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モチビックホモトピー論における
新・世界最後の日予想について

In serach of finteness phenomena

in the stable homotopy theory

Norihiko Minami (南　範彦)
Nagoya Institute of Technology

変換群のトポロジーとその周辺
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Contents of the talk

Adams filtration

§0, Background - untill early 90's

mod p <u>Adams filtration</u>of a spectra map $f : X \to Y$

$\overset{\text{Def.}}{\Longleftrightarrow}$ The maximum s $(0 \leq s \leq \infty)$ with such a lift $\tilde{f} : X \to F_s$

$$\begin{tikzcd}
& \vdots \arrow[d] & \\
& F_{s+1} \arrow[r] \arrow[d] & F_{s+1} \wedge H\mathbb{Z}/p \\
& F_s \arrow[r] \arrow[d] & F_s \wedge H\mathbb{Z}/p \\
& \vdots \arrow[d] & \\
& F_1 \arrow[r] \arrow[d] & F_1 \wedge H\mathbb{Z}/p \\
X \arrow[r, "f"] \arrow[uuur, dotted] & Y = F_0 \arrow[r] & F_0 \wedge H\mathbb{Z}/p
\end{tikzcd} \tag{1}$$

Adams spectral sequence

- <u>Adams spectral sequence</u> := the spectral sequence to compute $\{X, Y\}_*$ w.r.t. the Adams filtraion s
- $E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}^*}^{s,t}(H^*(Y;\mathbb{Z}/p), H^*(X;\mathbb{Z}/p)) \Longrightarrow \{X, Y\}_{t-s}$
  where $\mathcal{A}^* := \{\Sigma^* H\mathbb{Z}/p, \Sigma^* H\mathbb{Z}/p\}_{-*}$,
  the mod p <u>Steenrod 代数</u>
- Especially when $X = Y = \Sigma^\infty S^0$,

$$\boxed{E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}^*}^{s,t}(\mathbb{Z}/p, \mathbb{Z}/p) \Longrightarrow \pi_{t-s}^S(S^0)_p^\wedge}$$

- Elements in $\mathrm{Ext}_{\mathcal{A}^*}^{s,t}(\mathbb{Z}/p, \mathbb{Z}/p) \cong \mathrm{Ext}_{\mathcal{A}_*}^{s,t}(\mathbb{Z}/p, \mathbb{Z}/p)$ are represented by cobar complexes as $[\underbrace{\zeta_1|\zeta_2|\cdots|\zeta_s}_{s}]$, $a_i \in \mathcal{A}_*$, where

$$\mathcal{A}_* = \begin{cases} P(\xi_1, \xi_2, \ldots) & (p = 2) \\ P(\xi_1, \xi_2, \ldots) \otimes E(\tau_0, \tau_1, \ldots) & (p = \text{odd}) \end{cases}$$

$$\text{with} \begin{cases} |\xi_n| = 2^n - 1 & (p = 2) \\ |\xi_n| = 2(p^n - 1), \ |\tau_n| = 2p^n - 1 & (p = \text{odd}) \end{cases}$$

is the Milnor's <u>dual Steenrod algebra</u>.

The Steenrod algebra action on $\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathbb{Z}/p,\mathbb{Z}/p)$

- **The mod-p Steenrod algebra $\mathcal{A}^*$ also acts on the cohomology of the cocommutative Hopf algebras over $\mathbb{Z}/p$, in particular, on $\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathbb{Z}/p,\mathbb{Z}/p)$.**

- **HOWEVER, NEITHER Sq^0 $(p=2)$ OR $\mathcal{P}^0$ $(p=$ odd) ACT IDENTICALLY ON $\mathrm{Ext}^{s,*}_{\mathcal{A}_*}(\mathbb{Z}/p,\mathbb{Z}/p)$!**

$$\begin{aligned} Sq^0 : \mathrm{Ext}^{s,t}_{\mathcal{A}_*}(\mathbb{Z}/2,\mathbb{Z}/2) &\to \mathrm{Ext}^{s,2t}_{\mathcal{A}_*}(\mathbb{Z}/2,\mathbb{Z}/2) \\ [\zeta_1|\zeta_2|\cdots|\zeta_s] &\mapsto [\zeta_1^2|\zeta_2^2|\cdots|\zeta_s^2], \\ \mathcal{P}^0 : \mathrm{Ext}^{s,2t}_{\mathcal{A}_*}(\mathbb{Z}/p,\mathbb{Z}/p) &\to \mathrm{Ext}^{s,2pt}_{\mathcal{A}_*}(\mathbb{Z}/p,\mathbb{Z}/p) \\ [\zeta_1|\zeta_2|\cdots|\zeta_s] &\mapsto [\zeta_1^p|\zeta_2^p|\cdots|\zeta_s^p], \end{aligned}$$

$p=2$, filtraion $s=1$ case

$\mathrm{Ext}^{1,*}_{\mathcal{A}^*}(\mathbb{Z}/2,\mathbb{Z}/2)$
$=\mathbb{Z}/2\{h_i \big| h_i \in \mathrm{Ext}^{1,2^i}_{\mathcal{A}^*}(\mathbb{Z}/2,\mathbb{Z}/2)\ i\in\mathbb{Z}_{\geq 0}\}$
(infinite dimensional $\mathbb{Z}/2$ vector space!)

- $Sq^0(h_i)=h_{i+1}$
- **Only h_0,h_1,h_2,h_3 are the permanent cycle (Adams Hopf invariant one theorem)**
- **(J. Cohen; Doomsday Conjecture:) Fix a prime p. Then, for each filtration s , only finitely many of**

$$E_2^{s,t}=\mathrm{Ext}^{s,t}_{\mathcal{A}^*}(\mathbb{Z}/p,\mathbb{Z}/p)$$

are permanent cycles.
(Regard t as time ... As t increseas, or, as time passes, no one can survive!)

$p = 2$, filtraion $s = 2$ case

$\mathrm{Ext}_{\mathcal{A}_*}^{2,*}(\mathbb{Z}/2, \mathbb{Z}/2)$
$= \mathbb{Z}/2\{h_i h_j \big| 0 \le i \le j, j \ne i+1\}$

- $Sq^0(h_i h_j) = h_{i+1} h_{j+1}$
- (Mahowald) $^\forall j \ge 1$ $h_1 h_j$ is a permanent cycle.
  $\Longrightarrow$ Doomsday conjecture "doomed !"
- (Browder) $h_i^2 \in \mathrm{Ext}_{\mathcal{A}^*}^{2,2^{i+1}}(\mathbb{Z}/2, \mathbb{Z}/2)$ is a permanent cycle.
  $\Longleftrightarrow$ $^\exists$a Kervaire invariant one element in $\pi_{2^{i+1}-2}^S(S^0)$
  $\Longleftrightarrow$ $^\exists$an element in $\pi_{2^{i+1}-2}^S(S^0)$, which is never represented by a homotopy sphere as a framed bordism element.
- Until earlier 90's, experts believed that all the h_i^2 are permanent cycles.
  e.g. "An inductive approach to (constructing) the Kervaire invariante (one elements)"
- $\cdots$

§1, Newdoomsday Conjecture - untill 2009

Motiviational Question

Is there any systematic method to construct elements and permanent cycles of $\mathrm{Ext}_{\mathcal{A}_*}^{s,*}(\mathbb{Z}/p, \mathbb{Z}/p)$?

An Answer

Set $V_s := (\mathbb{F}_p)^s$ and consider the composite:

$$\begin{aligned}\pi_n^S(BV_s) &\to PH_n(BV_s) \ \left(\cong \ \mathrm{Ext}_{\mathcal{A}_*}^{0,n}(\mathbb{Z}/p,_n (BV_s))\right)\\ &\twoheadrightarrow \mathbb{Z}/p \otimes_{GL_s(\mathbb{F}_p)} PH_n(BV_s)\\ &\to \mathrm{Ext}_{\mathcal{A}_*}^{s,n+s}(\mathbb{Z}/p, \mathbb{Z}/p)\end{aligned}$$

M.

The stable Hurewicz map

$$\pi^S_{2n-s}(BV_s) \to H_{2n-s}(BV_s)$$

has a trivial image, when

$$\begin{cases} 2^{\nu_2(n)+2-2s} > \nu_2(n) + 2 + \left[\frac{5[\frac{s-1}{2}]}{2}\right] - \alpha_2\left(\left[\frac{s-1}{2}\right]\right) & \text{if } p = 2 \\ p^{\nu_p(n)+2-2s} > (p-1)\nu_p(n) + 1 + p\left[\frac{s-1}{2}\right] - \alpha_p\left(\left[\frac{s-1}{2}\right]\right) & \text{if } p = odd \end{cases}$$

where $\nu_p(n)$ is the index of the highest p-power dividing n.

Observe the following commutative diagram

$$\begin{tikzcd} & \pi^S_{2n-s}(BV_s) \arrow[rr] \arrow[d] & & PH_{2n-s}(BV_s) \arrow[d, hook] \\ BP_{2n-s}(BV_s)^{\phi^{p+1}} \arrow[r, hook] & BP_{2n-s}(BV_s) \arrow[rr] & & H_{2n-s}(BV_s) \end{tikzcd}$$

where ϕ^{p+1} is the BP-Adams operation.

Then the proof makes use of the computation of

$$BP_*(BV_s)$$

by Johnson-Wilson and Johnson-Wilson-Yan, and

$$\begin{cases} Sq^0 : PH_n(BV_s) \to PH_{P_{p,s}(n)}(BV_s) & (p=2) \\ t_{i_1} \otimes t_{i_2} \otimes \cdots \otimes t_{i_s} \mapsto t_{2i_1+1} \otimes t_{2i_2+1} \otimes \cdots \otimes t_{2i_s+1} \\ \mathcal{P}^0 : PH_{2n}(BT^s) \to PH_{2P_{p,s}(n)}(BT^s) & (p = \text{odd}) \\ y_{2i_1} \otimes y_{2i_2} \otimes \cdots \otimes y_{2i_s} \mapsto y_{2(pi_1+p-1)} \otimes y_{2(pi_2+p-1)} \otimes \cdots \otimes y_{2(pi_s+p-1)} \end{cases}$$

s.t. $\forall l \geq 0$, the following are isomorphisms:

$$\begin{cases} (Sq^0)^l : PH_{2^{s-1}t-s}(BV_s) \xrightarrow{\cong} PH_{2^{s+l-1}t-s}(BV_s) & (p=2) \\ (\mathcal{P}^0)^l : PH_{2(p^{s-1}t-s)}(BT^s) \xrightarrow{\cong} PH_{2(p^{s+l-1}t-s)}(BT^s) & (p = \text{odd}) \end{cases}$$ $\square$

Corollary

The stable iterated trasfer map

$$\pi^S_{2n-s}(BV_s) \to \mathrm{Ext}^{s,2n}_{\mathcal{A}_*}(\mathcal{A}/p, \mathcal{A}/p)$$

has a trivial image, when

$$\bigstar\begin{cases} 2^{\nu_2(n)+2-2s} > \nu_2(n) + 2 + \left[\dfrac{5[\frac{s-1}{2}]}{2}\right] - \alpha_2\left(\left[\dfrac{s-1}{2}\right]\right) & \text{if } p = 2 \\ p^{\nu_p(n)+2-2s} > (p-1)\nu_p(n) + 1 + p\left[\dfrac{s-1}{2}\right] - \alpha_p\left(\left[\dfrac{s-1}{2}\right]\right) & \text{if } p = odd \end{cases}$$

where $\nu_p(n)$ is the index of the highest p-power dividing n.

Barratt-Jones-Mahowald claimed $h_5^2 \in \mathrm{Ext}^{2,2\cdot 2^5}_{\mathcal{A}_*}(\mathcal{A}/2, \mathcal{A}/2) \Rightarrow \pi^s_{62}(S^0)^\wedge_2$ is a nontrivial permanent cycle, where $n = 2^5 = 32$ violates $\bigstar$.

Still, we may speculate the following conjecture:

Strong Conjecture

For each $s \geqq 1$, there exists some integer $n(s)$ such that, if $\nu_p(n) \geqq n(s)$, then no element in

$$\mathrm{Ext}^{s,2n}_{\mathcal{A}_*}(\mathbb{Z}/p, \mathbb{Z}/p)$$

is a notrivial permanent cycle.

In view of the commutative diagram:

$$\begin{array}{ccc} PH_n(BV_s) & \longrightarrow & \mathrm{Ext}^{s,n+s}_{\mathcal{A}_*}(\mathbb{Z}/p,\mathbb{Z}/p) \\ \mathcal{P}^0\big\downarrow(Sq^0 \text{ if } p{=}2) & & \mathcal{P}^0\big\downarrow(Sq^0 \text{ if } p{=}2) \\ PH_{p(n+s)-s}(BV_s) & \longrightarrow & \mathrm{Ext}^{s,p(n+s)}_{\mathcal{A}_*}(\mathbb{Z}/p,\mathbb{Z}/p) \end{array}$$

we may formulate the following slightly modest conjecture:

New Doomsday Conjecture

For each $s \geqq 1$, there exists some integer $n(s)$ such that no element in the image of $(\mathcal{P}^0)^{n(s)}$

$$(\mathcal{P}^0)^{n(s)}(\mathrm{Ext}^{s,2t}_{\mathcal{A}_*}(\mathbb{Z}/p,\mathbb{Z}/p)) \subseteqq \mathrm{Ext}^{s,2p^{n(s)}t}_{\mathcal{A}_*}(\mathbb{Z}/p,\mathbb{Z}/p)$$

is a nontrivial permanent cycle. Here, $\mathcal{P}^0 = Sq^0$ when $p = 2$.

- At any prime p, these conjectures are true for $s = 1$, by the Hopf invariant one theorem of Adams, Lieulevicius, Shimada-Yamanoshita (also follows from M. with the aid of the Kahn-Priddy theorem).

- When $p > 2$, "the odd primary Kervaire invariant one elements"
$b_j := \underbrace{\langle h_j, \ldots, h_j \rangle}_{p} \in \mathrm{Ext}^{2,2(p-1)p^{j+1}}_{\mathcal{A}_*}(\mathbb{Z}/p,\mathbb{Z}/p)$ enjoy $\mathcal{P}^0(b_j) = b_{j+1}$, and Ravenel showed, for $p \geqq 5$, b_j is not a permanent cycle for $j \geqq 1$.

- _Until 2009_ , the first unsetttled cases of these conjectures are:
  - At $p = 3$,, $b_j := \langle h_j, h_j, h_j \rangle \in \mathrm{Ext}^{2,4\cdot 3^{j+1}}_{\mathcal{A}_*}(\mathbb{Z}/3,\mathbb{Z}/3)$
  - At $p = 2$, $h_i^2 \in \mathrm{Ext}^{2,2^{i+1}}_{\mathcal{A}_*}(\mathbb{Z}/2,\mathbb{Z}/2)$

  $\Longrightarrow$ According to the traditional wisdom of the homotopy theory, the next good news was supposed to come at $p = 3$...

§2, The work of Hill-Hopkins-Ravenel on the Kervaire invariant one problem - the year 2009

Strangely enough, the next good news came at $p = 2$, bypassing the case $p = 3$...

Hill-Hopkins-Ravnel

At $p = 2$, for $i \geqq 7$, $h_i^2 \in \mathrm{Ext}_{\mathcal{A}_*}^{2,2^{i+1}}(\mathbb{Z}/2, \mathbb{Z}/2)$ are not permanent cycles.

- Since, at $p = 2$, $h_0^2, h_1^2, h_2^2, h_3^2$, h_4^2 (Barratt-Mahowald), h_5^2 (Barratt-Jones-Mahowald) are permanent cycles, the only unsettled case of the Kervaire invariant one element is one for $h_6^2 \in \mathrm{Ext}_{\mathcal{A}_*}^{2,128}(\mathbb{Z}/2, \mathbb{Z}/2) \implies \pi_{126}^s(S^0)_2^\wedge$.
- The method employed by Hill-Hopkins-Ravenel is *NOT* applicable to the case $p = 3$ for $b_j := \langle h_j, h_j, h_j \rangle \in \mathrm{Ext}_{\mathcal{A}_*}^{2,4\cdot 3^{j+1}}(\mathbb{Z}/3, \mathbb{Z}/3)$.

The Hill-Hopkins-Ravenel strategy of their proof - 1

Construct a multiplicative cohomology theory Ω as follows:

- Set $MU_{\mathbb{R}} :=$ the C_2-equivariant real bordism spectrum, consiting of real manifolds, i.e. stably almost complex manifolds equipped with a conjugate linear action of C_2 e.g. the space of complex points of a smooth variety defined over $\mathbb{R}$.
- Set $MU^{(C_8)} := MU_{\mathbb{R}} \wedge MU_{\mathbb{R}} \wedge MU_{\mathbb{R}} \wedge MU_{\mathbb{R}}$, the C_8-equivariant spectrum whose C_8-action is induced by

$$(a, b, c, d) \mapsto (\bar{d}, a, b, c)$$

- Choose an appropriate $D : S^{l\rho_8} \to MU^{(C_8)}$, representing a suitable C_8-manifold M, whose restricted C_2-action defines a real structure, of real dimension $8l$. (May take $l = 19$)
- Set $\Omega_{\mathcal{O}} := D^{-1}MU^{(C_8)}$, regarding D as an analogue of equivariant Bott periodicity class.
- Finally, set $\Omega := \Omega_{\mathcal{O}}^{hC^8}$, the homotopy fixed point spectrum.

The Hill-Hopkins-Ravenel strategy of their proof - 2

For the unit map $S^0 \to \Omega := \Omega_{\mathcal{O}}^{hC_8}$,

$$\begin{array}{ccc} \boxed{\text{Adams-Novikov s.s. for } S^0} & \longrightarrow & \boxed{\text{Hopkins-Miller homotopy } C_8 \text{ fixed point s.s. for } \Omega_{\mathcal{O}}^{hC_8}} \\ \downarrow & & \\ \boxed{\text{mod 2 Adams s.s. for } S^0} & & \end{array}$$

$$\Longrightarrow \quad \begin{array}{c} \mathrm{Ext}^{2,2^{i+1}}_{MU_*MU}(MU_*, MU_*) \xrightarrow{HM} H^2\left(C_8, (\Omega_{\mathcal{O}})_{2^{i+1}}\right) \\ T\downarrow \\ h_i^2 \in \mathrm{Ext}^{2,2^{i+1}}_{\mathcal{A}_*\mathcal{A}}(\mathbb{Z}/2, \mathbb{Z}/2) \end{array}$$

Algebraic detection theorem

$\forall x \in \mathrm{Ext}^{2,2^{i+1}}_{MU_*MU}(MU_*, MU_*)$, s.t. $T(x) = h_i^2 \in \mathrm{Ext}^{2,2^{i+1}}_{\mathcal{A}_*\mathcal{A}}(\mathbb{Z}/2, \mathbb{Z}/2)$

$$HM(x) \neq 0 \in H^2\left(C_8, (\Omega_{\mathcal{O}})_{2^{i+1}}\right) \Longrightarrow \pi_{2^{i+1}-2}(\Omega)$$

Since $d_2 : H^0\left(C_8, (\Omega_{\mathcal{O}})_{2^{i+1}-1}\right) \to H^2\left(C_8, (\Omega_{\mathcal{O}})_{2^{i+1}}\right)$ is trivial for $\pi^s_{\mathrm{odd}}(\Omega_{\mathcal{O}}) = 0$, the algebraic detection theorem implies

Any Kervaire invariant one element in $\pi^S_{2^{i+1}-2}(S^0)$ is detected in $\pi_{2^{i+1}-2}(\Omega)$ via the unit map $S^0 \to \Omega$.

The Hill-Hopkins-Ravenel strategy of their proof - 3

Now the proof follows from the following three theorems:

$$\begin{array}{c} \Omega_{\mathcal{O}}^{C_8} \\ \downarrow \\ S^0 \xrightarrow[\text{unit}]{} \Omega := \Omega_{\mathcal{O}}^{hC_8} \end{array}$$

Homotopy Fixed Point Theorem

$$\Omega_{\mathcal{O}}^{C_8} \xrightarrow{\simeq} \Omega_{\mathcal{O}}^{hC_8}$$

Periodicity Theorem

$\pi_k\left(\Omega_{\mathcal{O}}^{hC_8}\right)$ is $2^8 = 256$ periodic, i.e. depends on $k \mod 256$.

Gap Theorem

$$\pi_{-2}\left(\Omega_{\mathcal{O}}^{C_8}\right) = \pi_{-2}^{C_8}(\Omega_{\mathcal{O}}) = 0$$

In fact, the proof is completed by observing

$$^{\forall}i \geqq 7, \quad 2^{i+1} - 2 \equiv 2^8 - 2 \equiv -2 \mod 2^8 = 256 \qquad \square$$

Two remarks on the Hill-Hopkins-Ravenel proof

- Their proof makes use of the chromatic technology, e.g. the algebraic detection theorem, the periodicity theorem. *However, the chromatic technology, is not new in this business. It already appear in Ravnel's solution of the odd primary Kervaire invariant one problem for $p \geqq 5$.*
- Something really new is their proof of the gap theorem: $\pi_{-2}\left(\Omega_{\mathcal{O}}^{C_8}\right) = \pi_{-2}^{C_8}(\Omega_{\mathcal{O}}) = 0$ where they constructed a Postnikov type filtration in the equivariant stable homotopy theory, called the slice filtration, which is an analogue and the Voevodsky's slice filtration in the motivic stable homotopy category.

§3, Generalizing NDC, classically...
- from now on !

NDC problem for (nice) (-1)-connective ring spectra R - 1

In the classical mod p Adams spectral sequence

$$\mathrm{Ext}^{s,t}_{\mathcal{A}_*}(\mathbb{Z}/p, H_*(R)) \implies \pi_{t-s}(R)^\wedge_p$$

Strong Conjecture problem for R

For each $s \geqq 1$, does there exist some integer $n(s)$ s.t., if $\nu_p(n) \geqq n(s)$, then no element in

$$\mathrm{Ext}^{s,2n}_{\mathcal{A}_*}(\mathbb{Z}/p, H_*R)$$

is a notrivial permanent cycle?

New Doomsday Conjecture problem for R

For each $s \geqq 1$, there exists some integer $n(s)$ such that no element in the image of $(\mathcal{P}^0)^{n(s)}$

$$(\mathcal{P}^0)^{n(s)}(\mathrm{Ext}^{s,2t}_{\mathcal{A}_*}(\mathbb{Z}/p, H_*R)$$
$$\subseteqq \mathrm{Ext}^{s,2p^{n(s)}t}_{\mathcal{A}_*}(\mathbb{Z}/p, H_*R)$$

is a nontrivial permanent cycle. Here, $\mathcal{P}^0 = Sq^0$ when $p = 2$.

NDC problem for (nice) (-1)-connective ring spectra R - 2

- Of course, the case $R = S^0$ is our original case.
- The reason we restricted to (nice) (-1)-connective ring spectra is to avoid ambiguity arising from the shifts by suspensions $R \mapsto \Sigma^{t'} R$, by specifying the canonical granding coming from the unit map $S^0 \to R$, in addition to the sonstruction of $\mathcal{P}^0$ (Sq^0 $(p=2)$) at the E_2-term.
- The following (-1)-connective ring spectra R satisfy not only the Strong Conjecture, but also even the doomsday conjecture: $R = MU, BP, BP\langle n\rangle, bo, bu, H\mathbb{Z}, H\mathbb{Z}/p, tmf, \ldots$ *Note that many of these spectra have polynomial homotopy groups.*
- However, I am not sure if this holds for free...
  e.g. Express $(S^0)^\wedge_p = \mathrm{holim}_n X^n$ using the canonical Adams tower, and consider each finite stage X^n...
  Note X^n is NOT harmonic ! for any n.
- Considering the above example and that Hill-Hopkins-Ravenel proof made use of the chromatic technology, *we may have to impose the harmonic condition on R.*

§4, Generalizing NDC, motivically... - from now on !

Motivation

Since the essential new ingredient in the Hill-Hopkins-Ravenel is the slice filtration, which has the motivic origin, it might be natural to *speculate that the NDC has its origin in the motivic stable homotopy category.*

A summary of the motivic unstable homotopy theory - 1

S: the base scheme , which is Noetherian (i.e. locally Noetherian and quasi-compact) of finite dimension (i.e. the dimensions of the local rings are bounded). *Quite general!*

Morel-Voevodsky, Theorem 2.3.2

$\Delta^{op}Shv(Sm/S)_{Nis}$ is a model category with:
Weak equivalences: $\mathbb{A}^1$-weak equivalence
Cofibrations: monomorphisms
Fibrations: RLP w.r.t. trivial cofibrations

$\mathcal{H}(S)$: the homotopy category of $\Delta^{op}Shv(Sm/S)_{Nis}$ w.r.t. the above model structure

$*$: the simplicial sheaf (associated to) Δ^0, which is the final object in $\Delta^{op}Shv(Sm/S)_{Nis}$ and is called the point

$\Delta^{op}Shv_\bullet(Sm/S)_{Nis}$; the pointed analogue of $\Delta^{op}Shv(Sm/S)_{Nis}$. This is the category of the "based spaces" for Morel-Voevodsky!

$\mathcal{H}_\bullet(S)$: the pointed analogue of $\mathcal{H}(S)$.
This is the (pointed) motivic unstable homotopy category!

A summary of the motivic unstable homotopy theory - 2

Questions

From algebraists: Does $\mathcal{H}_\bullet(S)$ contain rich information?
From topologists: Is $\mathcal{H}_\bullet(S)$ friendly to deal with?

Answers

To algebraists: K-theory representability :

Morel-Voevodsky, Theorem 4.3.13, Morel, Example 3.1.11

If S is regular , $^\forall n \geqq 0$, $^\forall X \in Sm/S$,

$$K_n^Q(X) \xrightarrow{\cong} Hom_{\mathcal{H}_\bullet(S)}\left((X_+) \wedge S^n, \mathbb{Z} \times Gr\right)$$

To topologists: Homotopy purity :

Morel-Voevodsky, Theorem 3.2.23

Let $i : Z \to X$ be a closed embedding of smooth schemes over S. Denote by $\mathcal{N}_{X,Z} \to Z$ the normal vector bundle to i. Then there is a canonical isomorphism in $\mathcal{H}_\bullet(S)$ of the form

$$X\big/(X \setminus i(Z)) \cong Th(\mathcal{N}_{X,Z}).$$

A summary of the motivic stable homotopy theory - 1

Motivation

Topologists know unstable homotopy theory is hard...
Can we go stably?

two different kinds of circles in $\Delta^{op}Shv_\bullet(Sm/S)_{Nis}$

$S_s^1 := \Delta^1/\partial\Delta^1$, **which is called the simplicial circle**
$S_t^1 := \mathbb{G}_m$ **: the pointed k-scheme $(\mathbb{A}^1 \setminus \{0\}, 1)$, the Tate circle**

two different kinds of spheres and their mixture in $\Delta^{op}Shv_\bullet(Sm/S)_{Nis}$

$$S_s^n := \underbrace{S_s^1 \wedge \cdots \wedge S_s^1}_{n}, \quad S_t^n := \underbrace{S_s^1 \wedge \cdots \wedge S_t^1}_{n}$$

$S^{a+b,b} := S_s^a \wedge S_t^b$ **(equivariant homotopy theoretical notation!)**

$$\Longrightarrow S^{n,i} := S_s^{n-i} \wedge S_t^i$$

These suspensins allows us to define two distinct concepts of spectra w.r.t. "based spaces" $\Delta^{op}Shv_\bullet(Sm/S)_{Nis}$, but sometimes they lack practical applicability...

A summary of the motivic stable homotopy theory - 2

Set $T := \mathbb{A}^1/(\mathbb{A}^1 \setminus \{0\}) = \mathbb{A}^1/\mathbb{G}_m \in \Delta^{op}Shv_\bullet(Sm/S)_{Nis}$,
then, in $\mathcal{H}_\bullet(S)$,

- $T = \mathbb{A}^1/\mathbb{G}_m \overset{\mathbb{A}^1\text{-eq.}}{\simeq} */\mathbb{G}_m \simeq \Sigma_s\mathbb{G}_m = S^{2,1}$
- **From the elementary distinguished square** $\begin{tikzcd} \mathbb{G}_m \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[d] \\ \mathbb{A}^1 \arrow[r] & \mathbb{P}^1 \end{tikzcd}$,

$$T := \mathbb{A}^1/\mathbb{G}_m \overset{\text{Nis.}}{\cong} \mathbb{P}^1/\mathbb{A}^1 \overset{\mathbb{A}^1\text{-eq}}{\xleftarrow{\simeq}} \mathbb{P}^1$$

Thus,
$$\mathbb{P}^1 \simeq T \simeq \Sigma_s\mathbb{G}_m = S^{2,1}$$

From this, we have three concepts of "spectra" in $\Delta^{op}Shv_\bullet(Sm/S)_{Nis}$ **:**

S^1**-spectrum** $\left\{E_n, (\sigma_n : E_n^1 \wedge S_s \to E_{n+1})\right\}$
T**-spectrum** $\left\{E_n, (\sigma_n : E_n \wedge T \to E_{n+1})\right\}$
$\mathbb{P}^1$**-spectrum** : $\left\{E_n, (\sigma_n : E_n \wedge \mathbb{P}^1 \to E_{n+1})\right\}$

$Sp^{\mathbb{P}^1}(S)$ **: The (naive) category of $\mathbb{P}^1$-spectrra.**

Note: T-spectra may be naturally regarded as $\mathbb{P}^1$-spectra, by use of $\mathbb{P}^1 \to T$.

A summary of the motivic stable homotopy theory - 3

Let E be a $\mathbb{P}^1$-spectrum, $U \in Sm/S$ and $(n,m) \in (\mathbb{Z})^2$. We set

$$\tilde{\pi}_n(E)_m(U) := colim_{r \to +\infty} Hom_{\mathcal{H}_\bullet(S)}\left(S^{n+m} \wedge (U+) \wedge (\mathbb{P}^1)^{r-m}, E_r\right)$$

whose abelian group structure given by $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$ in $\mathcal{H}_\bullet(S)$.

Morel, An introduction, Definition 5.1.4

1. A morphism $f : E \to F$ of $\mathbb{P}^1$-spectra is called an <u>$\mathbb{A}^1$-stable weak equivalence</u> if and only if for any $U \in Sm/S$ and any pair $(n,m) \in (\mathbb{Z})^2$ the homomorphism :

$$\tilde{\pi}_n(E)_m(U) \to \tilde{\pi}_n(F)_m(U)$$

is an isomorphism.

2. A morphism $f : E \to F$ of $\mathbb{P}^1$-spectra is called a <u>cofibration</u> if and only if the morphisms

$$\begin{cases} E_0 & \to F_0 \\ E_{n+1} \wedge_{E_n \wedge \mathbb{P}^1} F_n \wedge \mathbb{P}^1 & \to F_{n+1} \ ({}^\forall n \geqq 0) \end{cases}$$

are cofibrations (= monomorphisms in $\Delta^{op}Shv_\bullet(Sm/S)_{Nis}$).

A summary of the motivic stable homotopy theory - 4

"The" model structure of $Sp^{\mathbb{P}^1}(S)$

$Sp^{\mathbb{P}^1}(S)$ becomes a model category, by
<u>Weak equivalences</u>: $\mathbb{A}^1$-stable weak equivalences
<u>Cofibrations</u>: cofibrations
<u>Fibrations</u>: RLP w.r.t. trivial cofibrations

The corresponding homotopy category is denoted by $\mathcal{SH}^{\mathbb{P}^1}(S)$ or even by $SH(S)$, which is <u>the motivic stable homotopy category</u>, and for $E, F \in \mathcal{SH}(S)$,

$$[E,F] := \mathrm{Hom}_{\mathcal{SH}(S)}(E,F)$$

Morel, An introduction, Definition 5.1.6

For any spectrum E, and for any integers $n, i \in \mathbb{Z}$ set

$$E(i)[n] := E \wedge S^{n,i}$$

For any $\mathcal{X} \in \Delta^{op}Shv_\bullet(Sm/S)_{Nis}$, and $\mathcal{Y} \in \Delta^{op}Shv(Sm/S)_{Nis}$, set

$$\tilde{E}^{n,i}(\mathcal{X}) := [\Sigma^\infty(\mathcal{X}), E(i)[n]],\ E^{n,i}(\mathcal{Y}) := [\Sigma^\infty(\mathcal{Y}_+), E(i)[n]] \cong \tilde{E}^{n,i}(\mathcal{X}_+)$$

Examples of $\mathbb{P}^1$-spectra - Morel, Example 5.1.2 - 1

$\mathbb{P}^1$-suspension spectrum

$^\forall \mathcal{X} \in \Delta^{op} Shv_\bullet(Sm/S)_{Nis}$ a "based space",

$$\Sigma^\infty_{\mathbb{P}^1}(\mathcal{X}) := \left\{ \mathcal{X} \wedge (\mathbb{P}^1)^{\wedge^n}, \left(\sigma_n : (\mathcal{X} \wedge (\mathbb{P}^1)^{\wedge^n}) \wedge \mathbb{P}^1 \xrightarrow{\cong} \mathcal{X} \wedge (\mathbb{P}^1)^{\wedge^{n+1}} \right) \right\}$$

example: $S^0 := \Sigma^\infty_{\mathbb{P}^1}(\mathrm{Spec}\, S_+)$, the sphere spectrum

algebraic K-theory $\mathbb{P}^1$-spectrum - Morel, Example 5.1.7

algebraic K-theory $\mathbb{P}^1$-spectrum $\mathbb{K}$ is given by

$$\mathbb{K} := \left\{ \mathbb{Z} \times Gr, \left(\sigma_n = \underline{\mathrm{Bottmap}} : (\mathbb{Z} \times Gr) \wedge \mathbb{P}^1 \to \mathbb{Z} \times Gr \right) \right\}$$

The Bott map induces the Bott periodicity in $\mathcal{SH}(S)$:

$$\mathbb{K} \wedge \mathbb{P}^1 \simeq \mathbb{K}$$

which is $(2,1)$-periodic. Thus, for $X \in Sm/S$ and $(n,i) \in (\mathbb{Z})^2$,

Good News! $\quad K^Q_{2i-n}(X) \cong \mathbb{K}^{n,i}(X)$

Examples of $\mathbb{P}^1$-spectra - Morel, Example 5.1.2 - 2

For $X \in Sm(S)$, the motivic Eilenberg-MacLane "space" $L[X]$ is the sheaf of abelian groups

$$U \mapsto c(U, X)$$

where $c(U,X)$ denotes the group of finite correspondences from U to X, i.e. the free abelian group generated by closed irreducible subsets of $U \times X$ which are finite over U and surjective over a connected component of U.

examples: For any morphism $f : U \to X$ in Sm/S, its graph $\Gamma(f)$ is an element of $c(U,X)$.
Researchers of the classical homotopy theorists might find it useful to regard this as an analogueq of the Dold-Thom infinite symmetric product construction: $T \to Sp^\infty T$.
The symmetric product construction shows up in the motivic analogue of the Steenrod algebra. Since the symmetric product construction takes us out of the smooth category, must resort to the resolution technologies...

Examples of $\mathbb{P}^1$-spectra - Morel, Example 5.1.2 - 3

The motivic cohomology spectrum, motivic Eilenberg spectrum $\mathbb{HZ}$ is a T-spectrum (thus a $\mathbb{P}^1$-spectrum), given by

$$\mathbb{HZ} := \left\{ L[\mathbb{A}^n] \big/ L[\mathbb{A}^n \setminus \{0\}]; \left(\sigma_n : L[\mathbb{A}^n] \big/ L[\mathbb{A}^n \setminus \{0\}] \wedge T \to L[\mathbb{A}^{n+1}] \big/ L[\mathbb{A}^{n+1} \setminus \{0\}]\right)\right\}$$

where σ_n is given by

$$\sigma_n : L[\mathbb{A}^n] \big/ L[\mathbb{A}^n \setminus \{0\}] \wedge T = \left(L[\mathbb{A}^n] \big/ L[\mathbb{A}^n \setminus \{0\}]\right) \wedge \left(\mathbb{A}^1/(\mathbb{A}^1 \setminus \{0\})\right)$$
$$\xrightarrow{1\wedge\Gamma(T)} \left(L[\mathbb{A}^n] \big/ L[\mathbb{A}^n \setminus \{0\}]\right) \wedge \left(L[\mathbb{A}^1] \big/ L[\mathbb{A}^1 \setminus \{0\}]\right)$$
$$\to L\left[(\mathbb{A}^n/(\mathbb{A}^n \setminus \{0\})) \wedge \left(\mathbb{A}^1/(\mathbb{A}^1 \setminus \{0\})\right)\right] \to L[\mathbb{A}^{n+1}] \big/ L[\mathbb{A}^{n+1} \setminus \{0\}]$$

Here the quotients /, where L shows up, are taken in the category of sheaves of abelian groups $\mathfrak{Ab}((Sm/k)_{Nis})$.

motivic cohomology - Suslin-Voevodsky, Voevodsky

For $X \in Sm/S,\ (n,i) \in (\mathbb{Z})^2$,

$$\mathbb{H}^{n,i}(X) \cong H^n\left(X, \mathbb{Z}(i)\right),$$

the Suslin-Voevodsky motivic cohomology group .

The motivic mod 2 cohomology and Steenrod algebra - 1

We now assume the base scheme $S = k$, a field of characteristic not equal to $\ell = 2$ (The case Voevodsky prove the Milnor conjecture).

- **Now, we have defined the motivic cohomology $\mathbb{H}^{n,i}(-)$ and so the motivic mod 2 cohomology $\mathbb{H}^{n,i}(-;\mathbb{Z}/2)$, just like the topological cohomology $H^*(-)$ and the topological mod p cohomology $H^*(X;\mathbb{Z}/2)$.**
- **In topology, the dual mod 2 Steenrod algebra $\mathcal{A}_*$ is a commutative Hopf algebra**

$$H_*(\mathbf{pt};\mathbb{Z}/2) = H^{-*}(\mathbf{pt};\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \text{if } * = 0 \\ 0 & \text{if } * \neq 0 \end{cases}$$
$$\mathcal{A}_* = H\mathbb{Z}/2_* H\mathbb{Z}/2 = H_*(\mathbf{pt};\mathbb{Z}/2)[\xi_k \mid k \geqq 1]$$
$$\Delta\xi_k = \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \xi_i \ (\xi_0 = 1)$$

inducing the action of the Steenrod algebra (w/ $Sq^0 \neq 1$) on $\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(H_*(\mathbf{pt};\mathbb{Z}/2), H_*(\mathbf{pt};\mathbb{Z}/2)) = \mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathbb{Z}/2,\mathbb{Z}/2)$, with

Adams S.S. $\quad E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}_*}(\mathbb{Z}/2,\mathbb{Z}/2) \Longrightarrow \pi^S_{t-s}(S^0)^\wedge_2$

The motivic mod 2 cohomology and Steenrod algebra - 2

The Question

How about the motivic case?

Bad News : **The dual motivic mod 2 Steenrod algebra $\mathcal{A}_{Mot_k}(*,*)$ is** not **a Hopf algebra, but a Hopf algebroid.**

$\Longrightarrow$ *May's machine to define the Steenrod algebra action on the* Ext *is* not *applicable to the case of $\mathcal{A}_{Mot_k}(*,*)$!*

Good News : *May use Bruner's machine....*

There is some similarity between the motivic $\mathcal{A}_{Mot_k(*,*)}$ and the topological $\mathcal{A}_*$. The main difference comes from $k_*^M(k)$, the mod 2 Milnor K-theory of the field k .

The Milnor K-theory $K_*^M(k)$ of a field k

Given a field k, treat $k^\times$ additively as a $\mathbb{Z}$-module...

$$K_*^M(k) := T_{\mathbb{Z}}(k^\times)\big/\left(\{a\}\otimes\{1-a\}\mid a\neq 0,1\right)$$

$$k_*^M(k) := K_*^M(k)/2 = T_{\mathbb{Z}/2}\left(k^\times/(k^\times)^2\right)\big/\left(\{a\}\otimes\{1-a\}\mid a\neq 0,1\right)$$

The motivic mod 2 cohomology and Steenrod algebra - 3

$H^{*,*}(\mathrm{Spec}\, k;\mathbb{Z}/2)$

Voevodsky computed (c.f. Milnor conjecture $H^{p,q}(\mathrm{Spec}\, k;\mathbb{Z}/2)\xrightarrow{\cong} H^p_{et}(\mathrm{Spec}\, k;\mu_2^{\otimes q})$ $(p\leqq q)$)

$$H_{*,*}(\mathrm{Spec}\, k;\mathbb{Z}/2) = k_*^M(k)[\tau]$$

where τ is the Tate twist with $|\tau| = (0,-1)$, $|k_1^M(k)| = (-1,-1)$

The algebra structure of $\mathcal{A}_{Mot_k(*,*)}$

Voevodsky computed

$$\mathcal{A}_{Mot_k(*,*)} = k_*^M(k)[\tau][\tau_0,\tau_1,\ldots,\xi_1,\xi_2,\ldots]/(\tau_i^2-\tau\xi_{i+1}-\rho(\tau_{i+1}+\tau_0\xi_{i+1}))$$

where $\rho=\{-1\}\in k_1^M(k)$ with

$$|\rho| = (-1,-1),$$
$$|\tau_i| = (2^i-1)(2,1)+(1,0) = (2^{i+1}-1, 2^i-1),$$
$$|\xi_i| = (2^i-1)(2,1) = (2^{i+1}-2, 2^i-1)$$

The motivic mod 2 cohomology and Steenrod algebra - 4

The Hopf algebroid structure of $\mathcal{A}_{Mot_k(*,*)}$

Voevodsky also computed the Hopf algebroid structure of $\mathcal{A}_{Mot_k(*,*)}$. While elements in $k_*^M(k)$ are primitive, τ is NOT primitive :

$$\eta_L\tau = \tau$$
$$\eta_R\tau = \tau + \rho\tau_0$$
$$\Delta\xi_k = \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \xi_i$$
$$\Delta\tau_k = \tau_k \otimes 1 + \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \tau_i.$$

The motivic mod 2 Adams spectral sequence for $S^0 := \Sigma_{\mathbb{P}}^\infty(\mathrm{Spec}\, k_+)$

Voevodsky also contructed the motivic Adams spectral sequence:

$$E_2^{s,(t,u)} = \mathrm{Ext}^{s,(t,u)}_{\mathcal{A}_{Mot_k(*,*)}}\left(k_*^M(k)[\tau], k_*^M(k)[\tau]\right) \Longrightarrow \pi^S_{t-s,u}(\mathrm{Spec}\, k_+)^\wedge_2$$

which has been to converge by Hu-Kriz-Ormsby.

The motivic mod 2 cohomology and Steenrod algebra - 5

The real reason why elements in $k_*^M(k)$ are primitive is explained by the following theorem of Morel-Hopkins:

(Hopkins-Morel) Milnor-Witt K-theory $K_*^{MW}(k)$ of k

$$K_*^{MW}(k) := \frac{\text{Free associative algebra ring on } \left(k^\times \amalg \{\eta\}\right)}{\substack{\{a\}\{1-a\} = 0, \{ab\} = \{a\} + \{b\} + \eta\{a\}\{b\}, \\ \eta\{a\} = \{a\}\eta,\ (2+\{-1\}\eta)\eta = 0}}$$

with $|\{a\}| = (-1,-1)$, $|\eta| = (1,1)$. Then,

$$\pi^S_{n,n}(\mathrm{Spec}\, k_+) = K^{MW}_{n(1,1)}(k)$$

Thus, by the mod 2 Hurewicz map,

$$\pi^S_{*(1,1)}(\mathrm{Spec}\, k_+) = K^{MW}_{*(1,1)}(k) \overset{\text{surjective}}{\twoheadrightarrow} k_*^M(k) \subsetneqq k_*^M(k)[\tau] = H_{*,*}(\mathrm{Spec}\, k; \mathbb{Z}/2)$$

The motivic mod 2 cohomology and Steenrod algebra - 6

Are you scared of the Minor K theory? Then, we might assume k is friendly enough to guarantee $k^M_*(k) = \mathbb{Z}/2$, which is the case for k algebraic closed.
In this case,

$$E_2^{s,(t,u)} = \mathrm{Ext}^{s,(t,u)}_{\mathcal{A}_{Mot_{k(*,*)}}}\left(k^M_*(k)[\tau], k^M_*(k)[\tau]\right) \Longrightarrow \pi^S_{t-s,u}(\mathrm{Spec}\, k_+)^\wedge_2$$

becomes closer to the classical topological situation.
In fact, using the solution of the Bloch-Kato conjecture for p odd, the correspong mod p odd E_2-term computation for algebraic closed field turns out to be the same as the classical E_2 term, with an extra grading.
$\Longrightarrow$ *Wish to export more classical insight about the classical Adams spectral sequence to the motivic Adams spectral sequence!*

The motivic mod ℓ cohomology and Steenrod algebra - 1

Now, let us consider more general base scheme S, which is Noetherian of finite dimension. From our viewpoint, we with to understand:

- $H^{p,q}(S, \mathbb{Z}/\ell)$. (c.f. $H^*(pt; \mathbb{Z}/p) = \mathbb{Z}/p$)
- $H\mathbb{Z}/\ell \wedge H\mathbb{Z}/\ell$ in $H(S)$. (c.f. $H\mathbb{Z}/p_* H\mathbb{Z}/p = \mathcal{A}_*$)

$H^{p,q}(S, \mathbb{Z}/\ell)$ may be interpreted as the Bloch Higher Chow group by the Nesterenko-Suslin theorem:

$$H^{p,q}(S, \mathbb{Z}/\ell) \cong CH^q(S, 2q-n; \mathbb{Z}/\ell) \qquad (2)$$

But the best tool to study $H^{p,q}(S, \mathbb{Z}/\ell)$ has been provided by the Voevodsky-Rost solution of the Bloch-Kato conjecture:

The motivic mod ℓ cohomology and Steenrod algebra - 2

Bloch-Kato conjecture, Voevodsky-Rost theorem

Suppose further that S is smooth scheme over a field k of characteristic $\neq \ell$, then the norm residue homomorphism

$$H^{p,q}(S;\mathbb{Z}/\ell) \to H^p_{et}(S;\mu_\ell^{\otimes q}) \quad (3)$$

is an isomrohpsim for $p \leqq q$ and a monomorphism for $p = q+1$. This recovers a theorem of Levine, which states under the same assumption,

$$H^{p,q}(S;\mathbb{Z}/\ell)[\tau^{-1}] \xrightarrow{\cong} H^p_{et}(S;\mu_\ell^{\otimes q}) \quad (4)$$

where $\tau \in H^{0,1}(Speck) \cong \mu_\ell(k)$, the primitive ℓ th root of unity.

Thus, up to nilpotency, $H^{p,q}(S;\mathbb{Z}/\ell)$ is extremely simple!!!

The motivic mod ℓ cohomology and Steenrod algebra - 3

For $H\mathbb{Z}/\ell \wedge H\mathbb{Z}/\ell$ (in $H(S)$),

arXiv13055690, The motivic Steenrod algebra in positive characteristic,

Marc Hoyois, Shane Kelly, Paul Arne Ostvar

made use of the Gabber's alteration to show (under some extra minor assumption):

$$H\mathbb{Z}/\ell*,*H\mathbb{Z}/\ell \cong H^{-*,-*}(S;\mathbb{Z}/\ell) \otimes_{H^{-*,-*}(k;\mathbb{Z}/\ell)} \mathcal{A}_{*,*}, \quad (5)$$

where $\mathcal{A}_{*,*}$ is the dual motivic Steenrod algebra for the base schme $Spec\ k$, defined by Voevodsky. $\mathcal{A}_{*,*}$ is very close to the topological Steenrod algebra!!!

The Motivic New Doomsday Conjecture

New Doomsday Conjecture

For each $s \geqq 1$, there exists some integer $n(s)$ such that no element in the image of $(\mathcal{P}^0)^{n(s)}$

$$(\mathcal{P}^0)^{n(s)}(\mathrm{Ext}^{s,(2t,u)}_{\mathcal{A}_{Mot_{k(*,*)}}}(S,S))$$
$$\subsetneqq \mathrm{Ext}^{s,(2p^{n(s)}t,u)}_{\mathcal{A}_{Mot_{k(*,*)}}}(S,,S)$$

is a nontrivial permanent cycle. Here, $\mathcal{P}^0 = Sq^0$ when $p = 2$. enditembox

多分定理???

The New Doomsday Conjecture implies the New Doomsday Conjecte.