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Some aspects of a finite T_0-G-space

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1 Introduction

The purpose of our presentation was to study actions of finite groups on finite T_0-spaces, i.e. topological spaces having finitely many points with the T_0-separation axioms. The definition of T_0-separation axiom is, for each pair of distinct points, there exists an open set containing one but not the other. A remarkable feature of a finite T_0-space is that it has the structure of a poset. Conversely, one can give any finite poset the structure of a finite T_0-space. The equvariant theory of finite T_0-spaces was first made by Stong [11]. After that, Kono and Ushitaki investigated the homeomorphism groups of finite spaces with group actions ([6], [7], [8]). Here a finite space is a topological space having finitely many points. In particular, they studied the homeomorphism groups of fixed point set X^G and G-actions on homeomorphism groups induced by given G-action on X, where X is a finite space with a G-action.

First we define a simplicail complex induced from a finite T_0-space. Recall that a finite T_0-space has a poset structure (see Proposition 2.2). Let X be a finite poset. The *order complex* $\Delta(X)$ of X is the abstract simplicial complex on the vertex set X whose faces are the chains of X, including the empty chain. The *dimension* of a simplex is defined to be the length of the chain, where the length of a chain is one less than its number of elements. In particular, the length of the empty chain is -1. When the dimension of a simplex σ is k, we write $\dim\sigma=k$. Next we shall define the geometric realization $|\Delta(X)|$ of $\Delta(X)$ by

$$|\Delta(X)|=\{m:X\rightarrow[0,1]\mid\sum_{x\in X}m(x)=1,\ \mathrm{supp}(m)\in\Delta(X)\},$$

where for a map $m:X\rightarrow[0,1]$, we mean that $\mathrm{supp}(m)=\{x\in X\mid m(x)>0\}$. The numbers $(m(x)\mid x\in X)$ are the *barycentric coordinates* of m. For a simplex $\sigma\in\Delta(X)$, we put

$$|\sigma|=\{m\in|\Delta(X)|\mid\mathrm{supp}(m)=\sigma\}.$$

We can define a metric topology on $|\Delta(X)|$. In details, we have a metric d on $|\Delta(X)|$ defined by

$$d(m_1,m_2)=\left(\sum_{x\in X}(m_1(x)-m_2(x))^2\right)^{\frac{1}{2}}.$$

Then we have $\overline{|\sigma|}=\{m\in|\Delta(X)|\mid\sum_{x\in\sigma}m(x)=1\}$, where $\overline{|\sigma|}$ indicates the closure of $|\sigma|$. Moreover a metric space $|\Delta(X)|$ is equipped with a CW-complex structure whose n-cell is a set $\{|\sigma| \,|\, \sigma \in \Delta(X),\ \dim\sigma = n\}$. Let $(p_x \,|\, x \in X)$ be a family of points in euclidean n-space $\mathbb{R}^n$. Consider the continuous map

$$f : |\Delta(X)| \to \mathbb{R}^n, \qquad m \mapsto \sum_{x\in X} m(x)p_x.$$

If f is an embedding, we call the image of f a *simplicial polyhedron* in $\mathbb{R}^n$ of type $\Delta(X)$, that is, $f(|\Delta(X)|)$ is a realization of $\Delta(X)$ as a polyhedron in $\mathbb{R}^n$.

Now, we shall introduce McCord's result [9, Theorem 2], which provides insight into understanding relations between finite T_0-spaces and simplicial complexes.

Proposition 1.1. *There exists a correspondence that assigns to each finite T_0-space X a finite simplicial complex $\Delta(X)$, whose vertices are the points of X, such that the map $\mu_X : |\Delta(X)| \to X$ induced from the correspondence above is a weak homotopy equivalence. Moreover, each map $\varphi : X \to Y$ of finite T_0-spaces is also a simplicial map $\Delta(X) \to \Delta(Y)$, and $\varphi\mu_X = \mu_Y|\varphi|$ where $|\varphi| : |\Delta(X)| \to |\Delta(Y)|$ is a continuous map induced by φ.*

Let G be a finite group. In this note, we focus on the equivariant order complex $\Delta(X)$ of a finite T_0-G-space X, that is, a finite T_0-space with a G-action, and then its orbit space $\Delta(X)/G$. In particular, we are interested in the following questions:

(i) Does $|\Delta(X)|$ has a G-CW-complex structure?
(ii) Is there the orbit space version of Proposition 1.1?

Our results related the above questions are the following.

Theorem A. Let X be a finite T_0-G-space. Then $|\Delta(X)|$ is a finite G-CW-complex.

We will prepare the following technical condition:

(C) If $g_0, g_1, \cdots, g_k$ are elements of G and $(x_0, x_1, \cdots, x_k)$ and $(g_0x_0, g_1x_1, \cdots, g_kx_k)$ are both simplices of K, then there exists an element g of G such that $gx_i = g_ix_i$ for all i. Here overlaps of some of x_i are allowed.

Theorem B. If $\Delta(X)$ satisfies property (C), there exists a weak homotopy equivalence $\tilde{\mu}_X : |\Delta(X)|/G \to X/G$.

The rest of this note is organized as follows. In section 2, we briefly review finite (T_0-)space theory. In section 3, we investigate an equivariant version of finite T_0-spaces and prove Theorem A. The last section studies orbit spaces of equivariant complexes and prove Theorem B.

2 Finite (T_0-)spaces

In this section, we survey well-known properties about finite (T_0-)spaces. General reference may be found in [2], [6] and [10]. Let X denote a finite space, i.e. a topological space having finitely many points. Let a set U_x be the minimal open set which contains a point x of X, that is, U_x is the intersection of all open sets containing x. It is easy to see that a set $\{U_x\}_{x\in X}$ constitute a basis for the topology of X. Now we can define a *preorder* on X by

$$x \leq y \quad \text{if} \quad x \in U_y.$$

In other words, every open set containing y also contains x if and only if $x \leq y$.

Proposition 2.1. *Let x and y be elements of a finite space X. Then X is T_0-space if and only if $U_x = U_y$ implies $x = y$.*

Proposition 2.2. *A finite T_0-space with the above preorder $\leq$ is a poset.*

If X is now a finite preordered set, one can define a topology on X given by the basis $\{y \in X \mid y \leq x\}_{x \in X}$. Note that if $y \leq x$, then y is contained in every basic set containing x, and therefore $y \in U_x$. Conversely, if $y \in U_x$, then $y \in \{z \in X \mid z \leq x\}$. After all, $y \leq x$ if and only if $y \in U_x$. This shows that these two applications, relating topologies and preorders on a finite set, are mutually inverse. Thus we have

Proposition 2.3. *A finite T_0-space corresponds to a finite poset.*

Example 2.4. Let $X = \{a, b, c\}$ be a finite space whose topology is $\{\emptyset, \{a, b, c\}, \{b, c\}, \{b\}, \{c\}\}$. This space is T_0. Immediately, $U_a = \{a, b, c\}$, $U_b = \{b\}$ and $U_c = \{c\}$. Therefore $b \leq a$ and $c \leq a$, but there exists no order relation between b and c.

Example 2.5. Let $X = \{a, b, c, d\}$ be a finite space whose topology is $\{\emptyset, \{a, b, c, d\}, \{b, c, d\}, \{b\}, \{b, c\}, \{b, d\}\}$. This space is also T_0. Immediately, $U_a = \{a, b, c, d\}$, $U_b = \{b\}, U_c = \{b, c\}$ and $U_d = \{b, d\}$. On the order relation, we see the following Hasse diagram:

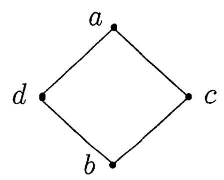


Figure 1.

Proposition 2.6. *Let X be a preordered set. A set $F_x = \{y \in X \mid x \leq y\}$ is a closed set of X. Moreover F_x is the closure of the set $\{x\}$.*

Definition 2.7. A subset U of a preordered set X is a *down-set* if for every $x \in U$ and $y \leq x$, it holds that $y \in U$. Dually, a subset F of a preordered set X is a *up-set* if for every $x \in F$ and $y \geq x$, it holds that $y \in F$. Open sets of finite spaces correspond to down-sets and closed sets to up-sets.

Proposition 2.8. *Let X and Y be finite spaces, and f be a map from X to Y. Then f is continuous if and only if f is an order-preserving map.*

Proposition 2.9. *Let X be a finite space, f a continuous map of X into itself. If f is either one-to-one or onto, then it is a homeomorphism.*

Next we state connectivity. First, for each U_x, we let $U_x \subset A \cup B$, where A and B are open sets of a finite space X. Then x is in one set, say $x \in A$, immediately $U_x \subset A$. Thus any finite space is locally connected.

Proposition 2.10. *Let x, y be two comparable points of a finite space X and $x \leq y$. Then there exists a path from x to y in X, that is, a map α from the unit interval I to X such that $\alpha(0) = x$ and $\alpha(1) = y$.*

Let X be a finite preordered set. A *fence* in X is a sequence $x_0, x_1, \cdots, x_n$ of points such that any two consecutive are comparable. X is *order-connected* if any two points $x, y \in X$ there exists a fence starting in x and ending in y.

Proposition 2.11. *Let X be a finite space. Then the following are equivalent:*
(i) X is a connected topological space.
(ii) X is an order-connected preordered set.
(iii) X is a path-connected topological space.

If X and Y are finite spaces, we can consider the finite set Y^X of continuous maps from X to Y with the pointwise order: $f \leq g$ if $f(x) \leq g(x)$ for every $x \in X$.

Proposition 2.12. *Let X and Y be two finite spaces. Then pointwise order on Y^X corresponds to the compact-open topology.*

Corollary 2.13. *Let $f, g : X \to Y$ be two maps between finite spaces. Then $f \simeq g$ if and only if there is a fence $f = f_0 \leq f_1 \geq f_2 \leq \cdots f_n = g$. Moreover, if $A \subset X$, then $f \simeq g$ rel A if and only if there exists a fence $f = f_0 \leq f_1 \geq f_2 \leq \cdots f_n = g$ such that $f_i|_A = f|_A$ for every $0 \leq i \leq n$.*

Any finite space is homotopy equivalent to a finite T_0-space.

Proposition 2.14. *Let X be a finite space. Let X_0 be the quotient $X/\sim$ where $x \sim y$ if $x \leq y$ and $y \leq x$. Then X_0 is T_0 and the quotient map $q : X \to X_0$ is a homotopy equivalence.*

Therefore , when studying homotopy types of finite spaces, we can restrict our attention to finite T_0-spaces.

Definition 2.15. A point x in a finite T_0-space X is a *down beat point* if x cover one and only one element of X. This is equivalent to saying that the set $\hat{U}_x = U_x \backslash \{x\}$ has a maximum. Dually, $x \in X$ is an *up beat point* if x is covered by a unique element or equivalently if $\hat{F}_x = F_x \backslash \{x\}$ has a minimum, where F_x denotes the closure of the set $\{x\}$. In any of these cases, we say that x is a *beat point* of X.

Proposition 2.16. *Let X be a finite T_0-space and let $x \in X$ be a beat point. Then $X \backslash \{x\}$ is a strong deformation retract of X.*

Definition 2.17. A finite T_0-space is a *minimal finite space* if it has no beat points. A *core* of a finite space X is a strong deformation retract which is a minimal finite space.

Proposition 2.18. *Let X be a minimal finite space. A map $f : X \to X$ is homotopic to the identity if and only if $f = 1_X$.*

Immediately, we have the following corollary.

Corollary 2.19. (Classification Theorem) *A homotopy equivalence between minimal finite spaces is a homeomorphism. In particular, the core of a finite space is unique up to homeomorphism and two finite spaces are homotopy equivalent if and only if they have homeomorphic cores.*

By the Classification Theorem, a finite space is contractible if and only if its core is a point. In fact, a one-point finite space has a core of the one-point. Therefore any contractible finite space has a point which is a strong deformation retract.This property is false in general for non-finite spaces.

3 Finite T_0-G-spaces

In this section, we treat an equivariant version of finite T_0-spaces. Let G be a topological group (a group, for short) and X a finite T_0-space. A G-invariant subspace $A \subset X$ is an *equivariant strong deformation retract* if there is an equivariant retraction $r : X \to A$ such that ir is homotopic to 1_X via a G-homotopy which is stationary at A. A finite T_0-space which is a G-space will be *a finite T_0-G-space.*

Remark If a topological group G acts on a finite topological space effectively, then it must be a finite topological group [7, Proposition 3.9]. Therefore, from now on, we assume that G is finite.

Proposition 3.1. *Let X be a finite T_0-G-space. Then there exists a core of X which is G-invariant and an equivariant strong deformation retract of X.*

Proposition 3.2. *A contractible finite T_0-G-space has a point which is fixed by the action of G.*

This proposition deduces Stong's result stated in introduction. Note that $A_p(G)$ is a finite T_0-G-space by conjugation. If $A_p(G)$ is contractible, $A_p(G)$ has exactly one point core which is G-invariant. Therefore $A_p(G)$ has a fixed point by the action of G. Consequently, G has a non-trivial normal p-subgroup.

Proposition 3.3. *Let X and Y be finite T_0-G-spaces and let $f : X \to Y$ be a G-map which is a homotopy equivalence. Then f is an equivariant homotopy equivalence.*

Let X be a finite T_0-G-space and x, y points of X. If $x \in U_y$, then $gx \in gU_y = U_{gy}$. Therefore a G-action on a finite T_0-space X preserves the order. Thus $\Delta(X)$ is a G-simplicial complex (in short, G-complex). Let $\mathbb{N}_0$ be the union set of natural numbers $\{1, 2, 3, \cdots\}$ and $\{0\}$.

Definition 3.4. Let G be a finite group. A CW-complex Z with a G-action is called a *G-CW-complex* if it satisfies the following conditions:
(i) The G-action determines a cellular map, that is, for any $g \in G$, $gZ^i \subset Z^i$ for each $i \in \mathbb{N}_0$, where Z^i denotes the union of cells of dimension $\leq i$ and is called the *i-skeleton* of Z.
(ii) If $g(e) = e$, then g is trivial on $\bar{e}$, that is, $Z^g \supset \bar{e}$, where $\bar{e}$ is the closure of e.

Proof of Theorem A.

Proof. For $g \in G$ and $m \in |\Delta(X)|$, we define a map $g(m) : X \to [0, 1]$ by

$$(g(m))(x) := m(g^{-1}(x)) \quad \text{for } x \in X.$$

Then we have

$$\sum_{x \in X}(g(m))(x) = \sum_{x \in X} m(g^{-1}(x)) = \sum_{g^{-1}(x) \in X} m(g^{-1}(x)) = 1,$$

on the other hand,

$$\begin{aligned}\mathrm{supp}(g(m)) &= \{x \in X \mid (g(m))(x) > 0\} \\ &= \{x \in X \mid m(g^{-1}(x)) > 0\} \\ &= \{x \in X \mid g^{-1}(x) \in \mathrm{supp}(m)\} \\ &= g(\mathrm{supp}(m)) \in \Delta(X).\end{aligned}$$

Therefore we have that $g(m) \in |\Delta(X)|$. Thus we can define a isometric map $g : |\Delta(X)| \to |\Delta(X)|$. For each $\sigma \in \Delta(X)$, it holds that $g(|\sigma|) = |g(\sigma)|$. In particular, a map g is a cellular map.

Let $g(|\sigma|) = |\sigma|$. Immediately, we have $g(\sigma) = \sigma$. Since g is an automorphism between totally ordered sets, it is an identity map. Therefore $g^{-1} : \sigma \to \sigma$ is also an identity map. Let m be any element of $\overline{|\sigma|}$.

Case $x \in \sigma$: It follows that $(g(m))(x) = m(g^{-1}(x)) = m(x)$.

Case $x \in X \backslash \sigma$: Since $g^{-1}(x) \in X \backslash g^{-1}(\sigma) = X \backslash \sigma$, we get that $(g(m))(x) = m(g^{-1}(x)) = 0 = m(x)$.

Therefore $g(m) = m$. Thus we obtain that $\overline{|\sigma|} \subset |\Delta(X)|^g$. □

Referring to [5, p.229], we now prepare the following technical properties concerning a G-complex K:

(P_1) For any $g \in G$ and simplex σ of K, g leaves $\sigma \cap g\sigma$ pointwise fixed.

(P_2) If $g_0, g_1, \cdots, g_k$ are elements of G and $(x_0, x_1, \cdots, x_k)$ and $(g_0x_0, g_1x_1, \cdots, g_kx_k)$ are both simplices of K, then there exists an element g of G such that $gx_i = g_ix_i$ for all i. Here overlaps of some of x_i are allowed.

(P_3) Let g be an element of G and σ a simplex of K. If $g(\sigma) = \sigma$, g leaves σ pointwise fixed.

Proposition 3.5. *It holds that* $(P_2) \Longrightarrow (P_1) \Longrightarrow (P_3)$.

Proposition 3.6. *Let X be a finite T_0-G-space. Then a G-complex $\Delta(X)$ holds both property (P_1) and property (P_3).*

On a G-complex, we can see a geometric simplex as a cell. One immediate consequence of this observation is the following.

Proposition 3.7. *Let $|K|$ be the geometric realization of a G-complex K with property (P_3). Then $|K|$ is a G-CW-complex.*

The following result is an equivariant version of Proposition 1.1 in a sense.

Proposition 3.8. *Let X be a finite T_0-G-space. For each subgroup H of G, it holds that $\Delta(X^H) = \Delta(X)^H$ and the map $\mu_X^H : |\Delta(X)|^H \to X^H$ is a weak homotopy equivalence.*

4 Orbit spaces

Next we will devote the study of the orbit space of a G-complex.

Proposition 4.1. *Let X be a finite T_0-G-space. Then the orbit space X/G is a finite T_0-space.*

Let X and Y be finite sets, and $\mathcal{P}(X)$ the power set of X. A map $f : X \to Y$ induces a map $\mathcal{P}(X) \to \mathcal{P}(Y)$, which we denote also by f. Let K be a simplicial complex such that X is the set of vertices of K. Then it is easy to see that the image $f(K)$ becomes a simplicial complex such that $f(X)$ is the set of vertices of $f(K)$. We apply this observation to our situation.

Let K be a G-complex and X be the set of vertices of K. Concerning the induced G-action on X. we consider its orbit space X/G and the orbit map $p : X \to X/G$. As observes above, p induced a map $\mathcal{P}(X) \to \mathcal{P}(X/G)$, which we denote by p as well and $p(K)$ becomes a simplicial complex such that X/G is the set of vertices of $p(K)$. For $s \in K$, we denote $p(s)$ by $\overline{s}$.

Next we consider another kind of orbit space. Let K be a G-complex. Denote by K/G the orbit space of the G-action on K and by $\pi : K \to K/G$ the orbit map. For $s \in K$, we denote $\pi(s)$ by $[s]$. Note that K/G is not a simplicial complex in general and K/G does not coincide with $p(K)$ in general.

Proposition 4.2. [5, **Lemma 5.10**] *Let K be a G-complex satisfying property (P_2) and X be the set of vertices of K. Then the orbit space K/G becomes a simplicial complex such that the set of vetrices K/G is X/G and K/G is naturally isomorphic to $p(K)$. Moreover the orbit map $\pi : K \to K/G$ is a simplicial map preserving dimension of simplexes.*

Corollary 4.3. *If K is a G-complex satisfying property (P_2), $|K|/G$ is homeomorphic to $|K/G|$.*

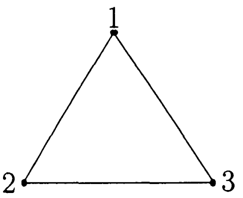
Furthermore, we add simplicial notion for both posets and (finite) cell complexes to investigate the simplicial structure of the orbit spaces in detail.

Definition 4.4. A *simplicial poset* P is a finite poset with a smallest element $\hat{0}$ such that every interval

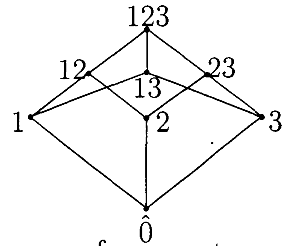
$$[\hat{0}, y] = \{x \in P \mid \hat{0} \leq x \leq y\}$$

for $y \in P$ is a boolean algebra, i.e., $[\hat{0}, y]$ is isomorphic to the set of all subsets of a finite set, ordered by inclusion. When a boolean algebra is the set of all subsets of a finite set consisting of n elements, we denote the boolean algebra by B_n. Let x be an element of P such that $[\hat{0}, x]$ is isomorphic to a boolean algebra B_n. Then the *dimension* of x is said to be $n - 1$, denoted by $\dim x = n - 1$. Remark that $\dim \hat{0} = -1$. Moreover, a simplicail poset P is *n-dimensional*, if it contains at least one point x such that $\dim x = n$ but no $(n + 1)$-dimensional points.

The set of all faces of a (finite) simplicial complex with empty set added forms a simplicial poset ordered by inclusion, where the empty set is the smallest element. Such a simplicial poset is called the *face poset* of a simplicial complex, and two simplicial complexes are isomorphic if and only if their face posets are isomorphic. Therefore, a simplicial poset can be thought of as a generalization of a simplicial complex. Figure 2 shows that a 2-simplicial complex and its face poset.



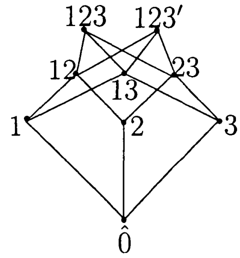
a 2-simplicial complex



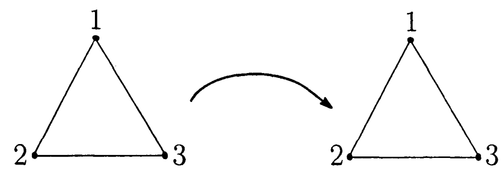
a face poset

Figure 2.

A CW-complex is said to be *regular* if all closed cells are homeomorphic to closed disks. Although a simplicial poset is not necessarilly the face poset of a simplicial complex, it is always the face poset of a regular CW-complex. Let P be a simplicial poset. To each element $y \in P\backslash\{\hat{0}\} = \overline{P}$, we assign a (geometric) simplex whose face poset is $[\hat{0}, y]$ and glue those geometric simplices according to the order relation in P. Then, we get the CW-complex in which the closure of each cell is identified with a simplex, the structure of faces being preserved; moreover, all characteristic mappings are embeddings. This CW-complex is called a *simplicial cell complex* associated to P and is denoted by $|P|$. For instance, if two 2-simplices are identified on their boundaries via the identity map, then it is not a simplicial complex but a CW-complex obtained from a simplicial poset (see Figure 3). Clearly, this CW-complex is homeomorphic to the 2-sphere S^2. The simplicial cell complex $|P|$ has a well-defined barycentric subdivision which is isomorphic to the order complex $\Delta(\overline{P})$ of the poset $\overline{P}$.



a simplicial poset P
(123′ is a copy of 123)



a simplicial cell complex $|P|$

Two 2-simplicial complexes are identified on their boundaries.

Figure 3.

By definition, we have the following proposition.

Proposition 4.5. *Let S is a finite cell complex. Then S is simplicial if and only if for each cell $\sigma \subset S$, the closure $\overline{\sigma}$ of σ is isomorphic to a simplex Δ of the same dimension with σ as a cell complex.*

In a word, a simplicial cell complex is a cell complex such that each closed cell is a geometric simplex. Obviously, the geometric realization of any finite simplicial complex is a simplicial cell complex.

Definition 4.6. Let S be a simplicial cell complex and $V(S)$ the set of all 0-cells of S. Let σ be a cell of S. We put $V(\sigma) = V(S) \cap \overline{\sigma}$. For each cell $\sigma \subset S$, there is an embedding

$$\varphi_\sigma : \Delta^{\dim \sigma}(V(\sigma)) \twoheadrightarrow \overline{\sigma} \subset S,$$

where $\Delta^{\dim \sigma}(V(\sigma))$ is the $\dim \sigma$-simplex whose vertex set is $V(\sigma)$. We say φ_σ a *characteristic map* of σ.

Proposition 4.7. *A simplicial poset corresponds to a simplicial cell complex.*

Let P be a simplicial poset and $x \in P$. A half-open interval $(\hat{0},\, x]$ is a subset $\{y \in P \,|\, \hat{0} \lneqq y \leq x\}$ of P.

Definition 4.8. Let P and Q be simplicial posets. A *simplicial poset map* $f : P \to Q$ is a map such that for any $x \in P$, $\dim f(x) \leq \dim x$ and $f((\hat{0},\, x]) = (\hat{0},\, f(x)]$.

For a simplicial poset P, we put $V(P) := \{x \in P \,|\, \dim x = 0\}$, which is called the *vertex set* of P. Similarly, for each $x \in P$, $V(x) := V([\hat{0}, x]) = [\hat{0}, x] \cap V(P)$, which is also called the *vertex set* of x. A simplicial poset map f is order-preserving and satisfies $f(V(x)) = V(f(x))$ for $x \in P$. Note that $V(P) = \bigcup\limits_{x \in P} V(x)$. Moreover we put

$$K_P := \{V(x) \,|\, x \in P\},$$

which is a simplicial complex whose vertex set is $V(P)$. Here we see K_P as a simplicial poset, so that a surjection $\varphi_P : P \to K_P$ defined by $\varphi_P(x) = V(x)$ is a simplicial poset map.

Definition 4.9. Let X and Y be simplicial cell complexes. A *simplicial cell complex map* $f : X \to Y$ is a cellular map such that for any cell $\sigma \in X$, $f(\sigma)$ is a cell of Y and $f\,|_{\overline{\sigma}} : \overline{\sigma} \to \overline{f(\sigma)} \subset Y$ extends linearly the map $f\,|_{V(\sigma)} : V(\sigma) \to V(f(\sigma)) \subset Y$. Note that $f(\overline{\sigma})$ is the compact set of a Hausdorff space Y.

Let X and Y be simplicial cell complexes. Let $\mathcal{F}(X)$ (*respectively*, $\mathcal{F}(Y)$) be a simplicial poset corresponding to X (*respectively*, Y). A simplicial cell complex map $f : X \to Y$ defines a simplicial poset map $\mathcal{F}(f) : \mathcal{F}(X) \to \mathcal{F}(Y)$ by $\sigma \mapsto f(\sigma)$ for each cell $\sigma \in X$. Conversely, we have the following.

Proposition 4.10. *For any simplicial poset map $\alpha : \mathcal{F}(X) \to \mathcal{F}(Y)$, there exists uniquely a simplicial cell complex map $f : X \to Y$ such that $\mathcal{F}(f) = \alpha$. In particular, if a simplicial poset map $\alpha : \mathcal{F}(X) \to \mathcal{F}(Y)$ is bijective, then f is an isomorphism from X to Y.*

Proposition 4.11. *For any simplicial poset P, there exists some simplicial cell complex X with $\mathcal{F}(X) \cong P$.*

From the above two propositions, there is uniquely an isomorphism class $[X]$ such that $\mathcal{F}(X) \cong P$. Then a simplicial cell complex X is said to be *a realization* of P, denoted by $|P|$ as well. Under this notation, we have a simplicail cell complex map $|\varphi_P| : |P| \to |K_P|$.

Let K be a G-complex. Now, we shall investigate the structure of the orbit space K/G. Let σ and τ be simplices of K. We define a partial ordering on K/G as follows:

$$\pi(\tau) \leq \pi(\sigma) \text{ if and only if there exists an element } g \in G \text{ such that } g(\tau) \subset \sigma,$$

where the map $\pi : K \to K/G$ is the orbit map. Note that the orbit space K/G has the minimum $\hat{0} = \pi(\emptyset)$. Moreover we denote the orbit map from $|K|$ to $|K|/G$ by π as well.

Proposition 4.12. *If a G-complex K has property (P_1), K/G is a simplicial poset. Moreover $|K|/G$ is a simplicial cell complex such that $\{\pi(|\sigma|) \mid \sigma \in K \backslash \{\emptyset\}\}$ is the set of all cells of $|K|/G$.*

Proposition 4.13. *If a G-complex K has property (P_1), it holds that $|K|/G \cong |K/G|$ as a simplicial cell complex.*

Corollary 4.14. *Let X be a finite T_0-G-space. The orbit space $|\Delta(X)|/G$ is a finite simplicial cell complex associated to a simplicial poset $\Delta(X)/G$. Moreover we have $|\Delta(X)|/G \cong |\Delta(X)/G|$.*

Let X be a finite T_0-G-space. Since the orbit map $p : X \to X/G$ is continuous, it is an order-preserving map. It determines a simplicial map

$$\Delta(p) : \Delta(X) \to \Delta(X/G),$$

and also a continuous map $|\Delta(p)| : |\Delta(X)| \to |\Delta(X/G)|$. Noting $|\Delta(X/G)|$ is a G-space with a trivial G-action, we have a continuous map $\tilde{p} : |\Delta(X)|/G \to |\Delta(X/G)|$ such that the following diagram commutes

$$\begin{array}{ccc} |\Delta(X)| & & \\ {\scriptstyle q}\downarrow & \searrow^{|\Delta(p)|} & \\ |\Delta(X)|/G & \xrightarrow[\tilde{p}]{} & |\Delta(X/G)| \end{array}$$

where q is the orbit map from $|\Delta(X)|$ to $|\Delta(X)|/G$.

Proposition 4.15. *Let X be a finite T_0-G-space. A simplicial complex $K_{\Delta(X)/G}$ concides with $\Delta(X/G)$.*

In consequence we have the following commutative diagram:

$$\begin{array}{ccc} |\Delta(X)|/G & \xrightarrow{\cong} & |\Delta(X)/G| \\ {\scriptstyle \tilde{p}}\downarrow & & \downarrow{\scriptstyle |\varphi_{\Delta(X)/G}|} \\ |\Delta(X/G)| & \xrightarrow[id]{} & |\Delta(X/G)|\,. \end{array}$$

A simplicial action of G on a simplicial complex K is called *regular in the sense of Bredon* if K possesses property (P_2) for the action of each subgroups of G. Now, we shall present an interesting example.

Example 4.16. Let n be an integer larger than one. Let X_{2n+2} be a set consisting of $2n + 2$ elements as follows:

$$X_{2n+2} =: \bigcup_{i=1}^{n+1} \{x_i,\ x_{-i}\}.$$

We set

$$\begin{cases} U(x_i) := \{x_i\} \bigcup_{j=1}^{i-1} \{x_j,\, x_{-j}\}, \quad \text{and} \\ U(x_{-i}) := \{x_{-i}\} \bigcup_{j=1}^{i-1} \{x_j,\, x_{-j}\}, \end{cases}$$

for $i = 1, 2, \cdots, n+1$. First note that each point x_i determines the smallest open set $U(x_i)$ on X_{2n+2}, that is, $U_{x_i} = U(x_i)$. Therefore we define a T_0-topology on X_{2n+2}. Let g be a map from X_{2n+2} to itself by $g(x_i) = x_{-i}$. We set $G := \langle g \rangle$(that is, a group is generated by g). Evidently, G is a cyclic group whose order is two. Since $|\Delta(X_{2n+2})|$ is homeomorphic to the n-sphere S^n, it holds that $|\Delta(X_{2n+2})|/G \cong \mathbb{R}P^n$, where $\mathbb{R}P^n$ is the n-dimensional real projective space. Note that $|\Delta(X_{2n+2})|/G$ is a simplicial cell complex by Proposition 4.12. On the other hand, X_{2n+2}/G is a totally ordered set with $n+1$ elements. Therefore $|\Delta(X_{2n+2}/G)|$ is homeomorphic to a n-simplex $\Delta^n(X_{2n+2}/G)$. Since the map $\tilde{p} : |\Delta(X_{2n+2})|/G \to |\Delta(X_{2n+2}/G)|$ is not a weak homotopy equivalence, $\tilde{p}$ is not an isomorphism between simplicial cell complexes. If $\Delta(X_{2n+2})/G$ is a simplicial complex, the map $|\varphi_{\Delta(X_{2n+2})/G}|$ is an isomorphism, and $\tilde{p}$ is also an isomorphism. This is a contradiction. Hence $\Delta(X_{2n+2})/G$ is not a simplicial complex, thereby G-action on $\Delta(X_{2n+2})$ is not regular in the sense of Bredon.

Proof of Theorem B.

Let X be a finite T_0-G-space. By Proposition 1.1, there is a weak homotopy equivalence $\mu_X : |\Delta(X)| \to X$. Then μ_X determines a continuous map $\tilde{\mu}_X : |\Delta(X)|/G \to X/G$ such that the following diagram commutes.

$$\begin{array}{ccc} |\Delta(X)|/G & \xrightarrow{\tilde{p}} & |\Delta(X/G)| \\ & \searrow^{\tilde{\mu}_X} & \downarrow{\scriptstyle \mu_{X/G}} \\ & & X/G \end{array}$$

Therefore $\tilde{p}$ is a weak homotopy equivalence if and only if $\tilde{\mu}_X$ is so. In general, $\tilde{\mu}_X$ is not a weak homotopy equivalence (see Example 4.16).

Remark that both $|\Delta(X)|/G$ and $|\Delta(X/G)|$ are CW-complexes. Therefore, we have
Claim 1. $\tilde{\mu}_X$ is a weak homotopy equivalence if and only if $\tilde{p}$ is a homotopy equivalence.

We consider the case where $\tilde{p}$ is a homeomorphism.

Claim 2. Let X be a finite T_0-G-space. Then the following conditions are equivalent:
(1) $\tilde{p}$ is a homeomorphism.
(2) $\Delta(X)/G$ is a simplicial complex.
(3) $\Delta(X)$ has property (P_2).

Proof. (1) $\Longrightarrow$ (2) Since $\tilde{p}$ is a homeomorphism, $\varphi_{\Delta(X)/G}$ is injective. Let U be a subset of X/G. Then there exists only one element s of $\Delta(X)/G$ at most with $V(s) = U$. Therefore $\Delta(X)/G$ is a simplicial complex. (2) $\Longrightarrow$ (1) Since $\Delta(X)/G$ is a simplicial complex, it holds that $|\Delta(X)/G| = |\Delta(X/G)|$. Noting that $\varphi_{\Delta(X)/G}$ is surjective, $\tilde{p}$ is also surjective.

By Proposition 2.9, $\tilde{p}$ is a homeomorphism. (2) $\Longrightarrow$ (3) Let $\sigma = \{x_i \mid i = 0, \cdots, k\}$ and $\tau = \{g_i x_i \mid g_i \in G,\ i = 0, \cdots, k\}$ be simplices of $\Delta(X)$. If $x_i = x_j$, then

$$g_j x_j = (g_j g_i^{-1})(g_i x_i) \in \tau \cap (g_j g_i^{-1})\tau.$$

Since a G-complex $\Delta(X)$ has property (P_1), we have $g_j x_j = (g_j g_i^{-1})^{-1}(g_j x_j) = g_i x_j$, so that $g_i x_i = g_i x_j = g_j x_j$. Hence we assume that each $x_i\,(i = 0, \cdots, k)$ is distinct, then both σ and τ are k-simplices of $\Delta(X)$. Therefore both $\pi(\sigma)$ and $\pi(\tau)$ are elements of $\Delta(X)/G$ such that $V(\pi(\sigma)) = V(\pi(\tau)) = \{\pi(x_i) \mid i = 0, \cdots, k\}$. By assumption, $\pi(\sigma) = \pi(\tau)$. In consequence there is some $g \in G$ such that $\tau = g(\sigma)$ and $g_i x_i = g x_i\,(i = 0, \cdots, k)$. (3) $\Longrightarrow$ (2) It follows from Proposition 4.2. □

Combining Claim 1 and Claim 2, we obtain Theorem B. □

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