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AN EQUIVARIANT TRANSVERSALITY THEOREM AND ITS APPLICATIONS

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To the memory of the late Professor Minoru Nakaoka

Abstract. Let G be a finite group. In this article, we recall an equivariant transversality theorem and discuss its applications to semifree G-actions on closed manifolds and to Smith-equivalent real G-modules.

1. Introduction

Unless otherwise stated, let G be a finite group. We mean by a *manifold* a paracompact smooth manifold. A *submanifold*, M say, of a manifold, N say, should be read as a regular smooth submanifold such that M is a closed subset of N. We mean by a *G-manifold* a smooth manifold with a smooth G-action. In particular, each connected component of a manifold in the present article is σ-compact, and an arbitrary G-manifold can be equipped with a G-invariant Riemannian metric.

Let M and N be manifolds, B a subset of M, Y a submanifold of N, and $f : M \to N$ a continuous map. We say that f is *transversal on B to Y in N* if f is smooth on a neighborhood of $f^{-1}(Y) \cap B$ in M and the linear map

$$T_x(M) \xrightarrow{df_x} T_y(N) \longrightarrow T_y(N)/T_y(Y)$$

is surjective for every $y \in Y$ and $x \in f^{-1}(y) \cap B$, where $T_x(M)$ stands for the tangent space of M at x. There have been obtained several versions of equivariant transversality theorems, e.g. A. Wasserman [19, Lemma 3.3], T. Petrie [16, §1,

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p.188], E. Bierstone [1, Theorem 1.3]. In this paper we will discuss applications of the next version.

Theorem 1.1. *Let M be a G-manifold, N a G-manifold with a G-invariant Riemannian metric, A a G-invariant closed subset of M, and Y a G-submanifold of N. Let $f : M \to N$ be a smooth G-map transversal on A to Y in N. Suppose the G-action on $M \setminus A$ is free. Then for an arbitrary G-invariant positive continuous function $\delta : M \to \mathbb{R}$, there exists a smooth G-map $g : M \to N$ satisfying the following conditions.*

(1) *g is transversal on M to Y in N.*
(2) $g|_A = f|_A$.
(3) *$d_N(f(x), g(x)) < \delta(x)$ for all $x \in M$, where d_N stands for the distance function on N induced from the Riemannian metric of N.*

We mean by a real (resp. complex) G-module a real (resp. complex) G-representation space of finite dimension. For a real G-module V (of finite dimension), let $S(V)$ denote the unit sphere of V with respect to some G-invariant inner product on V. The following two propositions have been known.

Proposition ([14, Lemma 2.1]). *If G is a group of order 2 and M is a connected closed G-manifold of positive dimension then $|M^G| \neq 1$.*

Proposition ([7, Lemma 2.2]). *If M is a connected closed orientable G-manifold of positive dimension such that the G-action on $M \setminus M^G$ is free, then $|M^G| \neq 1$.*

The latter proposition is generalized to the next result.

Theorem 1.2. *Let M be a connected closed oriented G-manifold of dimension $n+1$, and Σ an oriented homotopy sphere of dimension n. Suppose the G-action on M is semifree and preserves the orientation of M. If M^G is a finite set then the congruence*

$$\sum_{x \in M^G} \deg(f_x) \equiv 0 \mod |G|$$

holds, where $T_x(M)$ is the tangent space of M at x and f_x is an arbitrary (continuous) G-map $S(T_x(X)) \to \Sigma$ for each $x \in M^G$.

Theorem 1.3. *Let M be a connected closed oriented G-manifold of positive dimension such that the G-action on M is semifree and $M^G = \{a, b\}$. Then the spheres $S(T_a(M))$ and $S(T_b(M))$ are G-homotopy equivalent to each other.*

Corollary 1.4. *Let V and W be real G-representations satisfying* $\dim V = \dim W$. *If $S(V) \amalg S(W)$ is the boundary of a compact orientable G-manifold M such that the G-action on M is free then $S(V)$ and $S(W)$ are G-homotopy equivalent to each other.*

A homotopy sphere Σ with a G-action is called a *Smith sphere* if Σ^G consists of exactly 2 points. Two real G-modules V and W are said to be *Smith equivalent* if there exists a Smith sphere Σ such that $\Sigma^G = \{a, b\}$ with $V \cong T_a(\Sigma)$ and $W \cong T_b(\Sigma)$ as real G-modules.

Theorem 1.5. *Let G be a finite group and let V and W be Smith-equivalent real G-modules. For any normal subgroup H of G such that $|G/H|$ is a prime and a Sylow 2-subgroup H_2 of H is normal in H, $S(V^H)$ and $S(W^H)$ are G/H-homotopy equivalent to each other.*

2. Transversality of maps

In this section, let us recall classical transversality theorems. First, recall a result by A. Wasserman.

Lemma (A. Wasserman [19, Lemma 3.3]). *Let G be a compact Lie group, M and N G-manifolds, $f : M \to N$ a smooth G-map, $W \subset N$ a closed invariant submanifold, and C a closed subset of M^G. Suppose $f|_{M^G}$ is transversal on C to W^G in N^G. Then there exists a homotopy f_t such that $f_0 = f$, $f_t|_C = f|_C$, and $f_1|_{M^G}$ is transversal on M^G to W^G in N^G.*

T. Petrie gave several versions and the next one may be most basic.

Proposition (Petrie [16, §1, p.188]). *Let G be a compact Lie group. Let M, N and $Y \subset N$ be G-manifolds and $f : M \to N$ a proper G-map. Suppose $f : M \to N$ is transversal to Y on a G-neighborhood of a closed subset Z of M. Let K be a maximal closed subgroup such that $(M \smallsetminus Z)^K \neq \emptyset$. Then f is G-homotopic rel Z to a proper G-map $h : M \to N$ such that h^K is transversal on M^K to Y^K in Z^K.*

As its proof, T. Petrie wrote as follows. (Note that $N(K)/K$ acts freely on $M^K \setminus Z$.) "This uses the Thom Transversality Lemma [11] for the case of trivial group action and the G-homotopy extension lemma [19, Lemma 3.2]." Here the reference [11] should be replaced by an appropriate one.

Another version is obtained by E. Bierstone's theory, namely from the following three results.

Theorem (Bierstone [1, Theorem 1.3]). *Let G be a compact Lie group. If P is a closed G-submanifold of N, then the set of smooth equivariant maps $F : M \to N$ which are in general position with respect to P is open in Whitney topology.*

Theorem ([1, Theorem 1.4]). *Let G be a compact Lie group. If P is an invariant submanifold of N, then the set of smooth equivariant maps $F : M \to N$ which are in general position with respect to P is a countable intersection of open dense sets (in the Whitney of C^∞ topology).*

Proposition ([1, Proposition 6.3]). *If a smooth equivariant map $F : M \to N$ is in general position with respect to an invariant submanifold P of N, then it is stratumwise transversal to P. In other words, for every isotropy subgroup H of M, $F|_{M^H} : M_H \to N^H$ is transversal to P^H.*

Our Theorem 1.1 is an equivariant analogue of A. Hattori [6, Ch.6, §3, Theorem 3.6].

3. Maps between spheres

We mean by a *homotopy sphere* a closed manifold being homotopy equivalent to a sphere. Let X be a finite G-CW complex such that G acts freely on X. For a G-map $f : X \to X$, the Lefschetz number $L(f)$ is congruent to 0 mod $|G|$. In the case where X is a homotopy sphere of dimension n, we have

$$L(f) = 1 + (-1)^n \deg f \equiv 0 \mod |G|. \tag{3.1}$$

Using this property, we can prove the next fact without difficulties.

Lemma 3.1 ([17, 4, 9]). *Let X be a connected homotopy sphere with a free G-action. Then for any G-map $f : X \to X$, $\deg f$ is congruent to 1 mod $|G|$.*

In addition, by standard arguments using Steenrod's obstruction theory [18], we can prove the next fact.

Lemma 3.2 ([17, 4]). *Let X and Y be connected homotopy spheres of same dimension with free G-actions. Then the following conclusions hold.*

(1) *There exist a G-map $X \to Y$ and a G-map $Y \to X$.*
(2) *For any G-maps f_0, $f_1 : X \to Y$, $\deg f_0 \equiv \deg f_1 \mod |G|$.*
(3) *For any G-map $f_0 : X \to Y$ and any integer m, there exists a G-map $f_1 : X \to Y$ such that $\deg f_1 = \deg f_0 + m|G|$.*

These lemmas provide the next proposition.

Proposition 3.3. *Let X and Y be connected homotopy spheres of same dimension with free G-actions and let $f : X \to Y$ be a G-map. Then $\deg f$ is prime to $|G|$.*

Proof. By Lemma 3.2, there is a G-map $g : Y \to X$. Moreover by Lemma 3.1 we have

$$\deg(g \circ f) \equiv 1 \mod |G|$$

and $\deg(g \circ f) = \deg g \cdot \deg f$. Thus $\deg f$ is prime to $|G|$. □

4. Tangential representations

Let V be a real G-module such that the G-action on $V \setminus \{0\}$ is free. We adopt an orientation of the ambient space of V. Let Σ be an oriented homotopy sphere equipped with a free smooth G-action such that $\dim \Sigma = \dim S(V)$. Then there exists a smooth G-map $f_{V,\Sigma} : S(V) \to \Sigma$ and $\deg(f_{V,\Sigma})$ is prime to $|G|$.

Proof of Theorem 1.2. Let us fix an arbitrary point $a \in M^G$. The tangential representation $V = T_a(M)$ has the orientation inherited from that of M. Since the G-action on $S(V)$ preserves the orientation, $\dim S(V)$ is odd. Without any loss of generality, we can assume $\Sigma = S(V)$. Set $Y = S(\mathbb{R} \oplus V)$. There is a canonical orientation preserving G-diffeomorphism from the G-disk $D(V)$ to the upper hemisphere S_+ of Y. This diffeomorphism carries the center of $D(V)$ to the north pole $p_+ = (1, 0)$ of Y, where $1 \in \mathbb{R}$ and $0 \in V$. Take small G-disk neighborhoods D_x ($\cong D(T_x(M))$) of points $x \in M^G$ in M, respectively, so that $D_{x_1} \cap D_{x_2} = \emptyset$

for distinct x_1, $x_2 \in M^G$. For each point $x \in M^G$, there is a smooth G-map $f_x : \partial D_x \to S(V) = \partial S_+$. Let $Df_x : D_x \to D(V) = S_+$ denote the radial extension of the map f_x. Clearly Df_x is transversal on a color neighborhood of ∂D_x to p_+ in Y. In addition, it holds that

$$\deg(Df_x : (D_x, \partial D_x) \to (S_+, \partial S_+)) = \deg(f_x : \partial D_x \to \partial S_+).$$

Set $X = M \setminus \coprod_{x \in M^G} \mathrm{Int}(D_x)$. Then the G-action on X is free. Since S_+ is contractible, the G-map $\coprod_{x \in M^G} Df_x$ extends to a continuous G-map $f : M \to S_+$ such that f is smooth on X. We will regard f as a map $M \to Y$ as well. For a G-invariant positive function $\delta : M \to \mathbb{R}$, take a G-equivariant δ-approximation $g : M \to Y$ of f such that

(1) g is G-homotopic to f relatively to $\coprod_{x \in M^G} D_x$, and
(2) $g|_X$ is smooth and transversal on X to $\{p_+\}$ in Y.

Since the G-action on $g^{-1}(p_+) \cap X$ is free, each G-orbit in $g^{-1}(p_+) \cap X$ consists of $|G|$ points. Thus it holds that

$$\deg(g) \equiv \sum_{x \in M^G} \deg(f_x : \partial D_x = S(T_x(M)) \to S(V)) \mod |G|.$$

On the other hand, the equality $\deg(g) = \deg(f) = 0$ follows from the fact that $f : M \to Y$ is not a surjection. Hence we can conclude

$$\sum_{x \in M^G} \deg(f_x) \equiv 0 \mod |G|.$$

□

Proof of Theorem 1.3. Set $V = T_a(M)$ and $W = T_b(M)$. Then G freely acts on $S(V)$ and $S(W)$.

If $|G| = 2$ then V and W are isomorphic as real G-representations, and hence $S(V)$ and $S(W)$ are G-diffeomorphic.

Thus we consider the other case, namely one where $|G| \geq 3$. The real G-modules V and W have the inherited orientations from that of M, respectively. Since the G-action on $S(V)$ preserves the orientation, so does the G-action on M. Let $f_{V,V}$ be the identity map on $S(V)$ and take a smooth G-map $f_{W,V} : S(W) \to S(V)$. By

Theorem 1.2, we get

$$\deg(f_{V,V}) + \deg(f_{W,V}) = 1 + \deg(f_{W,V}) \equiv 0 \mod |G|,$$

and hence $\deg(f_{W,V}) \equiv -1 \mod |G|$. Thus there is a smooth G-map $f : S(W) \to S(V)$ satisfying $\deg(f) = -1$. On the other hand, there exists a smooth G-map $h : S(V) \to S(W)$. We have $\deg(h \circ f) \equiv 1 \mod |G|$ and hence $\deg(h) \equiv -1 \mod |G|$. There exists a smooth G-map $g : S(V) \to S(W)$ such that $\deg(g) = -1$. These f and g are G-homotopy inverses to each other. □

Proof of Theorem 1.5. Set $\Sigma^G = \{a, b\}$. If a connected component A of Σ^H containing either a or b has positive dimension then by Proposition 1 A contains both a and b. By Theorem 1.3, $S(V^H)$ and $S(W^H)$ are G/H-homotopy equivalent. If $\dim V^H = 0$ and $\dim W^H = 0$ both hold then $S(V^H)$ and $S(W^H)$ are the empty set and hence they are G/H-homotopy equivalent. □

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