod p decomposition of G. Observe that if $p \ge n$, each B_i has the homotopy type of $S^{2i+1}_{(p)}$ or a point. Then we can say that the p-local homotopy type of G degenerates as p gets larger. So it is natural to consider degeneration of the H-structure of $G_{(p)}$ as p gets larger. As for homotopy commutativity, the complete answer was given by McGibbon [M] as:

Theorem 1.2 (McGibbon [M]). *$G_{(p)}$ is homotopy commutative if and only if $p > 2n$.*

Later, this result was generalized by Kaji and Kishimoto [KaKi] and Kishimoto [Ki] to homotopy nilpotency.

Our object to study is a gauge group which is the topological group of all automorphisms of a principal bundle, i.e. self-maps of the total space which are compatible with the action of the fiber and cover the identity map of the base space. Recall that principal G-bundles over S^4 are classified by $\pi_4(BG) \cong \mathbb{Z}$. We write the gauge group of the principal G-bundle over S^4 corresponding to the integer $k \in \mathbb{Z} \cong \pi_4(BG)$ by $\mathcal{G}_k$. The homotopy theory of gauge groups has been studied in many directions (cf. [CS, Ko, KiKo]). In each work, we have seen that $\mathcal{G}_k$ has a close relation with G as is expected from definition. So we may expect that $\mathcal{G}_k$ possesses p-local properties analogous to G. As for the mod p decomposition, our expectation has been proved to be true.

The second author is partially supported by the Grant-in-Aid for Scientific Research (C)(No.25400087) from the Japan Society for Promotion of Sciences.

Theorem 1.3 (Kishimoto, Kono and Tsutaya [KKTs]). *There exist p-local spaces* $\mathcal{B}_1, \ldots, \mathcal{B}_{p-1}$ *satisfying*

$$\mathcal{G}_{k(p)} \simeq \mathcal{B}_1 \times \cdots \times \mathcal{B}_{p-1}$$

and homotopy fibrations

$$\Omega(\Omega_0^3 B_i) \to \mathcal{B}_i \to B_{i-2},$$

where we regard the spaces B_i *of Theorem 1.1 are indexed by* $\mathbb{Z}/(p-1)$. *Moreover, the homotopy fibrations are trivial if* $p \geq n+2$.

In particular, we can say that the p-local homotopy type of $\mathcal{G}_k$ degenerates as p gets larger, analogously to G. Now we naturally ask whether there is a gauge group version of Theorem 1.2. Let us state our main result.

Theorem 1.4. *Suppose* $n \geq 4$.

(1) *For* $p < 2n+1$, $\mathcal{G}_{k(p)}$ *is not homotopy commutative.*
(2) *For* $p > 2n+1$, $\mathcal{G}_{k(p)}$ *is homotopy commutative.*
(3) *For* $p = 2n+1$, $\mathcal{G}_{k(p)}$ *is homotopy commutative if and only if* p *divides* k.

Remark 1.5. Note that the integer k only appears in the border case $p = 2n+1$.

2. Noncommutativity

In this section, we give a sketch of the proof of the noncommutativity result on $\mathcal{G}_{k(p)}$. We first recall basic facts of gauge groups briefly. Let ϵ_i be a generator of $\pi_{2i-1}(G) \cong \mathbb{Z}$ for $i = 2, \ldots, n$. Recall that there is a natural homotopy equivalence

$$B\mathcal{G}_k \simeq \mathrm{map}(S^4, BG; k\bar{\epsilon}_2),$$

where $\mathrm{map}(X, Y; f)$ stands for the connected component of the space of maps from X to Y containing a map $f : X \to Y$ and $\bar{\epsilon}_2 : S^4 \to BG$ is the adjoint of ϵ_2. See [AB]. Then the evaluation map $\mathrm{map}(S^4, BG; k\bar{\epsilon}_2) \to BG$ induces a homotopy fibration

$$\mathcal{G}_k \xrightarrow{\pi} G \xrightarrow{\delta} \Omega_0^3 G, \tag{2.1}$$

where π is a loop map. The map δ is identified as:

Lemma 2.1 (Whitehead [W]). *The map* δ *is the adjoint of the Samelson product* $\langle \epsilon_2, 1_G \rangle$.

Hereafter, everything will be localized at the prime p.

We now sketch the proof of noncommutativity of $\mathcal{G}_k$. Suppose that there are $2 \leq i, j, \leq n$ such that

$$\langle \epsilon_2, \epsilon_i \rangle = 0, \quad \langle \epsilon_2, \epsilon_j \rangle = 0, \quad \langle \epsilon_i, \epsilon_j \rangle \neq 0. \tag{2.2}$$

Since $\delta \circ \epsilon_\ell$ is the adjoint of $\langle \epsilon_2, \epsilon_\ell \rangle$ by Lemma 2.1, $\delta \circ \epsilon_\ell$ is null homotopic for $\ell = i, j$. Then for $\ell = i, j$, ϵ_ℓ lifts to $\tilde{\epsilon}_\ell : S^{2\ell-1} \to \mathcal{G}_k$ through $\pi : \mathcal{G}_k \to G$. Consider the Samelson product $\langle \tilde{\epsilon}_i, \tilde{\epsilon}_j \rangle$. Since π is an H-map, we have

$$\pi \circ \langle \tilde{\epsilon}_i, \tilde{\epsilon}_j \rangle = \langle \pi \circ \tilde{\epsilon}_i, \pi \circ \tilde{\epsilon}_j \rangle = \langle \epsilon_i, \epsilon_j \rangle$$

which is nontrivial by assumption. Then in particular, we obtain that $\mathcal{G}_k$ is not homotopy commutative. So our task is to find $2 \leq i, j \leq n$ satisfying (2.2), which is easily done by the following classical result if $n \geq 4$.

Theorem 2.2 (Bott [B]). *If $2 \leq i, j \leq n$ and $i + j > n$, the order of the Samelson product $\langle \epsilon_i, \epsilon_j \rangle$ is a nonzero multiple of*

$$\frac{(i+j-1)!}{(i-1)!(j-1)!}.$$

3. Commutativity

In this section, we give a brief sketch of the proof of the commutativity result on $\mathcal{G}_k$. If the map π in the homotopy fibration (2.1) has a homotopy section, we have a decomposition

$$\mathcal{G}_k \simeq G \times \Omega(\Omega_0^3 G)$$

as spaces. If this decomposition is as H-spaces and G is homotopy commutative (i.e. $p > 2n$ by Theorem 1.2), we obtain that $\mathcal{G}_k$ is homotopy commutative as desired. Then we give a criterion for the decomposition being as H-spaces, where we omit the proof.

Lemma 3.1 (cf. [KiKo]). *If there is an H-map $\hat{s} : G \to \mathcal{G}_k$ such that $\pi \circ \hat{s}$ is a homotopy equivalence, then there is a homotopy equivalence as H-spaces*

$$\mathcal{G}_k \simeq G \times \Omega(\Omega_0^3 G).$$

In particular, if moreover $p > 2n$, $\mathcal{G}_k$ is homotopy commutative.

For the rest of this section, we assume $p > 2n$. Then in particular, $G \simeq S^3 \times S^5 \times \cdots \times S^{2n-1}$

Since G is homotopy commutative, it follows from Lemma 2.1 that π has a homotopy section $s : G \to \mathcal{G}_k$, not necessarily an H-map. We replace this homotopy section with an H-map. To this end, we employ the loop-suspension technique.

Theorem 3.2 (James [J]). *Consider a map $f : X \to Y$ where Y is a homotopy associative H-space. There is a unique (up to homotopy) H-map $\bar{f} : \Omega\Sigma X \to Y$ satisfying $\bar{f} \circ E \simeq f$ for the suspension map $E : X \to \Omega\Sigma X$, where $\bar{f}$ is called the extension of f.*

We put $A = S^3 \vee S^5 \vee \cdots \vee S^{2n-1}$ and let $i : A \to G$ be the inclusion of a wedge into a product. Let F be the homotopy fiber of the extension $\bar{i} : \Omega\Sigma A \to G$, and let $\lambda : F \to \Omega\Sigma$ be the fiber inclusion. By an easy diagram chasing, we can prove:

Lemma 3.3. *Consider a map $f : G \to Z$ where Z is a homotopy associative H-space. If the composite $F \xrightarrow{\lambda} \Omega\Sigma A \xrightarrow{\overline{f\circ i}} Z$ is null homotopic, there is an H-map $\hat{f} : G \to Z$ satisfying the homotopy commutative square*

$$\begin{array}{ccc} \Omega\Sigma A & \xrightarrow{\bar{i}} & G \\ \downarrow{\scriptstyle \overline{f\circ i}} & & \downarrow{\scriptstyle \hat{f}} \\ Z & = & Z. \end{array}$$

Suppose now that the composite $F \xrightarrow{\lambda} \Omega\Sigma A \xrightarrow{\overline{s\circ i}} \mathcal{G}_k$ is null homotopic. Then it follows from Lemma 3.3 that there is an H-map $\hat{s} : G \to \mathcal{G}_k$ satisfying the homotopy commutative diagram

$$\begin{array}{ccc} \Omega\Sigma A & \xrightarrow{\bar{i}} & G \\ \downarrow{\scriptstyle \overline{s\circ i}} & & \downarrow{\scriptstyle \hat{s}} \\ \mathcal{G}_k & = & \mathcal{G}_k. \end{array}$$

In particular, there is a chain of homotopies

$$\pi \circ \hat{s} \circ i \simeq \pi \circ \hat{s} \circ \bar{i} \circ E \simeq \pi \circ (\overline{s \circ i}) \circ E \simeq \pi \circ s \circ i \simeq i.$$

In the mod p homology, the map $i : A \to G$ induces the inclusion of ring generators. Then $\pi \circ \hat{s}$ turns out to be the identity map on ring generators in the mod p homology, hence since $\pi \circ \hat{s}$ is an H-map, it is an isomorphism in the mod p homology. So we obtain that $\pi \circ \hat{s}$ is a p-local homotopy equivalence. Then all we have to do is prove that the composite $F \xrightarrow{\lambda} \Omega\Sigma A \xrightarrow{\overline{s\circ i}} \mathcal{G}_k$ is null homotopic. To this end, we analyze the fiber inclusion λ.

Let F' be the homotopy fiber of the adjoint $\Sigma A \to BG$ of the inclusion $i : A \to G$. Since the extension $\bar{i} : \Omega\Sigma A \to G$ is the loop of the above adjoint, we get:

Lemma 3.4. *$F \simeq \Omega F'$ and the fiber inclusion $\lambda : \Omega F' \to \Omega\Sigma A$ is a loop map.*

Let L be the free Lie algebra generated by $\widetilde{H}_*(A; \mathbb{Z}/p)$. Then as in [CN], the induced map $\bar{i}_* : H_*(\Omega\Sigma A; \mathbb{Z}/p) \to H_*(G; \mathbb{Z}/p)$ is identified with the map between universal envelopes

$$U(L) \to U(L/[L, L])$$

induced from the abelianization $L \to L/[L, L]$. Moreover, there is a splitting

$$U(L) \cong U([L, L]) \otimes U(L/[L, L]),$$

hence the image of $\lambda_* : H_*(F; \mathbb{Z}/p) \to H_*(\Omega\Sigma A; \mathbb{Z}/p)$ is identified with $U([L, L]) \subset U(L)$. A little more consideration shows that the Lie algebra generators of $[L, L]$ are spherical and lift to F. So we obtain:

Theorem 3.5. *There is a wedge of spheres R such that $F' \simeq \Sigma R$, and the composite $R \xrightarrow{E} \Omega\Sigma R \xrightarrow{\lambda} \Omega\Sigma A$ is a wedge of iterated Samelson products of*

$$\mu_j : S^{2j-1} \xrightarrow{\text{incl}} A \xrightarrow{E} \Omega\Sigma A.$$

Corollary 3.6. *If $p > 2n+1$, the composite $F \xrightarrow{\lambda} \Omega\Sigma A \xrightarrow{\overline{s \circ i}} \mathcal{G}_k$ is null homotopic.*

Proof. Put $\bar{\mu}_j = (\overline{s \circ i}) \circ \mu_j$. We consider the Samelson product $\langle \bar{\mu}_{i_1}, \bar{\mu}_{i_2} \rangle$. Since π is an H-map and G is homotopy commutative, we have

$$\pi \circ \langle \bar{\mu}_{i_1}, \bar{\mu}_{i_2} \rangle = \langle \pi \circ \bar{\mu}_{i_1}, \pi \circ \bar{\mu}_{i_2} \rangle = 0.$$

Then $\langle \bar{\mu}_{i_1}, \bar{\mu}_{i_2} \rangle$ lifts to a map $S^{2i_1+2i_2-2} \to \Omega(\Omega_0^3 G)$ by the homotopy fibration $\Omega(\Omega_0^3 G) \to \mathcal{G}_k \xrightarrow{\pi} G$. Since $p > 2n+1$, we have $\pi_{2m}(\Omega(\Omega_0^3 G)) = 0$ for $m \le 2n-1$ by [To], implying that the above lift is null homotopic. Then we obtain $\langle \bar{\mu}_{i_1}, \bar{\mu}_{i_2} \rangle = 0$, hence

$$0 = \langle \bar{\mu}_{j_1}, \langle \cdots \langle \bar{\mu}_{j_{m-1}}, \bar{\mu}_{j_m} \rangle \cdots \rangle \rangle = (\overline{s \circ i}) \circ \langle \mu_{j_1}, \langle \cdots \langle \mu_{j_{m-1}}, \mu_{j_m} \rangle \cdots \rangle \rangle$$

since $\overline{s \circ i}$ is an H-map. Thus by Theorem 3.5, the composite $R \xrightarrow{E} \Omega\Sigma R \xrightarrow{\lambda} \Omega\Sigma A \xrightarrow{\overline{s \circ i}} \mathcal{G}_k$ is null homotopic. Therefore we obtain the desired result by the uniqueness of the extension and Lemma 3.4. □

4. The case $p = 2n+1$

Throughout this section, we assume $p = 2n+1$.

As in the previous section, it is sufficient for proving the commutativity result to show that the homotopy section $s : G \to \mathcal{G}_k$ is an H-map. This is equivalent to show that the adjoint

$$\bar{s} : \Sigma G \to B\mathcal{G}_k \simeq \mathrm{map}(S^4, BG : k\bar{\epsilon}_2)$$

extends to the projective plane $P^2 G$. By the exponential law, this is equivalent to existence of a map $\mu : S^4 \times P^2 G \to BG$ satisfying a homotopy commutative diagram

$$\begin{array}{ccc} S^4 \vee \Sigma G & \xrightarrow{k\bar{\epsilon}_2 \vee \bar{s}} & BG \\ \downarrow{\scriptstyle \mathrm{incl}} & & \| \\ S^4 \times P^2 G & \xrightarrow{\mu} & BG. \end{array}$$

Since $P^2 G$ is the cofiber of the Hopf construction $\Sigma G \wedge G \to \Sigma G$ and $\Sigma G \wedge G$ has the homotopy type of a wedge of spheres of dimension $\le 2n^2 - 1 = \frac{(p-1)^2}{2} - 1$, we see that the obstruction for existence of μ lies in $\pi_*(BG)$ for $* \le \frac{(p-1)^2}{2} + 3$. Since the obstruction is torsion in $\pi_*(BG)$, we see from [To] that it is of order at most p. Moreover, we also see that the obstruction is linear in k. Then we get:

Proposition 4.1. *If p divides k, the homotopy section s is an H-map, hence $\mathcal{G}_k$ is homotopy commutative.*

When p does not divide k, we can prove that the obstruction is nontrivial by looking at the Steenrod operation on the mod p cohomology of BG. Then we have:

Proposition 4.2. *If p does not divide k, the homotopy section s cannot be an H-map.*

Corollary 4.3. *If p does not divide k, $\mathcal{G}_k$ is not homotopy commutative.*

Proof. Suppose that $\mathcal{G}_k$ is homotopy commutative. Then the argument in the previous section ensures that there is an H-map $\hat{s} : G \to \mathcal{G}_k$ such that the composite $e = \pi \circ \hat{s}$ is a homotopy equivalence. If we put $s = \hat{s} \circ e^{-1}$, s is a homotopy section of π and is an H-map, which contradicts to Proposition 4.2. □

References

[AB] M.F. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Philos. Trans. Roy. Soc. London Ser. A **308** (1983), 523-615.

[B] R. Bott, *A note on the Samelson product in the classical Lie groups*, Comment. Math. Helv. **34** (1960), 245-256.

[CMN] F.R. Cohen, J.C. Moore, and J.A. Neisendorfer, *Torsion in Homotopy Groups*, Ann. of Math. **109** (1979), 121-168.

[CN] F.R. Cohen and J.A. Neisendorfer, *A construction of p-local H-spaces*, pp. 351-359. LNM **1051**, Springer, Berlin, 1984.

[CS] M.C. Crabb and W.A. Sutherland, *Counting homotopy types of gauge groups*, Proc. London Math. Soc. (3) **81** (2000), no. 3, 747-768.

[J] I.M. James, *Reduced Product Spaces*, Ann. of Math. **62** (1955), 170-197.

[M] C.A. McGibbon, *Homotopy commutativity in localized groups*, Amer. J. Math **106** (1984), 665-687.

[MNT] M. Mimura, G. Nishida and H. Toda, Mod p *decomposition of compact Lie groups*, Publ. Res. Inst. Math. Sci. **13** (1977/78), no. 3, 627-680.

[KaKi] S. Kaji and D. Kishimoto, *Homotopy nilpotency in p-regular loop spaces*, Math. Z. **264** (2010), no. 1, 209-224.

[Ki] D. Kishimoto, *Homotopy nilpotency in localized* SU(n), Homology, Homotopy Appl. **11** (2009), no. 1, 61-79.

[KiKo] D. Kishimoto and A. Kono, *Splitting of gauge groups*, Trans. Amer. Math. Soc. **362** (2010), 6715-6731.

[KKTs] D. Kishimoto, A. Kono and M. Tsutaya, *Mod p decompositions of gauge groups*, Algebr. Geom. Topol. **13** (2013), no. 3, 1757-1778.

[KKTh] D. Kishimoto, A. Kono and S. Theriault, Homotopy commutativity in p-localized gauge groups, Proc. Roy. Soc. Edinburgh Sect. A **143** (2013), 851-870.

[Ko] A. Kono, *A note on the homotopy type of certain gauge groups*, Proc. Roy. Soc. Edinburgh Sect. A **117** (1991), 295-297.

[Th] S.D. Theriault, *The odd primary H-structure of low rank Lie groups and its application to exponents*, Trans. Amer. Math. Soc. **359** (2007), no. 9, 4511-4535 (electronic).

[To] H. Toda, *Composition Methods in Homotopy Groups of Spheres*, Ann. of Math. Studies **49**, Princeton Univ. Press, Princeton N.J., 1962.

[W] G.W. Whitehead, *On products in homotopy groups*, Ann. of Math (2) **47**, (1946). 460-475.

Department of Mathematics, Kyoto University, Kyoto, 606-8502, Japan
E-mail address: `kishi@math.kyoto-u.ac.jp`