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SEARCHING FOR EVEN ORDER BORSUK-ULAM GROUPS

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Dedicated to the memory of Professor Doctor Minoru Nakaoka

ABSTRACT. A Borsuk-Ulam group G is a group which satisfies the Borsuk-Ulam inequality for every isovariant map. Except some cases, it is still unknown what kind of groups are Borsuk-Ulam groups. In this paper, we present some sufficient conditions for being a Borsuk-Ulam group when G has an even order. Moreover, we introduce a new family of Borsuk-Ulam groups for approaching an unsolved problem.

1. INTRODUCTION

Let G be a group. Suppose X and Y are G-spaces. A G-equivariant map $\varphi : X \to Y$ is called a *G-isovariant map* if $G_x = G_{\varphi(x)}$ holds for all $x \in X$, where G_x denotes the isotropy subgroup of G at x. As is well known, the Borsuk-Ulam theorem ([1]) is stated as follows:

Proposition 1.1. *Let C_2 be a cyclic group of order 2. Assume that C_2 acts on both S^m and S^n antipodally. If there exists a continuous C_2-map $f : S^m \to S^n$, then $m \leqq n$ holds.*

Since the actions on both spheres are free, f in the above proposition is an isovariant map. Several authors regard the Borsuk-Ulam theorem as a statement for equivariant maps, but we have been studying the Borsuk-Ulam type theorems in isovariant setting for this reason. An isovariant Borsuk-Ulam type theorem was introduced by Wasserman in 1991 ([5]). In his work, he introduced the Borsuk-Ulam groups. Let G be a compact Lie group. Let V and W be G-representations with the G-fixed point sets V^G and W^G respectively. The group G is called a Borsuk-Ulam group (BUG) if whenever there is a G-isovariant map $\varphi : V \to W$, then the Borsuk-Ulam inequality

$$\dim V/V^G \leqq \dim W/W^G,$$

that is,

$$\dim V - \dim V^G \leqq \dim W - \dim W^G$$

holds.

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Wasserman conjectured that all compact Lie groups are BUGs, but it is still unknown whether this conjecture is true or not. Wasserman gave a sufficient condition called the prime condition for being a BUG. In our previous work [3], we proved that it is not necessary, that is, we showed there are infinitely many finite groups which does not satisfy it. For the proof, we introduced a new sufficient condition called the Möbius condition. On the other hand, by using Wassermann's results proved in [5], we can easily see that every solvable group is a BUG. Thus, since every finite group of odd order is a BUG by the Feit-Thompson theorem, we have to give an insight into the finite groups of even order for the study of BUGs.

Let $\mathrm{Syl}_p(G)$ denote a p-Sylow subgroup of a finite group G. In this paper, we present our new result on BUGs of even order, that is :

Theorem A . *A finite group G which satisfies one of the following conditions is a BUG.*

(1) $\mathrm{Syl}_2(G)$ *is a cyclic group* C_{2^r} *of order* 2^r, *where* r *is a positive integer.*
(2) $\mathrm{Syl}_2(G)$ *is a diheadral group* D_{2^r} *of order* 2^r, *where* r *is an integer* $\geqq 2$.
(3) $\mathrm{Syl}_2(G)$ *is a diheadral group* Q_{2^r} *of order* 2^r, *where* r *is an integer* $\geqq 3$.
(4) $\mathrm{Syl}_2(G)$ *is abelian and* $\mathrm{Syl}_p(G)$ *is cyclic for every odd prime* p.

Remark 1.2. In Theorem A (2), D_4 means $C_2 \times C_2$.

Some fundamental properties about BUGs are still unknown. For example, it is unknown whether every subgroup of a BUG is a BUG or not. We say that a Borsuk-Ulam group G is a strong Borsuk-Ulam group (SBUG), if every subgroup of G is a BUG. For this problem, we obtained the following result.

Theorem B . *A finite group G which satisfies one of the following conditions is a SBUG.*

(1) *G is solvable.*
(2) *G satisfies the prime condition.*
(3) *G satisfies one of the conditions in Theorem A.*

This paper is organized as follows. In section 2, we review some properties of BUGs from [5] and our previous paper. In section 3, we give a part of the proof of Theorem A and Theorem B.

We would like to dedicate this article to the memory of Professor Minoru Nakaoka, who was our supervisor in our graduate school days. The first author leaned singular homology theory and a part of homotopy theory and the second author leaned the Borsuk-Ulam theorem by his lecture at Osaka University.

2. The Borsuk-Ulam groups and the strong Borsuk-Ulam groups

In this section, we review the Borsuk-Ulam groups from [5]. Let G be a compact Lie group. Let V and W be G-representations with the G-fixed point sets V^G and W^G respectively. As is easy to show that there exists a G-isovariant map $\varphi : V \to W$ if and only if there exists a G-isovariant map $\varphi' : V/V^G \to W/W^G$. The Borsuk-Ulam group (BUG) is defined as follows.

Definition 2.1. We say that G is a *Borsuk-Ulam group (BUG)* if whenever there exists a G-isovariant map $\varphi : V \to W$, then $\dim V/V^G \leqq \dim W/W^G$, that is,

$$\dim V - \dim V^G \leqq \dim W - \dim W^G$$

holds.

Example 2.2. Any cyclic group of prime order is a BUG. In fact, let C_p be a finite cyclic group of prime order p. Then, V/V^{C_p} and W/W^{C_p} are free C_p-representations. Hence, if $p = 2$, $\dim V/V^{C_2} \leqq \dim W/W^{C_2}$ holds by the Borsuk-Ulam theorem. Since the Borsuk-Ulam theorem also holds between the spheres with free C_p-actions for any odd prime p ([2]), the inequality $\dim V/V^{C_p} \leqq \dim W/W^{C_p}$ also holds.

The following two properties are fundamental for constructing BUGs.

Lemma 2.3 ([5]). *Let G be a BUG. If H is a closed normal subgroup of G, then G/H is a BUG.*

Lemma 2.4 ([5]). *Let H and K be BUGs. If $1 \to H \to G \to K \to 1$ is an exact sequence of compact Lie groups, then G is a BUG.*

The following proposition is an immediate consequence of Example 2.2 and Lemma 2.4.

Proposition 2.5 ([5]). *Any solvable compact Lie group is a BUG.*

Wasserman introduced the prime condition for positive integers and finite groups. This condition is also necessary for understanding our Theorem B.

Definition 2.6 ([5]). (1) An integer n is said to satisfy the prime condition if $\displaystyle\sum_{i=1}^{s} \frac{1}{p_i} \leqq 1$ holds, where $n = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$ is the prime factorization of n.

(2) A finite simple group G is said to satisfy the prime condition if, for each $g \in G$, $|g|$ satisfies the prime condition.

(3) Let G be a finite group, and $\{e\} = G_0 \lhd G_1 \lhd \cdots \lhd G_r = G$ a composition series of G. A finite group G is said to satisfy the prime condition if each component factor G_{i+1}/G_i of G satisfies the prime condition.

This condition gives a sufficient condition for being a BUG. In fact the following lemmas hold.

Proposition 2.7 ([5]). *If a finite group G satisfies the prime condition, then G is a BUG.*

Besides determining BUGs, the problem whether every subgroup of a BUG is a BUG or not is essential. Then, we define a new class of the Borsuk-Ulam groups called strong Borsuk-Ulam groups. We say that a Borsuk-Ulam group G is a strong Borsuk-Ulam group (SBUG), if every subgroup of G is a BUG. As BUGs, the following two properties hold:

Proposition 2.8. *Let G be a SBUG. If H is a closed normal subgroup of G, then G/H is a SBUG.*

Proposition 2.9. *Let H and K be SBUGs. If $1 \to H \to G \to K \to 1$ is an exact sequence of compact Lie groups, then G is a SBUG.*

The proofs of these statements will be written in our forthcoming article.

3. Proofs

In this section, we prove that a group with a cyclic 2-Sylow subgroup is a BUG and a SBUG. The proofs of the other statements which needs some deep results of the finite groups theory will be written in our forthcoming article. For proving Theorem A (1), we use the following fact (see page 144 in [4]).

Lemma 3.1. *Let G be a finite group, p the smallest prime divisor of $|G|$. If p-Sylow subgroup P of G is cyclic, then G has a normal subgroup N such that $G/N \cong P$.*

Proof of Theorem A (1)

By Lemma 3.1, if $\mathrm{Syl}_2(G) \cong C_{2^r}$, there exists a normal subgroup N of odd order such that the sequence

$$1 \to N \to G \to C_{2^r} \to 1 \tag{1}$$

is exact. Since N and C_{2^r} are solvable, Lemmas 2.4 and 2.5 yield that G is a BUG.

□

Proof of Theorem B (3)-1

By applying Theorem B (1) and Proposition 2.9 to the exact sequence (1), we obtain the result.

Remark 3.2. If G is a finite simple group with cyclic 2-Sylow subgroup, then $\mathrm{Syl}_2(G)$ is isomorphic to C_2.

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